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ON THE IMPULSIVE TEMPERED Ξ–HILFER FUZZY FRACTIONAL DIFFERENTIAL EQUATIONS WITH DELAY

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Abstract. In the present paper, we investigate the existence and uniqueness of solutions and derive the Ulam-Hyers type stability results for impulsive tempered Ξ-Hilfer fuzzy fractional differential equations with delay. The Banach contraction principle and a Gronwall inequality involving tempered Ξ-Riemann-Liouville fuzzy fractional integral are used. In addition, we offer three examples to clarify the results.

1. Introduction

The differential equation of fractional order is an extension of the differential equation of integer order. By the end of the sixteenth century (1695), the concept of fractional calculus (FC) was introduced. A generalisation of ordinary differentiation and integration to arbitrary order (non-integer) is known as FC. The advantages of fractional derivatives (FDs) are demonstrated in many other domains, such as the description of gases, liquids, minerals, and the mechanical and electrical properties of real materials, see [27, 47]. The FD is employed as a global operator in a variety of fields, including mathematics, physics, dynamics, fluid mechanics, control theory, chemistry, and mathematical biology, to simulate different processes and physical systems [4, 7, 14, 24]. It turns out that when compared to differential equations of integer order, fractional differential equations (FDEs) are more suited to explain real-world issues. Due to its importance and large number of applications, this area has attracted the attention of numerous mathematicians and researchers in the past few decades.

The generalised FD is a powerful tool for simulating complex real world problems due to its increased precision. An extension of the Riemann-Liouville (R-L) and Caputo fractional derivatives is the Hilfer fractional derivative (HFD) [14]. The ψ-Hilfer fractional derivative (ψ-HFD) with respect to another function was provided by Sousa and Oliveira [41].The freedom to choose the differentiation operator and the kernel function ψ is the advantage of the ψ-HFD.

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The generalised version of fractional calculus is known as tempered FC. When the FDs and integrals are multiplied by an exponential factor, the tempered FDs and integrals are obtained in [31, 40]. In [11, 26, 33, 35], the authors studied the existence, uniqueness, and stability for FDEs involving tempered FDs with respect to functions and established many results on the analytical theory in both R-L and Caputo types.

Recently, fuzzy fractional and differential calculus has been extensively developed across various domains. Bede et al. [8, 9] have demonstrated numerous applications of this calculus. This theory, along with the advantages of FC, has garnered significant attention from mathematicians, leading to research studies on existence, uniqueness, and stability. For instance, Allahiranloo et al. [5] used generalised Hukuhara Caputo differentiability to investigate the existence and uniqueness of solutions for fuzzy fractional differential equations (FFDEs). Mazandarani and Najariyan [34] focused on fundamental theories for numerical investigations. Ahmadian et al. [3] explored fuzzy solutions for fractional differential systems. Additionally, Hoa [15] investigated various results related to FFDEs with delays. The authors in [16] presented several fundamental theories on different initial value problems for implicit FFDEs, which involve various fuzzy FDs. For more recent developments on these concepts, [2, 17, 18, 19, 39] and the references therein include essential findings in the theory of FFDEs and methods for investigating their numerical solutions. Existence and uniqueness results for Hilfer-type FFDEs was studied by many authors [10, 21, 38] using successive approximation techniques. Vivek et al. [44] investigate the existence and stability of Ξ-Hilfer FFDEs with boundary conditions.

One of the very important branches of the fuzzy theory is fuzzy impulsive FDEs. As an advantage of the fuzzy impulsive FDE is, they are able to describe the solution of the model at the certain moments with more sharpness and rapid changes in their states. While the classical differential equations are not able to describe the mentioned behaviour. Fuzzy impulsive FDEs are usually hard to analytically and the exact solution is rather difficult to be obtained. But the idea of fuzzy impulsive FDEs has been studied by scientists and engineers such as Najafi and Allahviranloo [37]. The existence and stability outcomes for Ξ-Hilfer FFDEs with impulses is studied by Vivek et al [45].

A delay differential equation (DDE) is a type of differential equation where the derivative of an unknown function at a given time depends on its values at previous times. DDEs are widely used in numerous system models across physics, biology, and engineering. Furthermore, they are applied in various practical systems, including automatic control, traffic models, neuroscience, and lasers. References [13] and [25] provide a comprehensive study of DDEs and their applications.

On the other hand, modelling a dynamical phenomenon with a delay using ordinary DDEs are not always accurate. In general, the initial conditions or parameters of the equations are incomplete or vague. Fuzzy DDEs are studied as a result of this limitation. Numerous researchers have developed the theory and applications of fuzzy delay differential equations in recent years. Lupulescu introduced a DDE with a fuzzy case [32]. As a result, Khastan et al. [23] and Hoa et al. [20] have presented findings on the existence and uniqueness of solutions for generalized fuzzy DDEs. The existence and finite-time stability of Hilfer FFDEs with delay were studied in [6, 42, 43].

Stability analysis is a fundamental aspect of mathematical analysis that plays a crucial role in various engineering and science fields. Consequently, numerous researchers have investigated different types of stability problems for various fractional differential equations and FFDEs using different methods [1, 30, 36, 46, 48].

Motivated by [22, 28, 29], we study the existence, uniqueness, and Ulam-type stability of the following tempered Ξ-Hilfer-type fuzzy fractional impulsive differential equations with delay of the form:

$$\begin{cases} {}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)} v(t) = G(t,v_t), \quad t \in (0,T], \quad t \neq t_l, \quad l = 1,2,\cdots,m, \\ \Delta v(t_l) = \phi_l(v(t_l^-)), \quad l = 1,2,\cdots,m, \\ {}^{T}_{0^+}\mathscr{I}^{1-\gamma,\lambda}_{\Xi(t)} v(0) = v_0, \quad \gamma = p+q-pq, \\ v(t) = \rho(t), \quad t \in [-\sigma,0], \end{cases} \tag{1.1}$$

where $T > 0$, ${}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}(.)$ and ${}^{T}_{0^+}\mathscr{I}^{1-\gamma,\lambda}_{\Xi(t)}(.)$ are tempered Ξ-Hilfer fractional derivative (Ξ-HFD) of order $0 < p < 1$, type $0 \leqslant q \leqslant 1$, index $\lambda \in \mathbb{R}$, and tempered Ξ- type R-L fractional integral of order $1-\gamma$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $v(t_l) = v(t_l^+) - v(t_l^-)$, $v(t_l^+) = \lim\limits_{\omega \to 0^+} v(t_l + \omega)$ and $v(t_l^-) = \lim\limits_{\omega \to 0^-} v(t_l + \omega)$ represent the right and left limits of v at $t = t_l$, $l = 1,2,\cdots,m,$. The fuzzy functions $G : (0,T] \times E \to E$ and $\phi : E \to E$ are appropriate functions specified latter, where E is the space of fuzzy number. Further, $\rho \in C_\sigma := C([-\sigma,0],E)$-the space of all continuous fuzzy functions from $[-\sigma,0]$ to E. For each v defined on $[-\sigma,T]$ and for any $t \in [0,T]$ we denote by v_t the element of C_σ defined by $v_t(s) = w(t+s)$, $-\sigma \leqslant s \leqslant 0$, where $v_t(.)$ denotes the history of the state from time $-\sigma$ up to time t.

In this work, the existence and uniqueness of tempered Hilfer FFDE are provided. The Ulam-Hyers type stable for the proposed problem with the aid of the novel features of generalized Gronwall inequality is simulated. The main contribution of the investigation are complied as follows:

- To investigate the existence and uniqueness of the solution of impulsive tempered Hilfer fuzzy fractional DDE via Banach contraction principle.

- To investigate the Ulam-Hyers type stable of impulsive tempered Hilfer fuzzy fractional DDE by using the Gronwall inequality.

The following is the structure of the paper. Section 2 is related to some definitions, lemmas, and theorems. In Section 3, the main results of the paper are discussed. In Section 4, we give the existence and uniqueness of the solution for the problem (1.1). In Section 5, Ulam-type stability results for the proposed problem are established. In Section 6, we analyze the Ulam-Hyers-Mittag-Leffler stability of (1.1). In Section 7, we give three illustrative examples employing the tempered Caputo and R-L FD and comparing the outcomes obtained. Conclusion is drawn in Section 8.

2. Preliminaries

In this section, we provide some definitions, theorems, and properties that will be useful throughout this work.

Take $\mathscr{J} = (0,T]$ and $\mathscr{J}^* = [0,T]$. We assume that $\Xi \in C^1(\mathscr{J}^*, E)$ be an increasing function with $\Xi'(t) \neq 0$, for all $t \in \mathscr{J}^*$. The space of the continuous fuzzy function v on $\mathscr{J}^*$ with the norm is defined by

$$\|v\|_{C(\mathscr{J}^*,E)} = \max_{t\in\mathscr{J}^*} D_0[v(t), \widehat{0}],$$

where the Hausdroff distance D_0 is given by

$$D_0[v_1, v_2] = \sup_{\beta\in[0,1]} \{|\underline{v}_1(\beta) - \underline{v}_2(\beta)|, |\overline{v}_1(\beta) - \overline{v}_2(\beta)|\}, \quad v_1, v_2 \in E.$$

The weighted space $C_{1-\gamma,\Xi}(\mathscr{J}^*, E)$ of continuous v on $\mathscr{J}$ is defined by

$$C_{1-\gamma,\Xi}(\mathscr{J}^*,E) = \{h : \mathscr{J} \to E | (\Xi(t) - \Xi(a))^{1-\gamma} v(t) \in C(\mathscr{J}^*, E)\}, \quad 0 \leqslant \gamma < 1,$$

with the norm

$$\|v\|_{C_{1-\gamma,\Xi}(\mathscr{J}^*,E)} = \max_{t\in\mathscr{J}^*} D_0[(\Xi(t) - \Xi(a))^{1-\gamma} v(t), \widehat{0}].$$

Denote by $PC_{1-\gamma,\Xi}(\mathscr{J}^*,E)$ the space of piecewise continuous fuzzy functions as follows:

$$\begin{aligned} PC_{1-\gamma,\Xi}(\mathscr{J}^*,E) = \{ & v : \mathscr{J} \to E : v \in C_{1-\gamma,\Xi}((t_l,t_{l+1}],E), \quad l = 0,1,\cdots,m, \\ & v(t_l^+), v(t_l^-) \quad \text{exists and} \quad v(t_l^-) = v(t_l) \quad \text{for} \quad l = 1,2,\cdots,m\}. \end{aligned}$$

with respect to the uniform norm

$$\|v\|_{PC_{1-\gamma,\Xi}(\mathscr{J}^*,E)} = \sup_{t\in\mathscr{J}^*} D_0[(\Xi(t) - \Xi(0))^{1-\gamma} v(t), \widehat{0}].$$

Let $L(\mathscr{J}^*,E)$ be the set of all fuzzy functions $v; \mathscr{J}^* \to E$ such that $t \longmapsto D_0[v(t), \widehat{0}]$ belong to $L^1(\mathscr{J}^*)$.

DEFINITION 1. [39] Let $\mathscr{M}(\mathbb{R}^d)$ be the family of all nonempty, compact and convex subsets of $\mathbb{R}^d$. The addition and scalar multiplication in $\mathscr{M}(\mathbb{R}^d)$ are defined as usual, i.e., for $\mathscr{A}, \mathscr{B} \in \mathscr{M}(\mathbb{R}^d)$ and $\tau \in \mathbb{R}$,

$$\mathscr{A} + \mathscr{B} = \{a + b | a \in \mathscr{A}, b \in \mathscr{B}\}, \quad \tau\mathscr{A} = \{\tau a | a \in \mathscr{A}\}.$$

The Hausdorff distance d_H in $\mathscr{M}(\mathbb{R}^d)$ is defined as follows:

$$d_H(\mathscr{A},\mathscr{B}) = \max\Big\{\sup_{y\in\mathscr{A}} \inf_{z\in\mathscr{B}} \|y - z\|_{\mathbb{R}^d}, \sup_{z\in\mathscr{B}} \inf_{y\in\mathscr{A}} \|y - z\|_{\mathbb{R}^d}\Big\},$$

where $\|.\|_{\mathbb{R}^d}$ represents the usual Euclidean norm in R^d. It is known that $\mathscr{M}(\mathbb{R}^d)$ is a complete, separable and locally compact metric space with respect to d_H.

DEFINITION 2. [39] Let E denote the set of fuzzy subsets of the real axis, if $v:\mathbb{R}\to[0,1]$, satisfying the following conditions:

(1) v is normal, that is, there exists $y_0\in\mathbb{R}$ such that $v(y_0)=1$;

(2) v is fuzzy convex, that is, for $\tau\in[0,1]$

$$v(\tau y_1+(1-\tau)y_2)\geqslant\min\{v(y_1,v(y_2))\},\quad\text{for any}\quad y_1,y_2\in\mathbb{R};$$

(3) v is upper semicontinuous on $\mathbb{R}$;

(4) $[v]^0=cl\{y\in\mathbb{R}:v(y)>0\}$ is compact, where cl denotes the closure in $(\mathbb{R},|.|)$.

Then E is called the space of fuzzy numbers.

Let $v\in E$. Then for each $\beta\in(0,1]$, the set

$$[v]^\beta=\{y\in\mathbb{R};\quad v(y)\geqslant\beta\},$$

is called the β-level set of v.

DEFINITION 3. [17] Let $v_1,v_2\in E$ and $\tau\in\mathbb{R}$. Based on Zadeh's extension principle, v_1+v_2 and $\tau.v_1$ are in E and defined as follows:

$$\begin{aligned}[v_1+v_2]^\beta&=[v_1]^\beta+[v_2]^\beta,\\ [\tau v]^\beta&=\tau[v]^\beta,\quad\text{for all}\quad\beta\in[0,1],\end{aligned}$$

where $[v_1]^\beta+[v_2]^\beta$ indicates that two intervals of $\mathbb{R}$ have been added and $\tau[v_1]^\beta$ is the product from a scalar to a real interval number. The diameter of the β-level set of $v\in E$ is presented by $d([v]^\beta)=\overline{v}(\beta)-\underline{v}(\beta)$.

DEFINITION 4. [17] The distance $D_0[v_1,v_2]$ between two fuzzy numbers is presented as follows:

$$D_0[v_1,v_2]=\sup_{\beta\in[0,1]}\{d_H([v_1]^\beta,[v_2]^\beta)\},\quad v_1,v_2\in E,$$

where $d_H([v_1]^\beta,[v_2]^\beta)=\max\left\{|\underline{v_1}(\beta)-\underline{v_2}(\beta)|,|\overline{v_1}(\beta)-\overline{v_2}(\beta)|\right\}$ is the Hausdorff distance between $[v_1]^\beta,[v_2]^\beta$.

DEFINITION 5. [9] The generalized Hukuhara difference (gH-difference) of two fuzzy numbers $v_1,v_2\in E$ is given by

$$v_1\ominus_{gH}v_2=v_3\iff\begin{cases}(i)\quad v_1=v_2+v_3,\quad\text{or}\\(ii)\quad v_2=v_1+(-1)v_3.\end{cases}$$

DEFINITION 6. [17] Let $v:[a,b]\to E$ be a fuzzy function. Then, for each $\beta\in[0,1]$, the function $t\longmapsto d([v(t)]^{\beta})$ is non-decreasing (non-increasing) on $[a,b]$, we say that v is d-increasing (d-decreasing) on $[a,b]$. If v is d-increasing or d-decreasing on $[a,b]$, then v is called d-monotone on $[a,b]$.

DEFINITION 7. [9] Let $v:(a,b)\to E$ and $t\in(a,b)$. We say that the fuzzy function v is generalized Hukuhara differentiable (gH-differentiable) at t if there exists an element $v'_{gH}(t)\in E$ such that the following statement is true:

$$v'_{gH}(t_0)=\lim_{h\to 0}\frac{v(t_0+h)\ominus_{gH}v(t_0)}{h},$$

if $v'_{gH}(t_0)\in E$, we say that v is generalized Hukuhara differentiable (gH-differentiable) at t_0.

Moreover, we say that v is $[(i)-gH]$-differentiable at t_0 if

$$\begin{aligned}[v'_{gH}(t_0)]^{\beta}&=\left[\left[\lim_{h\to 0}\frac{\underline{v}(t_0+h)\ominus_{gH}\underline{v}(t_0)}{h}\right]^{\beta},\left[\lim_{h\to 0}\frac{\overline{v}(t_0+h)\ominus_{gH}\overline{v}(t_0)}{h}\right]^{\beta}\right]\\&=[(\underline{v})'(\beta,t_0),(\overline{v})'(\beta,t_0)],\end{aligned}$$

and that v is $[(ii)-gH]$-differentiable at t_0 if

$$[v'_{gH}(t_0)]^{\beta}=[(\overline{v})'(\beta,t_0),(\underline{v})'(\beta,t_0)].$$

DEFINITION 8. [33] Let $p\in(0,1)$ and $\lambda\in\mathbb{R}$. Then the tempered Ξ-type R-L fractional integral of v is defined by

$$\begin{aligned}({}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)}v)(t)&=e^{-\lambda(\Xi(t)-\Xi(0))}{}_{0^+}\mathscr{I}^{p}_{\Xi(t)}(e^{\lambda(\Xi(t)-\Xi(0))}v(t))\\&=\frac{1}{\Gamma(p)}\int_0^t\Xi'(s)(\Xi(t)-\Xi(s))^{p-1}e^{-\lambda(\Xi(t)-\Xi(s))}v(s)ds,\quad t\in\mathscr{J}^*.\end{aligned}$$

DEFINITION 9. [33] The tempered Ξ-type R-L fractional derivative of v is defined by

$$\begin{aligned}\left({}^{RL}_{0^+}\mathscr{D}^{p,\lambda}_{\Xi(t)}v\right)(t)&=e^{-\lambda(\Xi(t)-\Xi(0))}{}^{RL}_{0^+}\mathscr{D}^{p}_{\Xi(t)}(e^{\lambda(\Xi(t)-\Xi(0))}v(t))\\&=(e^{-\lambda(\Xi(t)-\Xi(0))})\left(\frac{1}{\Xi'(t)}\frac{d}{dt}\right){}_{0^+}\mathscr{I}^{1-p}_{\Xi(t)}(e^{\lambda(\Xi(t)-\Xi(0))}v(t)).\end{aligned}$$

For $v\in L(\mathscr{J}^*,E)$, we define the tempered Ξ-type R-L fractional integral of order p and index λ of the fuzzy function v:

$$v_{p,\Xi}(t):={}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)}v(t)=\frac{1}{\Gamma(p)}\int_0^t\Xi'(s)(\Xi(t)-\Xi(s))^{p-1}e^{-\lambda(\Xi(t)-\Xi(s))}v(s)ds.$$

Since $[v(t)]^{\beta}=[\underline{v}(t,\beta),\overline{v}(t,\beta)]$, we can define the tempered Ξ-type fuzzy R-L fractional integral of fuzzy function v based on lower and upper functions

$$[{}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)}v(t)]^{\beta}=[{}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)}\underline{v}(t,\beta),{}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)}\overline{v}(t,\beta)],\quad t\geqslant 0,$$

where

$${}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)}\underline{v}(t,\beta)=\frac{1}{\Gamma(p)}\int_0^t\Xi'(s)(\Xi(t)-\Xi(s))^{p-1}e^{-\lambda(\Xi(t)-\Xi(s))}\underline{v}(s,\beta)ds,$$

and

$${}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)}\overline{v}(t,\beta)=\frac{1}{\Gamma(p)}\int_0^t\Xi'(s)(\Xi(t)-\Xi(s))^{p-1}e^{-\lambda(\Xi(t)-\Xi(s))}\overline{v}(s,\beta)ds,$$

It follows that the operator $v_{p,\Xi}(t)$ is linear and bounded from $C(\mathscr{J}^*,E)$ to $C(\mathscr{J}^*,E)$.

DEFINITION 10. [26] The tempered fuzzy Ξ-HFD of order $p\in(n-1,n)$, type $q\in[0,1]$ and index $\lambda\in\mathbb{R}$ of a function v is defined by

$$\begin{aligned}{}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}v(t)&={}^{T}_{0^+}\mathscr{I}^{q(1-p),\lambda}_{\Xi(t)}\left(\frac{1}{\Xi'(t)}\cdot\frac{d}{dt}+\lambda\right)^n{}^{T}_{0^+}\mathscr{I}^{(1-q)(1-p),\lambda}_{\Xi(t)}v(t)\\&=e^{-\lambda(\Xi(t)-\Xi(0))}{}^{H}_{0^+}\mathscr{D}^{p,q}_{\Xi(t)}(e^{\lambda(\Xi(t)-\Xi(0))}v(t)),\end{aligned}$$

where ${}^{H}_{0^+}\mathscr{D}^{p,q}_{\Xi(t)}(.)$ represents the usual Ξ-HFD. If the gH-derivative $v'_{1-p,\Xi}(t)$ exists for $t\in\mathscr{J}^*$, where

$$\begin{aligned}v_{1-p,\Xi}(t)&={}^{T}_{0^+}\mathscr{I}^{1-p,\lambda}_{\Xi(t)}v(t)\\&=\frac{1}{\Gamma(1-p)}\int_0^t\Xi'(s)(\Xi(t)-\Xi(s))^{-p}e^{-\lambda(\Xi(t)-\Xi(s))}v(s)ds.\end{aligned}$$

REMARK 1. We note that, if $\lambda=0$ and $\Xi(t)=t$ in Definitions 8–9, then ${}_{0^+}\mathscr{I}^{p}_{\Xi(t)}$ and ${}_{0^+}\mathscr{D}^{p}_{\Xi(t)}$ reduces to the classical fractional integral and derivative.

REMARK 2. In view of Definition 10, we observe that:

- If $q=0$, then ${}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}$ reduces to ${}^{RL}_{0^+}\mathscr{D}^{p,\lambda}_{\Xi(t)}$.
- If $q=1$, then ${}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}$ reduces to ${}^{C}_{0^+}\mathscr{D}^{p,\lambda}_{\Xi(t)}$.
- If $\lambda=0$, then ${}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}$ reduces to ${}^{H}_{0^+}\mathscr{D}^{p,q}_{\Xi(t)}$.
- If $q=0,\ \lambda=0$, then ${}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}$ reduces to ${}^{RL}_{0^+}\mathscr{D}^{p}_{\Xi(t)}$.
- If $q=1,\ \lambda=0$ then ${}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}$ reduces to ${}^{C}_{0^+}\mathscr{D}^{p}_{\Xi(t)}$.

- If $\Xi(t)=t$, then ${}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}$ reduces to ${}^{H}_{0^+}\mathscr{D}^{p,q,\lambda}_{t}$.
- If $q=0$, $\Xi(t)=t$ then ${}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}$ reduces to ${}^{RL}_{0^+}\mathscr{D}^{p,\lambda}_{t}$.
- If $q=1$, $\Xi(t)=t$, then ${}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}$ reduces to ${}^{RL}_{0^+}\mathscr{D}^{p,\lambda}_{t}$.
- If $\Xi(t)=t$, $\lambda=0$, then ${}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}$ reduces to ${}^{H}_{0^+}\mathscr{D}^{p,q}_{t}$.
- If $q=0$, $\Xi(t)=t$, $\lambda=0$, then ${}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}$ reduces to ${}^{RL}_{0^+}\mathscr{D}^{p,\lambda}_{t}$.
- If $q=1$, $\Xi(t)=t$, $\lambda=0$, then ${}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}$ reduces to ${}^{C}_{0^+}\mathscr{D}^{p,\lambda}_{t}$.

LEMMA 1. [41] *Let $p>0$, $q>0$, $\gamma=p+q-pq$, $\lambda\in\mathbb{R}$ and $v\in C^{\gamma}_{1-\gamma,\Xi}(\mathscr{J}^*,E)$. Then*

$$ {}^{T}_{0^+}\mathscr{I}^{\gamma,\lambda}_{\Xi(t)}\,{}^{T}_{0^+}\mathscr{D}^{\gamma,\lambda}_{\Xi(t)}v={}^{T}_{0^+}\mathscr{I}^{\gamma,\lambda}_{\Xi(t)}\,{}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}v, $$

and

$$ {}^{T}_{0^+}\mathscr{D}^{\gamma,\lambda}_{\Xi(t)}\,{}^{T}_{0^+}\mathscr{I}^{\gamma,\lambda}_{\Xi(t)}v={}^{T}_{0^+}\mathscr{D}^{q(1-p),\lambda}_{\Xi(t)}v. $$

Here we note that $C^{\gamma}_{1-\gamma,\Xi}(\mathscr{J}^*,E)\subset C^{p,q}_{1-\gamma,\Xi}(\mathscr{J}^*,E)$

LEMMA 2. [41] *Let $v\in C^{1}(\mathscr{J}^*,E)$, $0<p<1$, $0\leqslant q\leqslant 1$ and $\mu\in\mathbb{R}$. Then*

(i) ${}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)}\,{}^{T}_{0^+}\mathscr{I}^{p_1,\lambda}_{\Xi(t)}v(t)={}^{T}_{0^+}\mathscr{I}^{p+p_1,\lambda}_{\Xi(t)}v(t)$,

(ii) ${}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}\,{}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)}v(t)=v(t)$.

(iii) ${}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}(\Xi(t)-\Xi(0))^{\eta-1}=\widehat{0},\quad 0<\eta<1$.

LEMMA 3. [26] *Let $p,\eta>0$. Then*

$$ {}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)}(\Xi(t)-\Xi(0))^{\eta-1}=\frac{\Gamma(\eta)}{\Gamma(p+\eta)}(\Xi(t)-\Xi(0))^{p+\eta-1}e^{-\lambda(\Xi(t)-\Xi(0))},\quad t>0. $$

LEMMA 4. [41] *Let $0<p<1$, $0\leqslant q\leqslant 1$, $\lambda\in\mathbb{R}$ $v\in C_{1-\gamma,\Xi}(\mathscr{J}^*,E)$ and ${}^{T}_{0^+}\mathscr{I}^{1-\gamma,\lambda}_{\Xi(t)}v\in C^{1}_{1-\gamma,\Xi}(\mathscr{J}^*,E)$. Then, we have*

$$ \left({}^{T}_{0^+}\mathscr{I}^{\gamma,\lambda}_{\Xi(t)}\,{}^{T}_{0^+}\mathscr{D}^{\gamma,\lambda}_{\Xi(t)}v\right)(t)=v(t)\ominus_{gH}\frac{{}^{T}_{0^+}\mathscr{I}^{1-\gamma,\lambda}_{\Xi(t)}v(0)}{\Gamma(\gamma)}(\Xi(t)-\Xi(0))^{\gamma-1}e^{-\lambda(\Xi(t)-\Xi(0))} $$

REMARK 3. If $v(t) = (u_1(t), u_2(t), u_3(t))$ is a triangular fuzzy number valued function, then

(i) If v is $[(i)-gH]$-differentiable at $t \in \mathscr{J}^*$ then

$$\left({}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}v\right)(t) = \left({}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}u_1(t), {}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}u_2(t), {}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}u_3(t)\right). \tag{2.1}$$

(ii) If v is $[(ii)-gH]$-differentiable at $t \in \mathscr{J}^*$ then

$$\left({}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}v\right)(t) = \left({}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}u_3(t), {}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}u_2(t), {}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}u_1(t)\right). \tag{2.2}$$

DEFINITION 11. Let $v : \mathscr{J}^* \to E$ be $[(i)-gH]^{TH}_{p,q,\lambda}$-differentiable at $t \in \mathscr{J}^*$. We say that v is $[(i)-gH]^{TH}_{p,q,\lambda}$-differentiable at $t \in \mathscr{J}^*$ if

$$\left[\left({}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}v\right)(t)\right]^\beta = \left[{}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}\underline{v}(t,\beta), {}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}\overline{v}(t,\beta)\right], \quad \beta \in [0,1]$$

and that v is $[(ii)-gH]^{TH}_{p,q,\lambda}$-differentiable at t if

$$\left[\left({}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}v\right)(t)\right]^\beta = \left[{}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}\overline{v}(t,\beta), {}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}\underline{v}(t,\beta)\right], \quad \beta \in [0,1].$$

where ${}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}\underline{v}(t,\beta)$ and ${}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}\overline{v}(t,\beta)$ defined in Definition 10.

DEFINITION 12. Let $w : \mathscr{J}^* \to E$ be a fuzzy function. A point $t \in (0,b)$ is said to be a switching point for the $[gH]^{TH}_{p,q,\lambda}$-differentiability of w, if in any neighborhood $\mathscr{V}$ of t there exist points $t_1 < t < t_2$ such that

type (I) at t_1 (2.1) holds while (2.2) does not hold and at t_2 (2.2) holds and (2.1) does not holds, or

type (II) at t_1 (2.2) holds while (2.1) does not hold and at t_2 (2.1) holds and (2.2) does not holds.

THEOREM 1. [12] (Banach's contraction principle) *Let S be a Banach space and I be a non-empty closed subset of S. If $T : I \to I$ is a contraction, then there exists a unique fixed point of T.*

THEOREM 2. [22] (Gronwall's Inequality) *Let $V \in PC_{1-\gamma,\Xi}(\mathscr{J}^*, \mathbb{R})$ satisfy the following inequality:*

$$V(t) \leqslant W(t) + g(t)\int_0^t \Xi'(s)(\Xi(t)-\Xi(s))^{p-1}e^{-\lambda(\Xi(t)-\Xi(s))}V(s)ds + \sum_{0<t_l<t} q_l V(t_l^-), \ \ t > 0,$$

where g is a continuous function, $W \in PC_{1-\gamma,\Xi}(\mathscr{J}^*, \mathbb{R})$ is nonnegative, $q_l > 0$ for $l = 1, 2, \cdots, m$. Then, we have

$$V(t) \leqslant W(t)\left[\prod_{i=1}^{l}\left\{1 + q_l E_p\Big(g(t)\Gamma(p)(\Xi(t_i) - \Xi(0))^p\Big)\right\}\right]$$
$$\times E_p\Big(g(t)\Gamma(p)(\Xi(t) - \Xi(0))^p\Big), \quad t \in (t_l, t_{l+1}].$$

3. Main results

LEMMA 5. *Let* $0 < p < 1$ *and* $G : \mathscr{J}^* \to E$ *be continuous. A fuzzy function* $v \in C_{1-\gamma,\Xi}(\mathscr{J}^*, E)$ *is a solution of the fuzzy fractional integral equation*

$$v(t) = \left[\frac{v_0 \ominus {}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)} G_a}{(\Xi(a) - \Xi(0))^{\gamma-1} e^{-\lambda(\Xi(a)-\Xi(0))}}\right] (\Xi(t) - \Xi(0))^{\gamma-1} e^{-\lambda(\Xi(t)-\Xi(0))} + {}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)} G(t, v_t), \tag{3.1}$$

if and only if, v *is the solution of the following Cauchy problem*

$$\begin{cases} {}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)} v(t) = G(t, v_t), & t \in \mathscr{J}^*, \\ v(a) = v_0, & a > 0, \end{cases}$$

in which ${}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)} G_a = {}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)} G(t, v_t)|_{t=a}$.

Proof. Applying ${}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}$ on both sides of the Eqn. (3.1). Then using Lemma 4, we arrive at

$$v(t) - c_1(\Xi(t) - \Xi(0))^{\gamma-1} e^{-\gamma(\Xi(t)-\Xi(0))} = {}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)} G(t, v_t), \tag{3.2}$$

where c_1 is an arbitrary constant. Thus (3.2) reduces to

$$v(t) = c_1(\Xi(t) - \Xi(0))^{\gamma-1} e^{-\gamma(\Xi(t)-\Xi(0))} + {}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)} G(t, v_t). \tag{3.3}$$

In (3.3), the boundary condition $v(a) = v_0$ leads to

$$c_1 = \left[\frac{v_0 \ominus {}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)} G_a}{(\Xi(a) - \Xi(0))^{\gamma-1} e^{-\lambda(\Xi(a)-\Xi(0))}}\right].$$

We insert c_1 in (3.3), we get (3.1).

On the other hand, assume v satisfying (3.1), taking ${}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}$ on both sides of (3.1), and using Lemma 4 and Lemma 2, we get

$${}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)} v(t) = G(t, v_t).$$

Hence, the proof is complete. □

LEMMA 6. *A fuzzy function* $v : [-\sigma, T] \to E$ *is a solution to problem* (1.1) *on* $[-\sigma, T]$ *if and only if* v *is a continuous fuzzy function and it satisfies to one of the following fuzzy fractional integral equations:*

(L1) *If v is $[(i)-gH]^{TH}_{p,q,\lambda}$ differentiable, then*

$$v(t)=\begin{cases}\rho(t) \quad t\in[-\sigma,0],\\ \dfrac{e^{-\lambda(\Xi(t)-\Xi(0))}(\Xi(t)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)}v_0+{}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)}G(t,v_t), \quad t\in[0,t_1]\\ \left(\dfrac{v_0}{\Gamma(\gamma)}+\dfrac{\phi_l(v(t_1^-))}{e^{-\lambda(\Xi(t_1)-\Xi(0))}(\Xi(t_1)-\Xi(0))^{\gamma-1}}\right)e^{-\lambda(\Xi(t)-\Xi(0))}(\Xi(t)-\Xi(0))^{\gamma-1}\\ \quad +{}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)}G(t,v_t), \quad t\in(t_1,t_2],\\ \quad\vdots\\ \left(\dfrac{v_0}{\Gamma(\gamma)}+\sum\limits_{0<t_l<t}\dfrac{\phi_l(v(t_l^-))}{e^{-\lambda(\Xi(t_l)-\Xi(0))}(\Xi(t_l)-\Xi(0))^{\gamma-1}}\right)e^{-\lambda(\Xi(t)-\Xi(0))}(\Xi(t)-\Xi(0))^{\gamma-1}\\ \quad +{}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)}G(t,v_t), \quad t\in(t_l,t_{l+1}].\end{cases} \tag{3.4}$$

(L2) *If v is ${}^{TH}[(ii)-gH]^{TH}_{p,q,\lambda}$ differentiable in the sense that the Hukuhara difference exist, then*

$$v(t)=\begin{cases}\rho(t) \quad t\in[-\sigma,0]\\ \dfrac{e^{-\lambda(\Xi(t)-\Xi(0))}(\Xi(t)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)}v_0\ominus(-1)^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)}G(t,v_t), \quad t\in[0,t_1]\\ \left(\dfrac{v_0}{\Gamma(\gamma)}+\dfrac{\phi_l(v(t_1^-))}{e^{-\lambda(\Xi(t_1)-\Xi(0))}(\Xi(t_1)-\Xi(0))^{\gamma-1}}\right)e^{-\lambda(\Xi(t)-\Xi(0))}(\Xi(t)-\Xi(0))^{\gamma-1}\\ \quad \ominus(-1)^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)}G(t,v_t), \quad t\in(t_1,t_2],\\ \quad\vdots\\ \left(\dfrac{v_0}{\Gamma(\gamma)}+\sum\limits_{0<t_l<t}\dfrac{\phi_l(v(t_l^-))}{e^{-\lambda(\Xi(t_l)-\Xi(0))}(\Xi(t_l)-\Xi(0))^{\gamma-1}}\right)e^{-\lambda(\Xi(t)-\Xi(0))}(\Xi(t)-\Xi(0))^{\gamma-1}\\ \quad \ominus(-1)^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)}G(t,v_t), \quad t\in(t_l,t_{l+1}].\end{cases} \tag{3.5}$$

Proof. Assume that $v\in PC_{1-\gamma,\Xi}(\mathscr{J}^*,E)$ satisfies (1.1).
When $t\in[0,t_1]$, we consider

$$\begin{cases} {}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}v(t)=G(t,v_t), \quad t\in(0,t_1],\\ {}^{T}_{0^+}\mathscr{I}^{1-\gamma,\lambda}_{\Xi(t)}v(0)=v_0.\end{cases} \tag{3.6}$$

Then, the problem (3.6) is equivalent to the following integral equation:

$$v(t)=\frac{e^{-\lambda(\Xi(t)-\Xi(0))}(\Xi(t)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)}v_0+{}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)}G(t,v_t), \quad t\in[0,t_1]. \tag{3.7}$$

When $t \in (t_1, t_2]$, we consider

$$\begin{cases} {}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)} v(t) = G(t, v_t), \quad t \in (t_1, t_2], \\ v(t_1^+) - v(t_1^-) = \phi_1(v(t_1^-)). \end{cases} \tag{3.8}$$

By Lemma 5, we have

$$\begin{aligned} v(t) &= \left[\frac{v_0 \ominus {}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)} G_a}{(\Xi(a) - \Xi(0))^{\gamma-1} e^{-\lambda(\Xi(a)-\Xi(0))}} \right] (\Xi(t) - \Xi(0))^{\gamma-1} e^{-\lambda(\Xi(t)-\Xi(0))} \\ &\quad + {}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)} G(t, v_t) \\ &= \left[\frac{v(t_1^+) \ominus {}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)} G(t, v_t)|_{t=t_1}}{(\Xi(t_1) - \Xi(0))^{\gamma-1} e^{-\lambda(\Xi(t_1)-\Xi(0))}} \right] (\Xi(t) - \Xi(0))^{\gamma-1} e^{-\lambda(\Xi(t)-\Xi(0))} \\ &\quad + {}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)} G(t, v_t). \end{aligned}$$

This implies that

$$\begin{aligned} v(t) &= \left[\frac{\big(v(t_1^-) + \phi_1(v(t_1^-))\big) \ominus {}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)} G(t, v_t)|_{t=t_1}}{(\Xi(t_1) - \Xi(0))^{\gamma-1} e^{-\lambda(\Xi(t_1)-\Xi(0))}} \right] (\Xi(t) - \Xi(0))^{\gamma-1} \\ &\quad \times e^{-\lambda(\Xi(t)-\Xi(0))} + {}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)} G(t, v_t), \quad t \in (t_1, t_2]. \end{aligned} \tag{3.9}$$

Now, from (3.7), we have

$$v(t_1^-) \ominus {}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi} G(t, v_t)|_{t=t_1} = \frac{e^{-\lambda(\Xi(t_1)-\Xi(0))} (\Xi(t_1) - \Xi(0))^{\gamma-1}}{\Gamma(\gamma)} v_0, \quad t \in (t_1, t_2]. \tag{3.10}$$

Using (3.10) in (3.9), we get

$$\begin{aligned} v(t) &= \left(\frac{v_0}{\Gamma(\gamma)} + \frac{\phi_l(v(t_1^-))}{e^{-\lambda(\Xi(t_1)-\Xi(0))} (\Xi(t_1) - \Xi(0))^{\gamma-1}} \right) e^{-\lambda(\Xi(t)-\Xi(0))} (\Xi(t) - \Xi(0))^{\gamma-1} \\ &\quad + {}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)} G(t, v_t), \quad t \in (t_1, t_2]. \end{aligned}$$

Repeating the same process for $t \in [t_l, t_{l+1}]$, we obtain (3.5).

Conversely, Let v satisfies (3.5) then, for $t \in \mathscr{J}^*$, we have

$$\begin{aligned} v(t) &= \left(\frac{v_0}{\Gamma(\gamma)} + \frac{\phi_l(v(t_1^-))}{e^{-\lambda(\Xi(t_1)-\Xi(0))} (\Xi(t_1) - \Xi(0))^{\gamma-1}} \right) e^{-\lambda(\Xi(t)-\Xi(0))} (\Xi(t) - \Xi(0))^{\gamma-1} \\ &\quad + {}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)} G(t, v_t). \end{aligned}$$

Applying ${}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}$ on both sides of the above equation and using the result ${}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)} (\Xi(t) - \Xi(0))^{\gamma-1} = 0$, $0 < \gamma < 1$ and Lemma 4, we get

$${}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)} v(t) = G(t, v_t), \quad t \in \mathscr{J}.$$

Further, from (3.7), we have

$$ {}_{0^+}^{T}\mathscr{I}_{\Xi(t)}^{1-\gamma,\lambda} v(t) = v_0 + {}_{0^+}^{T}\mathscr{I}_{\Xi(t)}^{1-\gamma+p,\lambda} G(t,v_t), $$

which gives

$$ {}_{0^+}^{T}\mathscr{I}_{\Xi(t)}^{1-\gamma,\lambda} v(0) = v_0. $$

And we can easily prove that $\Delta v(t_l) = \phi_l(v(t_l^-))$, $l = 1,2,\cdots,m$. □

4. Existence and uniqueness results

We impose the following hypotheses.

$(H1)$ There exists a constant $L_G > 0$ such that the functional $G : \mathscr{J} \times PC([-\sigma,0],E) \to E$ satisfies locally

$$ D_0[G(t,v_t),G(t,w_t)] \leqslant L_G\left(\sup_{s\in[-\sigma,t]} D_0[(\Xi(s)-\Xi(0))^{1-\gamma}u_s,(\Xi(s)-\Xi(0))^{1-\gamma}w_s]\right), $$

for almost every $t \in \mathscr{J}^*$ and for all $v,w \in PC^1([-\sigma,T],E)$, with $v_s(\zeta) = v(s+\zeta)$ and $w_s(\zeta) = w(s+\zeta)$, $\zeta \in [-\sigma.0]$.

$(H2)$ There is $\vartheta > 0$ such that

$$ D_0[\phi_l(v(t_l^-)),\phi_l(w(t_l^-))] \leqslant \vartheta D_0[v(t_l),w(t_l)], \quad \text{for all} \quad v,w \in PC^1([-\sigma,T],E). $$

$(H3)$ Let $\mu \in C(\mathscr{J}^*,\mathbb{R}^+)$ be an increasing function, then there exist $\omega_\mu > 0$ such that

$$ {}_{0^+}^{T}\mathscr{I}_{\Xi(t)}^{p,\lambda}\mu(t) \leqslant \omega_\mu \mu(t), \quad \text{for all} \quad t \in \mathscr{J}. $$

THEOREM 3. *If $(H1)$, $(H3)$ are met, and*

$$ \Upsilon := \vartheta m + \frac{L_G}{\Gamma(p+1)} e^{-\lambda(\Xi(T)-\Xi(0))}(\Xi(T)-\Xi(0)^{1-q(1-p)} < 1, \tag{4.1} $$

then the problem (1.1) *has a unique solution in* $PC_{1-\gamma,\Xi}([-\sigma,T],E)$.

Proof. From Lemma 6 we obtain that (1.1) is equivalent to the following system:

$$ v(t) = \begin{cases} \rho(t), \quad t \in [-\sigma,0], \\ \left(\dfrac{v_0}{\Gamma(\gamma)} + \displaystyle\sum_{0<t_l<t} \dfrac{\phi_l(v(t_l^-))}{e^{-\lambda(\Xi(t_l)-\Xi(0))}(\Xi(t_l)-\Xi(0))^{\gamma-1}}\right) e^{-\lambda(\Xi(t)-\Xi(0))} \\ \quad \times(\Xi(t)-\Xi(0))^{\gamma-1} \ominus (-1)\,{}_{0^+}^{T}\mathscr{I}_{\Xi(t)}^{p,\lambda} G(t,v_t), \quad t \in \mathscr{J}^*. \end{cases} \tag{4.2} $$

The existence of a solution for (4.2) can be turned into a fixed point problem in $PC_{1-\gamma,\Xi}([-\sigma,T],E)$ for the operator T_G defined by

$$(T_G v)(t) = \begin{cases} \rho(t), & t \in [-\sigma, 0], \\ \left(\dfrac{v_0}{\Gamma(\gamma)} + \displaystyle\sum_{0<t_l<t} \dfrac{\phi_l(v(t_l^-))}{e^{-\lambda(\Xi(t_l)-\Xi(0))}(\Xi(t_l)-\Xi(0))^{\gamma-1}} \right) e^{-\lambda(\Xi(t)-\Xi(0))} \\ \quad \times (\Xi(t)-\Xi(0))^{\gamma-1} \ominus (-1)^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)} G(t,v_t), & t \in \mathscr{J}^*. \end{cases} \tag{4.3}$$

Note that for any continuous fuzzy function G, T_G is also continuous. Indeed,

$$\begin{aligned} & D_0[T_G(v)(t), T_G(v)(t_0)] \\ & = D_0\Bigg[\left(\frac{v_0}{\Gamma(\gamma)} + \sum_{0<t_l<t} \frac{\phi_l(v(t_l^-))}{e^{-\lambda(\Xi(t_l)-\Xi(0))}(\Xi(t_l)-\Xi(0))^{\gamma-1}} \right) e^{-\lambda(\Xi(t)-\Xi(0))}(\Xi(t)-\Xi(0))^{\gamma-1} \\ & \quad + {}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)} G(t,v_t), \left(\frac{v_0}{\Gamma(\gamma)} + \sum_{0<t_l<t} \frac{\phi_l(v(t_l^-))}{e^{-\lambda(\Xi(t_l)-\Xi(0))}(\Xi(t_l)-\Xi(0))^{\gamma-1}} \right) \\ & \quad \times e^{-\lambda(\Xi(t_0)-\Xi(0))}(\Xi(t_0)-\Xi(0))^{\gamma-1} + {}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)} G(t_0, v_{t_0})\Bigg] \\ & \to 0, \end{aligned}$$

as $t \longrightarrow t_0$.

Next, we show T_G defined in (4.3) is a contraction mapping on $PC_{1-\gamma,\Xi}([-\sigma,T],E)$. Consider $T_G : PC_{1-\gamma,\Xi}([-\sigma,T],E) \to PC_{1-\gamma,\Xi}([-\sigma,T],E)$ defined in (4.3). For $t \in [-\sigma,0]$, we have

$$\begin{aligned} & D_0[(\Xi(t)-\Xi(0))^{1-\gamma}(T_G v)(t), (\Xi(t)-\Xi(0))^{1-\gamma}(T_G \bar{v})(t)] \\ & \leqslant \vartheta \sum_{0<t_l<t} (\Xi(t_l)-\Xi(0))^{1-\gamma} e^{\lambda(\Xi(t_l)-\Xi(0))} D_0[v(t_l^-), \bar{v}(t_l^-)] \\ & \quad + L_G D_0[(\Xi(t)-\Xi(0))^{1-\gamma} v, (\Xi(t)-\Xi(0))^{1-\gamma}\bar{v}] \\ & \quad \times \frac{1}{\Gamma(p)} \int_0^t \Xi'(s)(\Xi(t)-\Xi(s))^{p-1} e^{-\lambda(\Xi(t)-\Xi(s))} ds \\ & \leqslant \left(\vartheta m + \frac{L_G}{\Gamma(p+1)} e^{-\lambda(\Xi(T)-\Xi(0))}(\Xi(T)-\Xi(0)^{1-q(1-p)} \right) D_0[v,\bar{v}] \\ & \leqslant \Upsilon D_0[v,\bar{v}], \end{aligned}$$

where

$$m = \sum_{0<t_l<t} (\Xi(t_l)-\Xi(0))^{1-\gamma} e^{\lambda(\Xi(t_l)-\Xi(0))}.$$

Since $\Upsilon < 1$, T_G is a contraction. By Theorem 1, the operator T_G has a unique fixed point, which is a solution of problem (1.1). □

5. Stability theory

In this section, we discuss the Ulam-Hyers-Rassias (U-H-R) stable and generalized Ulam-Hyers-Rassias (G-U-H-R) stable of the system (1.1).

Let ε, $\delta > 0$ and $\mu \in PC(\mathscr{J}^*, \mathbb{R}^+)$ be a nondecreasing function. Consider the following inequality:

$$\begin{cases} D_0[{}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}\bar{v}(t), G(t,\bar{v}_t)] \leqslant \varepsilon\mu(t), & t \in \mathscr{J}^* \\ D_0[\Delta v(t_l), \phi_l(v(t_l^-))] \leqslant \varepsilon\delta, & l = 1,2,\cdots,m. \end{cases} \tag{5.1}$$

In the fuzzy concept, we have the following definitions.

DEFINITION 13. The problem (1.1) is U-H-R stable with respect to (μ,δ), if there exists $C_{G,m,\mu} > 0$ such that, for each $\varepsilon > 0$ and every solution $\bar{v} \in PC_{1-\gamma,\Xi}(\mathscr{J}^*, E)$ of inequality (5.1), there is a solution $v \in PC_{1-\gamma,\Xi}(\mathscr{J}^*, E)$ of Eqn. (1.1) with

$$D_0[(\Xi(t)-\Xi(0))^{1-\gamma}\bar{v}(t), (\Xi(t)-\Xi(0))^{1-\gamma}v(t)] \leqslant C_{G,m,\mu}\varepsilon(\mu(t)+\delta), \quad t \in \mathscr{J}^*.$$

DEFINITION 14. The problem (1.1) is G-U-H-R stable with respect to (μ,δ) if there exists $C_{G,m,\mu} > 0$ such that, for each solution $\bar{v} \in PC_{1-\gamma,\Xi}(\mathscr{J}^*, E)$ of inequality (5.1), there is a solution $v \in PC_{1-\gamma,\Xi}(\mathscr{J}^*, E)$ of Eqn. (1.1) with

$$D_0[(\Xi(t)-\Xi(0))^{1-\gamma}\bar{v}(t), (\Xi(t)-\Xi(0))^{1-\gamma}v(t)] \leqslant C_{G,m,\mu}(\mu(t)+\delta), \quad t \in \mathscr{J}^*.$$

REMARK 4. A fuzzy function $\bar{v} \in PC_{1-\gamma,\Xi}(\mathscr{J}^*, E)$ is a solution of the inequality (5.1) if and only if there is a function $f \in PC_{1-\gamma,\Xi}(\mathscr{J}^*, E)$ and a sequence $\{f_l\}$, $l = 1,2,\cdots,m$, such that

(i) $D_0[f(t),\widehat{0}] \leqslant \varepsilon\mu(t)$ and $D_0[f_l,\widehat{0}] \leqslant \varepsilon\delta$, $\quad l = 1,2,\cdots,m$;

(ii) ${}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi}\bar{v}(t) = G(t,\bar{v}_t) + f(t), \quad t \in \mathscr{J}^*$;

(iii) $\Delta\bar{v}(t_l) = \phi_l(\bar{v}(t_l^-)) + f_l, \quad l = 1,2,\cdots,m.$

THEOREM 4. *Assume that* $(H1)$, $(H2)$, $(H3)$ *are satisfied. Then, the problem* (1.1) *is U-H-R stable with respect to* (μ,δ).

Proof. Let $\bar{v} \in PC_{1-\gamma,\Xi}([-\sigma,T],E)$ be a solution of the inequality (5.1). Then by using Lemma 6 and Remark 4, we have

$$v(t) = \left(\frac{v_0}{\Gamma(\gamma)} + \sum_{0<t_l<t} \frac{\phi_l(v(t_l^-))}{e^{-\lambda(\Xi(t_l)-\Xi(0))}(\Xi(t_l)-\Xi(0))^{\gamma-1}}\right)e^{-\lambda(\Xi(t)-\Xi(0))} \\ \times (\Xi(t)-\Xi(0))^{\gamma-1} + {}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)}G(t,v_t), \quad t \in \mathscr{J}. \tag{5.2}$$

Let $v \in PC_{1-\gamma,\Xi}([-\sigma,T],E)$ be the unique solution of the problem (1.1). Then, by using (5.2) and in view of Remark 4, we have for $t \in [-\sigma,0]$ that $D_0[\bar{v}(t),v(t)] = 0$. For $t \in \mathscr{J}$, yields

$$
\begin{aligned}
&D_0[\bar{v}(t),v(t)] \\
&\quad \leqslant \sum_{0<t_l<t} \frac{D_0[\phi_l(\bar{v}(t_l^-)),\phi_l(v(t_l^-))]}{e^{-\lambda(\Xi(t_l)-\Xi(0))}(\Xi(t_l)-\Xi(0))^{\gamma-1}} e^{-\lambda(\Xi(t)-\Xi(0))}(\Xi(t)-\Xi(0))^{\gamma-1} \\
&\qquad + e^{-\lambda(\Xi(t)-\Xi(0))}(\Xi(t)-\Xi(0))^{\gamma-1} \sum_{0<t_l<t} \frac{D_0[f_l,\widehat{0}]}{e^{-\lambda(\Xi(t_l)-\Xi(0))}(\Xi(t_l)-\Xi(0))^{\gamma-1}} \\
&\qquad + \frac{1}{\Gamma(p)} \int_0^t \Xi'(s)(\Xi(t)-\Xi(s))^{p-1} e^{-\lambda(\Xi(t)-\Xi(s))} D_0[f(s),\widehat{0}]ds \\
&\qquad + \frac{1}{\Gamma(p)} \int_0^t \Xi'(s)(\Xi(t)-\Xi(s))^{p-1} e^{-\lambda(\Xi(t)-\Xi(s))} D_0[G(s,v_s),G(s,\bar{v}_s)]ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
&D_0[\bar{v}(t),v(t)] \\
&\quad \leqslant \vartheta \sum_{0<t_l<t} (\Xi(t_l)-\Xi(0))^{1-\gamma} e^{\lambda(\Xi(t_l)-\Xi(0))} D_0[\bar{v}(t_l^-),v(t_l^-)] e^{-\lambda(\Xi(t)-\Xi(0))} \\
&\qquad \times (\Xi(t)-\Xi(0))^{\gamma-1} + \frac{e^{-\lambda(\Xi(t)-\Xi(0))}(\Xi(t)-\Xi(0))^{\gamma-1}}{e^{-\lambda(\Xi(T)-\Xi(0))}(\Xi(T)-\Xi(0))^{\gamma-1}} m\varepsilon\delta + \varepsilon\omega_\mu\mu(t) \\
&\qquad + \frac{L_G}{\Gamma(p)} \int_0^t \Xi'(s)(\Xi(t)-\Xi(s))^{p-1} e^{-\lambda(\Xi(t)-\Xi(s))} \\
&\qquad \times \left(\sup_{s\in[-\sigma,t]} D_0[(\Xi(s)-\Xi(0))^{1-\gamma} v_s, (\Xi(s)-\Xi(0))^{1-\gamma} \bar{v}_s] \right) ds.
\end{aligned}
$$

Hence, if

$$
Q_{\mu,m} = \max\left\{ \frac{m}{e^{-\lambda(\Xi(T)-\Xi(0))}(\Xi(T)-\Xi(0))^{\gamma-1}}, \frac{\omega_\mu}{e^{-\lambda(\Xi(T)-\Xi(0))}(\Xi(T)-\Xi(0))^{\gamma-1}} \right\},
$$

we obtain

$$
\begin{aligned}
&D_0[(\Xi(t)-\Xi(0))^{1-\gamma}\bar{v}(t),(\Xi(t)-\Xi(0))^{1-\gamma}v(t)] \\
&\quad \leqslant \frac{L_G}{\Gamma(p)e^{-\lambda(\Xi(T)-\Xi(0))}(\Xi(T)-\Xi(0))^{\gamma-1}} \int_0^t \Xi'(s)(\Xi(t)-\Xi(s))^{p-1} e^{-\lambda(\Xi(t)-\Xi(s))} \\
&\quad \times D_0[\bar{v}_s,v_s]ds + Q_{\mu,m}\varepsilon(\mu(t)+\delta) + \vartheta \sum_{0<t_l<t} e^{\lambda(\Xi(t_l)-\Xi(0))}(\Xi(t_l)-\Xi(0))^{1-\gamma} \\
&\quad \times D_0[\bar{v}(t_l^-),v(t_l^-)].
\end{aligned}
$$

Choose

$$
Q_{\mu,m} = \max\left\{ \frac{m}{e^{-\lambda(\Xi(T)-\Xi(0))}(\Xi(T)-\Xi(0))^{\gamma-1}}, \frac{\omega_\mu}{e^{-\lambda(\Xi(T)-\Xi(0))}(\Xi(T)-\Xi(0))^{\gamma-1}} \right\}
$$

and

$$u(t) = \sup_{s\in[-\sigma,t]} \{D_0[(\Xi(s)-\Xi(0))^{1-\gamma}\overline{v}_s, (\Xi(s)-\Xi(0))^{1-\gamma}v_s]\},$$

we have $u \in PC(\mathscr{J}^*, \mathbb{R}^+)$ and

$$u(t) \leqslant Q_{\mu,m}\varepsilon(\mu(t)+\delta) + \vartheta \sum_{0<t_l<t} u(t_l^-) + \frac{L_G}{\Gamma(p)e^{-\lambda(\Xi(T)-\Xi(0))}(\Xi(T)-\Xi(0))^{\gamma-1}} \times \int_0^t \Xi'(s)(\Xi(t)-\Xi(s))^{p-1}e^{-\lambda(\Xi(t)-\Xi(s))}u(s)ds.$$

The function $a^*(t) = Q_{\mu,m}\varepsilon(\mu(t)+\delta)$ is nonnegative, non-decreasing and bounded in $\mathscr{J}^*$, given the assumptions about $Q_{\mu,m}$, μ and δ.

The function $g(t) = \frac{L_G}{\Gamma(p)}e^{\lambda(\Xi(T)-\Xi(0))}(\Xi(T)-\Xi(0))^{1-\gamma}$ is continuous and non-negative at $\mathscr{J}^*$, in addition to being bounded. Since $q_l = \vartheta \geqslant 0$, all the hypotheses of Theorem 2 are satisfied and therefore

$$\begin{aligned} u(t) &\leqslant Q_{\mu,m}\varepsilon(\mu(t)+\delta)\left\{\prod_{i=1}^{l}\left[1+\vartheta E_p\left(\frac{L_G(\Xi(t_i)-\Xi(0))^p e^{-\lambda(\Xi(t_i)-\Xi(0))}}{(\Xi(T)-\Xi(0))^{\gamma-1}e^{-\lambda(\Xi(T)-\Xi(0))}}\right)\right]\right\} \\ &\quad \times E_p\left(\frac{L_G(\Xi(t)-\Xi(0))^p e^{-\lambda(\Xi(t)-\Xi(0))}}{(\Xi(T)-\Xi(0))^{\gamma-1}e^{-\lambda(\Xi(T)-\Xi(0))}}\right) \\ &\leqslant C_{G,m,\mu}\varepsilon(\mu(t)+\delta), \end{aligned}$$

where

$$C_{G,m,\mu} = Q_{\mu,m}\left\{\prod_{i=1}^{l}\left[1+\vartheta E_p\left(\frac{L_G(\Xi(t_i)-\Xi(0))^p e^{-\lambda(\Xi(t_i)-\Xi(0))}}{(\Xi(T)-\Xi(0))^{\gamma-1}e^{-\lambda(\Xi(T)-\Xi(0))}}\right)\right]\right\} \times E_p\left(\frac{L_G(\Xi(t)-\Xi(0))^p e^{-\lambda(\Xi(t)-\Xi(0))}}{(\Xi(T)-\Xi(0))^{\gamma-1}e^{-\lambda(\Xi(T)-\Xi(0))}}\right),$$

and $E_p(.)$ is a one parameter Mittag-Leffler (M-L) function.

In particular, as

$$D_0[(\Xi(t)-\Xi(0))^{1-\gamma}\overline{v}(t), (\Xi(t)-\Xi(0))^{1-\gamma}v(t)] \leqslant u(t)$$

we obtain that

$$D_0[(\Xi(t)-\Xi(0))^{1-\gamma}\overline{v}(t), (\Xi(t)-\Xi(0))^{1-\gamma}v(t)] \leqslant C_{G,m,\mu}\varepsilon(\mu(t)+\delta), \quad t \in \mathscr{J}^*,$$

which means that the problem (1.1) is U-H-R stable. Furthermore, assuming $\varepsilon = 1$, we can conclude that the problem (1.1) is G-U-H-R stable. □

6. Particular cases: Ulam-Hyers (U-H) and Ulam-Hyers-Mittag-Leffler (U-H-M-L) stabilities

The Ulam-Hyers (U-H) stability is a particular case of U-H-R stability, in the case of the fuzzy concept, where $\mu(t) = 1$ and $\delta = 1$. To prove that the problem (1.1) is U-H stable, we have to show that the function μ satisfies $(H3)$.

Consider the following inequality:

$$\begin{cases} D_0[{}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)}\bar{v}(t), G(t,\bar{v}_t)] \leqslant \varepsilon, & t \in \mathscr{J} \\ D_0[\Delta\bar{v}(t_l), \phi_l(\bar{v}(t_l^-))] \leqslant \varepsilon, & l = 1,2,\cdots,m. \end{cases} \tag{6.1}$$

subject to the initial conditions

$$\begin{cases} {}^{T}_{0^+}\mathscr{I}^{1-\gamma,\lambda}_{\Xi(t)}v(0) = v_0, & \gamma = p + q(1-p), \\ v(t) = \rho(t), & t \in [-\sigma, 0]. \end{cases}$$

DEFINITION 15. The problem (1.1) is U-H stable if there exists $C_{G,m} > 0$, such that for each $\varepsilon > 0$ and for each solution $\bar{v} \in PC_{1-\gamma,\Xi}(\mathscr{J}^*, E)$ of (6.1) with

$$D_0[(\Xi(t)-\Xi(0))^{1-\gamma}\bar{v}(t), (\Xi(t)-\Xi(0))^{1-\gamma}v(t)] \leqslant C_{G,m}\varepsilon, \quad t \in \mathscr{J}^*. \tag{6.2}$$

THEOREM 5. *Suppose that the hypotheses $(H1)$, $(H2)$ and $(H3)$ are satisfied. Then, the problem (1.1) is U-H stable.*

Proof. Indeed,

$${}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)}\mu(t) = \frac{(\Xi(T)-\Xi(0))^p}{\Gamma(p+1)}e^{-\lambda(\Xi(T)-\Xi(0))} \times 1 = \omega_\mu \mu(t),$$

and μ satisfies $(H3)$, since μ is an increasing and integrable function in $\mathscr{J}^*$. Let $\delta = 1$, then by using Theorem 4, we have

$$D_0[(\Xi(t)-\Xi(0))^{1-\gamma}\bar{v}(t), (\Xi(t)-\Xi(0))^{1-\gamma}v(t)] \leqslant C_{G,m}\varepsilon, \quad t \in \mathscr{J}^*,$$

where,

$$C_{G,m} = 2Q_{m,1}\left\{\prod_{i=1}^{l}\left[1 + E_p\left(\frac{L_G(\Xi(t_i)-\Xi(0))^p e^{-\lambda(\Xi(t_i)-\Xi(0))}}{\Gamma(p)(\Xi(T)-\Xi(0))^{\gamma-1}e^{-\lambda(\Xi(T)-\Xi(0))}}\right)\right]\right\} \times E_p\left(\frac{L_G(\Xi(t)-\Xi(0))^p e^{-\lambda(\Xi(t)-\Xi(0))}}{\Gamma(p)(\Xi(T)-\Xi(0))^{\gamma-1}e^{-\lambda(\Xi(T)-\Xi(0))}}\right).$$

Consequently, the problem (1.1) is U-H stable. □

For $\bar{v} \in PC_{1-\gamma,\Xi}([-\sigma,T],E)$ and $\varepsilon, \delta > 0$, consider the following inequality:

$$\begin{cases} D_0[{}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi}\bar{v}(t), G(t,\bar{v}_t)] \leqslant \varepsilon E_p((\Xi(t)-\Xi(0))^p), & t \in \mathscr{J}' \\ D_0[\Delta\bar{v}(t_l), \phi_l(\bar{v}(t_l^-))] \leqslant \varepsilon\delta, & l = 1,2,\cdots,m. \end{cases} \tag{6.3}$$

DEFINITION 16. The problem (1.1) is U-H-M-L stable with respect to $(E_p((\Xi(t) - \Xi(0))^p, \delta)$, $E_p(.)$ an one parameter M-L function, with $0 < p < 1$, if there exists $C_{E_p} > 0$ such that, for each $\varepsilon > 0$ and every solution $\bar{v} \in PC_{1-\gamma,\Xi}([-\sigma, T], E)$ of (6.3), there exists a solution $v \in PC_{1-\gamma,\Xi}([-\sigma, T], E)$ satisfying (1.1) with

$$D_0[(\Xi(t) - \Xi(0))^{1-\gamma}\bar{v}(t), (\Xi(t) - \Xi(0))^{1-\gamma}v(t)] \leqslant C_{E_p}\varepsilon(E_p((\Xi(t) - \Xi(0))^p) + \delta), \quad t \in [-\sigma, T].$$

THEOREM 6. *If hypotheses $(H1)$, $(H2)$ are satisfied. Then, the Eqn. (6.3) is U-H-M-L stable.*

Proof. Since $E_p((\Xi(t) - \Xi(0))^p)$ is an increasing and integrable function in $\mathscr{J}^*$. If the function $E_p((\Xi(t) - \Xi(0))^p)$ satisfies $(H3)$. With the aid of Theorem 4, the Eqn. (6.3) is U-H-M-L stable. Indeed By $(H3)$, we have

$$\begin{aligned}
&{}^{T}_{0^+}\mathscr{I}^{p,\lambda}_{\Xi(t)} E_p((\Xi(t) - \Xi(0))^p) \\
&= \frac{1}{\Gamma(p)} \int_0^t \Xi'(s)(\Xi(t) - \Xi(s))^{p-1} e^{-\lambda(\Xi(t)-\Xi(s))} E_p((\Xi(s) - \Xi(0))^p) ds \\
&= \frac{1}{\Gamma(p)} \sum_{j=0}^{\infty} \frac{1}{\Gamma(jp+1)} \int_0^t \Xi'(s)(\Xi(t) - \Xi(s))^{p-1} \\
&\quad \times e^{-\lambda(\Xi(t)-\Xi(s))}(\Xi(s) - \Xi(0))^{jp} ds \\
&\leqslant \sum_{j=0}^{\infty} \frac{(\Xi(t) - \Xi(0))^{jp}}{\Gamma(jp+1)} e^{-\lambda(\Xi(t)-\Xi(0))} \\
&= e^{-\lambda(\Xi(t)-\Xi(0))} E_p((\Xi(t) - \Xi(0))^p).
\end{aligned}$$

Therefore, Eqn. (6.3) is U-H-M-L stable. □

7. Examples

EXAMPLE 1. Consider the impusive tempered Ξ-Hilfer tye FFDE with delay

$$\begin{cases}
{}^{TH}_{0^+}\mathscr{D}^{p,q,\lambda}_{\Xi(t)} v(t) = \dfrac{(\Xi(t) - \Xi(0))^3}{10} v\left(t - \dfrac{1}{4}\right), \quad 0 < \lambda < 1, \quad t \in (0, 1] \backslash \left\{\dfrac{1}{4}\right\}, \\
\Delta v(\frac{1}{4}^-) = \phi(v(\frac{1}{4}^-)) = \dfrac{v(\frac{1}{4}^-)}{7 + v(\frac{1}{4}^-)}, \\
v(t) = 1, \quad t \in [-1, 0],
\end{cases} \tag{7.1}$$

Define $G : (0, 1] \times E \to E$ by

$$G(t, v_t) = \frac{(\Xi(t) - \Xi(0))^3}{10} v_t.$$

and $\phi_1 : E \to E$ by

$$\phi_1\left(v\left(\frac{1}{4}^-\right)\right) = \frac{v(\frac{1}{4}^-)}{7+v(\frac{1}{4}^-)}.$$

Then G satisfies $(H1)$ indeed for $v,w \in E$ and for $t \in (0,1]$, we have

$$\begin{aligned}
&D_0[G(t,v_t),G(t,w_t)] \\
&\quad = \frac{(\Xi(t)-\Xi(0))^3}{10} D_0[v_s,w_s] \\
&\quad \leqslant \frac{(\Xi(t)-\Xi(0))^{2+\gamma}}{10}\left(D_0[(\Xi(s)-\Xi(0))^{1-\gamma}v_s,(\Xi(s)-\Xi(0))^{1-\gamma}w_s]\right) \\
&\quad \leqslant \frac{(\Xi(1)-\Xi(0))^{2+\gamma}}{10}\left(\sup_{s\in[-\sigma,t]} D_0[(\Xi(s)-\Xi(0))^{1-\gamma}v_s,(\Xi(s)-\Xi(0))^{1-\gamma}w_s]\right).
\end{aligned}$$

This implies that G satisfies the hypotheses $(H1)$ with $L_G = \frac{1}{10}$. Further, for any $v,w \in E$,

$$\begin{aligned}
D_0\left[\phi\left(v\left(\frac{1}{4}^-\right)\right),\phi\left(w\left(\frac{1}{4}^-\right)\right)\right] &\leqslant D_0\left[\frac{v(\frac{1}{4}^-)}{7+v(\frac{1}{4}^-)},\frac{w(\frac{1}{4}^-)}{7+w(\frac{1}{4}^-)}\right] \\
&\leqslant \frac{1}{7}D_0\left[v\left(\frac{1}{4}^-\right),w\left(\frac{1}{4}^-\right)\right].
\end{aligned}$$

This shows that ϕ_1 satisfy the hypothesis $(H2)$ with $\vartheta = \frac{1}{7}$, $m=1$. Thus, by using Theorem 3, the problem (7.1) has a unique solution, if the condition

$$\begin{aligned}
\Upsilon &= \vartheta m + \frac{L_G}{\Gamma(p+1)}e^{-\lambda(\Xi(1)-\Xi(0))}(\Xi(1)-\Xi(0)^{1-q(1-p)} \\
&= \frac{1}{7} + \frac{(\Xi(1)-\Xi(0))^{2+p}e^{-\lambda(\Xi(1)-\Xi(0))}}{10\Gamma(p+1)} < 1,
\end{aligned}$$

is satisfied and it is equivalent to the bellows that

$$(\Xi(1)-\Xi(0))^{2+p}e^{-\lambda(\Xi(1)-\Xi(0))} < \frac{60}{7}\Gamma(p+1). \tag{7.2}$$

Take $p = \frac{5}{2}$, $q=0$, $\lambda = 0$ and $\Xi(t) = \log(t)$. In this case, from (7.2), $\Upsilon = 0.142 < 1$.

So, verification of $H3$ depends on the selected Ξ function and the fractional order p.

7.1. Particular case: Tempered R-L

Taking $q=0$ and $\Xi(t)=t$, we obtain a particular case of problem (1.1) reduces to the tempered R-L impulsive FFDE with delay of the form:

$$\begin{cases} {}^{RL}_{0^+}\mathscr{D}_t^{p,\lambda} v(t) = G(t,v_t), \quad 0<\lambda<1, \quad t\in \mathscr{J}\backslash\{t_1,t_2,\cdots,t_m\}, \\ \Delta v(t_l) = v(t_l^+) - v(t_l^-) = \phi_l(v(t_l^-)), \quad l=1,2,\cdots,m, \\ {}^{T}_{0^+}\mathscr{I}_t^{1-p,\lambda} v(0) = v_0, \\ v(t) = \rho(t), \quad t\in[-\sigma,0], \end{cases} \tag{7.3}$$

and it is equivalent to the integral equation

$$v(t) = \begin{cases} \rho(t), \quad t\in[-\sigma,0], \\ \left(\dfrac{v_0}{\Gamma(\gamma)} + \displaystyle\sum_{0<t_l<t} \dfrac{\phi_l(v(t_l^-))}{t_l^{p-1}e^{-\lambda t_l}}\right) t^{p-1}e^{-\lambda t} \\ \quad + {}^{T}_{0^+}\mathscr{I}_t^{p,\lambda} G(t,v_t), \quad t\in\mathscr{J}. \end{cases} \tag{7.4}$$

THEOREM 7. *Suppose that the function G satisfied with*

$$D_0[G(t,v_t),G(t,w_t)] \leqslant L_G\left(\sup_{s\in[-\sigma,t]} D_0[s^{1-\gamma}v_s, s^{1-\gamma}w_s]\right)$$

for all $v,w\in PC_{1-p,\Xi}([-\sigma,T],E)$. If

$$\vartheta m + \frac{L_G T e^{-\lambda T}}{\Gamma(p+1)} < 1$$

and ϑ is given by

$$D_0[\phi_l(v(t_l^-)),\phi_l(w(t_l^-))] \leqslant \vartheta D_0[v(t_l),w(t_l)], \quad l=1,2,\cdots,m.$$

Then, the problem (7.3) has a unique solution in $PC_{1-p,\Xi}([-\sigma,T],E)$ and is U-H and U-H-M-L stable.

Proof. The proof can be performed parallel to the Theorems 3, 5 and 6 with $q=0$ and $\Xi(t)=t$. $\square$

EXAMPLE 2. In according to Example 1, we will take $\Xi(t)=t$, $p=\frac{1}{2}$, $q=0$, $\lambda=1$, we have $\gamma=\frac{1}{2}$ and the problem (1.1) reduces to the tempered R-L impulsive

FFDE with delay

$$\begin{cases} {}^{RL}_{0^+}\mathscr{D}_t^{\frac{1}{2},1} v(t) = \frac{t^3}{10} v\left(t - \frac{1}{4}\right), \quad t \in (0,1]\backslash\{\frac{1}{4}\}, \\ \Delta v(\frac{1}{4}^-) = \phi(v(\frac{1}{4}^-)) = \frac{v(\frac{1}{4}^-)}{7 + v(\frac{1}{4}^-)}, \\ {}^{T}_{0^+}\mathscr{I}_t^{1-\frac{1}{2},1} v(0) = v_0, \\ v(t) = 1, \quad t \in [-1,0]. \end{cases} \tag{7.5}$$

It is observed that the hypotheses of $(H1)$ and $(H2)$ are satisfied. In this case, from (7.2),

$$\begin{aligned}(\Xi(1) - \Xi(0))^{2+p} e^{-\lambda(\Xi(1)-\Xi(0))} &= (1-0)^{2+\frac{1}{2}} e^{-(1-0)} = 0.678 \\ &< \frac{60}{7}\Gamma(\frac{1}{2}+1) \approx 7.596.\end{aligned}$$

Hence, the problem (7.5) has a unique solution and is U-H and U-H-M-L stable.

7.2. Particular case: Tempered Caputo

Taking $q = 1$ and $\Xi(t) = t$, we obtain a particular case of problem (1.1) reduces to the tempered Caputo impulsive FFDE with delay of the form:

$$\begin{cases} {}^{C}_{0^+}\mathscr{D}_t^{p,\lambda} v(t) = G(t, v_t), \quad 0 < \lambda < 1, \quad t \in \mathscr{J}\backslash\{t_1, t_2, \cdots, t_m\}, \\ \Delta v(t_l) = v(t_l^+) - v(t_l^-) = \phi_l(v(t_l^-)), \quad l = 1, 2, \cdots, m, \\ v(t) = \rho(t), \quad t \in [-\sigma, 0], \end{cases} \tag{7.6}$$

and it is equivalent to the integral equation

$$v(t) = \begin{cases} \rho(t) \quad t \in [-\sigma, 0] \\ \left(v(0^+) + \sum_{0<t_l<t} \phi_l(v(t_l^-))\right) + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} e^{-\lambda(t-s)} G(s, v_s) ds, \quad t \in \mathscr{J}. \end{cases} \tag{7.7}$$

THEOREM 8. *Suppose that the function G satisfies the condition*

$$D_0[G(t, v_t), G(t, w_t)] \leqslant L_G\left(\sup_{s\in[-\sigma,t]} D_0[v_s, w_s]\right)$$

for each $v, w \in PC_{1-p,\Xi}([-\sigma, T], E)$. If

$$\vartheta m + \frac{L_G T^p e^{-\lambda T}}{\Gamma(p+1)} < 1$$

and ϑ is given by

$$D_0[\phi_l(v(t_l^-)),\phi_l(w(t_l^-))] \leqslant \vartheta D_0[v(t_l),w(t_l)], \quad l=1,2,\cdots,m.$$

Then, the problem (7.6) *has a unique solution in* $PC_{1-p,\Xi}([-\sigma,T],E)$ *and is U-H and U-H-M-L stable.*

Proof. Let $q=1$ and $\Xi(t)=t$ within Theorems 3, 5 and 6; then, the proof is completed. □

EXAMPLE 3. In according to Example 1, we will take $\Xi(t)=t$, $p=\frac{1}{3}$, $q=1$, $\lambda=1$, and the problem (1.1) reduces to the tempered Caputo impulsive FFDE with delay

$$\begin{cases} {}^{C}_{0^+}\mathscr{D}_t^{\frac{1}{3},1}v(t)=\dfrac{t^3}{10}v\left(t-\dfrac{1}{4}\right), \quad t\in(0,1]-\left\{\dfrac{1}{4}\right\}, \\ \Delta v(\frac{1}{4}^-)=\phi(v(\frac{1}{4}^-))=\dfrac{v(\frac{1}{4}^-)}{7+v(\frac{1}{4}^-)}, \\ v(t)=1, \quad t\in[-1,0]. \end{cases} \tag{7.8}$$

It is observed that the hypotheses $(H1)$ are satisfied. In this case, from (7.2),

$$\begin{aligned}(\Xi(1)-\Xi(0))^{2+p}e^{-\lambda(\Xi(1)-\Xi(0))} &= (1-0)^{2+\frac{1}{3}}e^{-(1-0)}=0.678 \\ &< \frac{60}{7}\Gamma(\frac{1}{3}+1)\approx 7.6534.\end{aligned}$$

Hence, the problem (7.8) has a unique solution and is U-H and U-H-M-L stable.

8. Conclusion

With the aid of Lemma 5 obtained in [28], we have given the main results for the solution of our proposed problem (1.1). The existence and uniqueness of solutions are obtained via Banach contraction principle, while the U-H type stability result is derived by using generalized Gronwall inequality. Since tempered Ξ-HFD operators incorporates various well-known fractional derivative operators including tempered Caputo and R-L derivatives. The acquired results have been justified by three examples.

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