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POSITIVE SOLUTIONS FOR A RIEMANN–LIOUVILLE FRACTIONAL SYSTEM WITH ρ–LAPLACIAN OPERATORS

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Abstract. This paper studies the existence and uniqueness of positive solutions for Riemann-Liouville fractional differential equations with ρ-Laplacian operators and coupled nonlocal boundary conditions involving the Riemann-Stieltjes integrals. By means of an interesting fixed point theorem, some new sufficient conditions guaranteeing the existence and uniqueness of positive solutions are presented, and the unique positive solution can be the limit of a sequence constructed for any given initial point in a special set. To demonstrate the conclusion, a good example is given.

1. Introduction

This article investigates a system of Riemann-Liouville fractional differential equations with ρ_1-Laplacian and ρ_2-Laplacian operators, which are governed by coupled nonlocal boundary conditions

$$\begin{cases} D_{0+}^{\gamma_1}(\varphi_{\rho_1}(D_{0+}^{\delta_1}u(t)))+f(t,v(t))=0, \;\; t\in(0,1), \\ D_{0+}^{\gamma_2}(\varphi_{\rho_2}(D_{0+}^{\delta_2}v(t)))+g(t,u(t))=0, \;\; t\in(0,1), \\ u^{(i_1)}(0)=0, \;\; D_{0+}^{\delta_1}u(0)=0, \;\; D_{0+}^{\alpha_0}u(1)=\sum\limits_{j=1}^{n}\int_0^1 D_{0+}^{\alpha_j}v(\tau)d\mathfrak{H}_j(\tau), \\ v^{(i_2)}(0)=0, \;\; D_{0+}^{\beta_0}v(1)=\sum\limits_{j=1}^{m}\int_0^1 D_{0+}^{\beta_j}u(\tau)d\mathfrak{K}_j(\tau), \end{cases} \tag{1.1}$$

$i_1=0,\dots,p-2$; $i_2=0,\dots,q-2$; $D_{0+}^{\delta_2}v(0)=0$, where $\gamma_1,\gamma_2\in(0,1]$, $\delta_1\in(p-1,p]$, $\delta_2\in(q-1,q]$, $p,q\geqslant 3$, $p,q,n,m\in\mathbb{N}$, $\alpha_j,\beta_j\in\mathbb{R}$ for all $j=0,1,\dots,n$, and $j=0,1,\dots,m$, $0\leqslant\alpha_1<\alpha_2<\cdots<\alpha_n\leqslant\beta_0<\delta_2-1$, $\beta_0\geqslant 1$, $0\leqslant\beta_1<\beta_2<\cdots<\beta_m\leqslant\alpha_0<\delta_1-1$, $\alpha_0\geqslant 1$, the functions $f,g:[0,1]\times\mathbb{R}_+\to\mathbb{R}_+$, $\rho_1,\rho_2>1$, $\varphi_{\rho_i}(\zeta)=|\zeta|^{\rho_i-2}\zeta$, $\varphi_{\rho_i}^{-1}=\varphi_{\varrho_i}$, $\varrho_i=\frac{\rho_i}{\rho_i-1}$, $i=1,2$. The integrals in the nonlocal boundary conditions are Riemann-Stieltjes integrals with $\mathfrak{H}_j$, $j=1,\dots,n$ and $\mathfrak{K}_j$, $j=1,\dots,m$ are

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functions of bounded variation, and D^{k}_{0+} denotes the Riemann-Liouville derivative of order k, here $k=\gamma_1,\delta_1,\gamma_2,\delta_2;\ \alpha_j;\ j=0,1,\ldots,n;\ \beta_j;\ j=0,1,\ldots m$. As we know, fractional differential equation has a history of more than 300 years, and fractional calculus was favored by many scholars, see [1,3,5,6,9,11,12,18,20,24,29,31–36,38,40] for instance.

In [30], Yang and Zhu discussed a nonlinear fractional system with p-Laplacian operators:

$$\begin{cases} D^{\alpha_1}_{0^+}(\varphi_{p_1}(D^{\beta_1}_{0^+}u(t)))+\lambda f(t,u(t),v(t))=0,\ \ 0<t<1,\\ D^{\alpha_2}_{0^+}(\varphi_{p_2}(D^{\beta_2}_{0^+}v(t)))+\mu g(t,u(t),v(t))=0,\ \ 0<t<1,\\ u(0)=u(1)=u'(0)=u'(1)=0,\ \ D^{\beta_1}_{0^+}u(0)=0,\ \ D^{\beta_1}_{0^+}u(1)=b_1D^{\beta_1}_{0^+}u(\eta_1),\\ v(0)=v(1)=v'(0)=v'(1)=0,\ \ D^{\beta_2}_{0^+}v(0)=0,\ \ D^{\beta_2}_{0^+}v(1)=b_2D^{\beta_2}_{0^+}v(\eta_2), \end{cases} \tag{1.2}$$

where $\alpha_i\in(1,2]$, $\beta_i\in(3,4]$, $D^{\alpha_i}_{0^+}$ and $D^{\beta_i}_{0^+}$ are the Riemann-Liouville derivatives, $\eta_i\in(0,1)$, $b_i\in(0,\eta_i^{(1-\alpha_i)/(p_i-1)})$, $i=1,2$, and $f,\ g\in C([0,1]\times[0,+\infty)\times[0,+\infty),[0,+\infty))$ and λ and μ are two positive parameters. They got the existence and uniqueness of positive solutions with respect to two parameters. It is well-known that the differential equation with p-Laplacian operator is mainly derived from the non-Newtonian fluid theory and the turbulence theory of porous medium gas, and then it has been widely used in many fields, can also see [2,4,7,8,10,13–17,19,21,23,25–28,39,41].

In [37], Zhai and Wang studied a class of Hadamard fractional differential equations with integral conditions:

$$\begin{cases} {}^HD^pu(t)+f(t,v(t))=a,\ \ 1<p\leqslant 2,\ \ t\in(1,e),\\ {}^HD^qv(t)+g(t,u(t))=b,\ \ 1<q\leqslant 2,\ \ t\in(1,e),\\ u(1)=0,{}^HD^{p-1}u(e)=\sum\limits_{i=1}^{m}\mu_i{}^HI^{\alpha_i}v(\eta),\\ v(1)=0,{}^HD^{q-1}v(e)=\sum\limits_{j=1}^{n}\sigma_j{}^HI^{\beta_j}u(\xi), \end{cases} \tag{1.3}$$

where ${}^HD$ is Hadamard fractional derivative, ${}^HI$ is Hadamard fractional integral, $f,g\in C([1,e)\times(-\infty,+\infty),\ (-\infty,+\infty))$, a,b are constants, $\mu_i,\sigma_j>0,\ i=1,\ldots,m,\ j=1,\ldots,n;\ \eta,\xi>0$. The authors got the existence and uniqueness of solutions easily by a fixed-point method.

In [22], the authors used the Guo-Krasnosel'skii fixed point theorem to discuss the existence of a positive solution to the equations

$$\begin{cases} D^{\alpha_1}_{0+}(\varphi_{\rho_1}(D^{\beta_1}_{0+}x(t)))+\lambda f(t,x(t),y(t))=0,\quad t\in(0,1),\\ D^{\alpha_2}_{0+}(\varphi_{\rho_2}(D^{\beta_2}_{0+}y(t)))+\mu g(t,x(t),y(t))=0,\quad t\in(0,1), \end{cases} \tag{1.4}$$

with the coupled nonlocal boundary conditions

$$\begin{cases} x^{(j)}(0)=0, \quad j=0,\dots,n-2; \quad D_{0+}^{\beta_1}x(0)=0, \\ D_{0+}^{\gamma_0}x(1)=\sum\limits_{i=1}^{p}\int_0^1 D_{0+}^{\gamma_i}y(t)dH_i(t), \\ y^{(j)}(0)=0, \quad j=0,\dots,m-2; \quad D_{0+}^{\beta_2}y(0)=0, \\ D_{0+}^{\delta_0}y(1)=\sum\limits_{i=1}^{q}\int_0^1 D_{0+}^{\delta_i}x(t)dK_i(t), \end{cases} \tag{1.5}$$

where $\alpha_1,\alpha_2\in(0,1]$, $\beta_1\in(n-1,n]$, $\beta_2\in(m-1,m]$, $n,m\geqslant 3$, $n,m,p,q\in\mathbb{N}$, $\gamma_i,\delta_i\in\mathbb{R}$ for all $i=0,1,\dots,p$, $0\leqslant\gamma_1<\gamma_2<\cdots<\gamma_p\leqslant\delta_0<\beta_2-1$, $\delta_0\geqslant 1$, and $i=0,1,\dots,q$, $0\leqslant\delta_1<\delta_2<\cdots<\delta_q\leqslant\gamma_0<\beta_1-1$, $\gamma_0\geqslant 1$, $\rho_1,\rho_2>1$, $\lambda,\mu>0$, the functions $f,g\in C([0,1]\times\mathbb{R}_+\times\mathbb{R}_+,\mathbb{R}_+)$, the integrals from (1.5) are Riemann-Stieltjes integrals with H_i, $(i=1,\dots,p)$ and K_i, $(i=1,\dots,q)$ are functions of bounded variation, and D_{0+}^{k} denotes the Riemann-Liouville derivative of order k (for $k=\alpha_1,\beta_1,\alpha_2,\beta_2,\gamma_i$ for $i=0,1,\dots,p$; δ_i for $i=0,1,\dots,q$). However, the uniqueness has not been considered. So we discuss the similar system (1.1) in a new way by modifying it appropriately. Motivated by [22,30,37], we take an interesting approach to deal with the system (1.1), and we intend to establish the existence and uniqueness of solutions for the system (1.1). Moreover, it is easy to see that the boundary conditions in (1.1) are more complex than ones in (1.2).

2. Preliminaries and previous results

We present here some definitions and related properties of Riemann-Liouville fractional derivatives and integrals. Some auxiliary results which will be used to prove our main results are also listed.

DEFINITION 2.1. ([18]) For a continuous function $f:(0,+\infty)\to(-\infty,+\infty)$, the Riemann-Liouville fractional integral of order $\alpha>0$ of is given by

$$I_{0^+}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds,$$

provided the right-hand side is pointwise defined on $(0,+\infty)$.

DEFINITION 2.2. ([18]) For a continuous function $f:(0,+\infty)\to(-\infty,+\infty)$, the Riemann-Liouville fractional derivative of order $\alpha>0$ of is given by

$$D_{0^+}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t (t-s)^{n-\alpha-1}f(s)ds,$$

where $n=[\alpha]+1,[\alpha]$ denotes the integer part of the number α, provided that the righthand side is pointwise defined on $(0,+\infty)$.

For $\widetilde{h},\widetilde{k}\in C[0,1]$, we study the system of fractional differential equations

$$\begin{cases} D_{0+}^{\gamma_1}(\varphi_{\rho_1}(D_{0+}^{\delta_1}u(t)))+\widetilde{h}(t)=0, \quad t\in(0,1), \\ D_{0+}^{\gamma_2}(\varphi_{\rho_2}(D_{0+}^{\delta_2}v(t)))+\widetilde{k}(t)=0, \quad t\in(0,1), \end{cases} \tag{2.1}$$

under the coupled boundary conditions

$$\begin{cases} u^{(i_1)}(0)=0,\ D_{0+}^{\delta_1}u(0)=0,\ D_{0+}^{\alpha_0}u(1)=\sum\limits_{j=1}^{n}\int_0^1 D_{0+}^{\alpha_j}v(\tau)d\mathfrak{H}_j(\tau),\\ v^{(i_2)}(0)=0,\ D_{0+}^{\delta_2}v(0)=0,\ D_{0+}^{\beta_0}v(1)=\sum\limits_{j=1}^{m}\int_0^1 D_{0+}^{\beta_j}u(\tau)d\mathfrak{K}_j(\tau),\end{cases} \tag{2.2}$$

with $i_1=0,\ldots,p-2$; $i_2=0,\ldots,q-2$. Let

$$\Delta_1=\sum_{i=1}^{n}\frac{\Gamma(\delta_2)}{\Gamma(\delta_2-\alpha_i)}\int_0^1\tau^{\delta_2-\alpha_i-1}d\mathfrak{H}_i(\tau),$$
$$\Delta_2=\sum_{i=1}^{m}\frac{\Gamma(\delta_1)}{\Gamma(\delta_1-\beta_i)}\int_0^1\tau^{\delta_1-\beta_i-1}d\mathfrak{K}_i(\tau),$$
$$\Delta=\frac{\Gamma(\delta_1)\Gamma(\delta_2)}{\Gamma(\delta_1-\alpha_0)\Gamma(\delta_2-\beta_0)}-\Delta_1\Delta_2.$$

LEMMA 2.1. ([10,22]) *If* $\Delta\neq 0$, *then the unique solution* $(u,v)\in(C[0,1])^2$ *of* (2.1), (2.2) *is given by*

$$\begin{cases} u(t)=\int_0^1\mathfrak{G}_1(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1}\widetilde{h}(\zeta))d\zeta+\int_0^1\mathfrak{G}_2(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2}\widetilde{k}(\zeta))d\zeta,\\ v(t)=\int_0^1\mathfrak{G}_3(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1}\widetilde{h}(\zeta))d\zeta+\int_0^1\mathfrak{G}_4(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2}\widetilde{k}(\zeta))d\zeta,\end{cases} \tag{2.3}$$

$t\in[0,1]$, *where*

$$\mathfrak{G}_1(t,\zeta)=\mathfrak{g}_1(t,\zeta)+\frac{t^{\delta_1-1}\Delta_1}{\Delta}(\sum_{j=1}^{m}\int_0^1\mathfrak{g}_{1j}(\tau,\zeta)d\mathfrak{K}_j(\tau)),$$
$$\mathfrak{G}_2(t,\zeta)=\frac{t^{\delta_1-1}\Gamma(\delta_2)}{\Delta\Gamma(\delta_2-\beta_0)}\sum_{j=1}^{n}\int_0^1\mathfrak{g}_{2j}(\tau,\zeta)d\mathfrak{H}_j(\tau),$$
$$\mathfrak{G}_3(t,\zeta)=\frac{t^{\delta_2-1}\Gamma(\delta_1)}{\Delta\Gamma(\delta_1-\alpha_0)}\sum_{j=1}^{m}\int_0^1\mathfrak{g}_{1j}(\tau,\zeta)d\mathfrak{K}_j(\tau),$$
$$\mathfrak{G}_4(t,\zeta)=\mathfrak{g}_2(t,\zeta)+\frac{t^{\delta_2-1}\Delta_2}{\Delta}(\sum_{j=1}^{n}\int_0^1\mathfrak{g}_{2j}(\tau,\zeta)d\mathfrak{H}_j(\tau))$$

for all $(t,\zeta)\in[0,1]\times[0,1]$ *and*

$$\mathfrak{g}_1(t,\zeta)=\frac{1}{\Gamma(\delta_1)}\begin{cases} t^{\delta_1-1}(1-\zeta)^{\delta_1-\alpha_0-1}-(t-\zeta)^{\delta_1-1},\ 0\leqslant\zeta\leqslant t\leqslant 1,\\ t^{\delta_1-1}(1-\zeta)^{\delta_1-\alpha_0-1},\ 0\leqslant t\leqslant\zeta\leqslant 1,\end{cases}$$
$$\mathfrak{g}_{1j}(\tau,\zeta)=\frac{1}{\Gamma(\delta_1-\beta_j)}\begin{cases} \tau^{\delta_1-\beta_j-1}(1-\zeta)^{\delta_1-\alpha_0-1}-(\tau-\zeta)^{\delta_1-\beta_j-1},\ 0\leqslant\zeta\leqslant\tau\leqslant 1,\\ \tau^{\delta_1-\beta_j-1}(1-\zeta)^{\delta_1-\alpha_0-1},\ 0\leqslant\tau\leqslant\zeta\leqslant 1,\end{cases}$$

$$\mathfrak{g}_2(t,\zeta)=\frac{1}{\Gamma(\delta_2)}\begin{cases}t^{\delta_2-1}(1-\zeta)^{\delta_2-\beta_0-1}-(t-\zeta)^{\delta_2-1}, & 0\leqslant\zeta\leqslant t\leqslant 1,\\ t^{\delta_2-1}(1-\zeta)^{\delta_2-\beta_0-1}, & 0\leqslant t\leqslant\zeta\leqslant 1,\end{cases}$$

$$\mathfrak{g}_{2j}(\tau,\zeta)=\frac{1}{\Gamma(\delta_2-\alpha_j)}\begin{cases}\tau^{\delta_2-\alpha_j-1}(1-\zeta)^{\delta_2-\beta_0-1}-(\tau-\zeta)^{\delta_2-\alpha_j-1}, & 0\leqslant\zeta\leqslant\tau\leqslant 1,\\ \tau^{\delta_2-\alpha_j-1}(1-\zeta)^{\delta_2-\beta_0-1}, & 0\leqslant\tau\leqslant\zeta\leqslant 1,\end{cases}$$

for all $j=1,\dots,m$ *and* $j=1,\dots,n$.

LEMMA 2.2. ([10,22]) *Suppose that* $\Delta>0$, $\mathfrak{H}_j$ $(j=1,\dots,n)$, $\mathfrak{K}_j$ $(j=1,\dots,m)$ *are nondecreasing functions. The functions* $\mathfrak{G}_i$ $(i=1,\dots,4)$ *have the following properties:*

(1) $\mathfrak{G}_i:[0,1]\times[0,1]\to\mathbb{R}_+$ $(i=1,\dots,4)$ *are continuous functions;*

(2) $\mathfrak{G}_1(t,\zeta)\leqslant\mathfrak{J}_1(\zeta)$ *for all* $(t,\zeta)\in[0,1]\times[0,1]$, *where*

$$\mathfrak{J}_1(\zeta)=\mathfrak{h}_1(\zeta)+\frac{\Delta_1}{\Delta}\left(\sum_{j=1}^{m}\int_0^1\mathfrak{g}_{1j}(\tau,\zeta)d\mathfrak{K}_j(\tau)\right),\ \forall\zeta\in[0,1]$$

$$\mathfrak{h}_1(\zeta)=\frac{1}{\Gamma(\delta_1)}(1-\zeta)^{\delta_1-\alpha_0-1}(1-(1-\zeta)^{\alpha_0}),\ \forall\zeta\in[0,1];$$

(3) $\mathfrak{G}_1(t,\zeta)\geqslant t^{\delta_1-1}\mathfrak{J}_1(\zeta)$ *for all* $(t,\zeta)\in[0,1]\times[0,1]$;

(4) $\mathfrak{G}_2(t,\zeta)\leqslant\mathfrak{J}_2(\zeta)$ *for all* $(t,\zeta)\in[0,1]\times[0,1]$, *where*

$$\mathfrak{J}_2(\zeta)=\frac{\Gamma(\delta_2)}{\Delta\Gamma(\delta_2-\beta_0)}\sum_{j=1}^{n}\int_0^1\mathfrak{g}_{2j}(\tau,\zeta)d\mathfrak{H}_j(\tau),\ \forall\zeta\in[0,1];$$

(5) $\mathfrak{G}_2(t,\zeta)=t^{\delta_1-1}\mathfrak{J}_2(\zeta)$ *for all* $(t,\zeta)\in[0,1]\times[0,1]$;

(6) $\mathfrak{G}_3(t,\zeta)\leqslant\mathfrak{J}_3(\zeta)$ *for all* $(t,\zeta)\in[0,1]\times[0,1]$, *where*

$$\mathfrak{J}_3(\zeta)=\frac{\Gamma(\delta_1)}{\Delta\Gamma(\delta_1-\alpha_0)}\sum_{j=1}^{m}\int_0^1\mathfrak{g}_{1j}(\tau,\zeta)d\mathfrak{K}_j(\tau),\ \forall\zeta\in[0,1];$$

(7) $\mathfrak{G}_3(t,\zeta)=t^{\delta_2-1}\mathfrak{J}_3(\zeta)$ *for all* $(t,\zeta)\in[0,1]\times[0,1]$;

(8) $\mathfrak{G}_4(t,\zeta)\leqslant\mathfrak{J}_4(\zeta)$ *for all* $(t,\zeta)\in[0,1]\times[0,1]$,*where*

$$\mathfrak{J}_4(\zeta)=\mathfrak{h}_2(\zeta)+\frac{\Delta_2}{\Delta}\left(\sum_{j=1}^{n}\int_0^1\mathfrak{g}_{2j}(\tau,\zeta)d\mathfrak{H}_j(\tau)\right),\ \forall\zeta\in[0,1]$$

$$\mathfrak{h}_2(\zeta)=\frac{1}{\Gamma(\delta_2)}(1-\zeta)^{\delta_2-\beta_0-1}(1-(1-\zeta)^{\beta_0}),\ \forall\zeta\in[0,1];$$

(9) $\mathfrak{G}_4(t,\zeta)\geqslant t^{\delta_2-1}\mathfrak{J}_4(\zeta)$ *for all* $(t,\zeta)\in[0,1]\times[0,1]$.

Now, in order to reach the main conclusion, we give some concepts, notations and conclusions in abstract spaces.

Let $(E,\|\cdot\|)$ be a real Banach space which is partially ordered by a cone $P\subset E$. An operator $A:E\to E$ is increasing, if $x\leqslant y$ implies $Ax\leqslant Ay$. For any $x,y\in E$, the notation $x\sim y$ means that there exist $\lambda>0$ and $\mu>0$ such that $\lambda x\leqslant y\leqslant\mu x$. It shows that $\sim$ is an equivalence relation. Given $h>\theta$ (i.e., $h\geqslant\theta$ and $h\neq\theta$), we denote P_h by the set $P_h=\{x\in E\mid x\sim h\}$, $P_h\subset P$. For $h_1,h_2\in P$ with $h_1,h_2\neq\theta$. Suppose $h=(h_1,h_2)$, then $h\in\overline{P}:=P\times P$. If P is normal, then $\overline{P}=P\times P$ is normal.

LEMMA 2.3. ([34]) *For $h_1, h_2 \in P$ with $h_1, h_2 \neq \theta$, let $h = (h_1, h_2)$, then $\overline{P_h} = P_{h_1} \times P_{h_2}$.*

LEMMA 2.4. ([35]) *Let E be a real Banach space and P be normal in E, $h > \theta$, and $A : P \to P$ be an increasing operator, satisfying:*

(i) there is $h_0 \in P_h$ such that $Ah_0 \in P_h$;

(ii) for any $x \in P$ and $t \in (0,1)$, there exists $\varphi(t) \in (t,1)$ such that $A(tx) \geqslant \varphi(t)Ax$.

Then:

(1) the operator equation $Ax = x$ has a unique solution x^* in P_h;

(2) take any initial value $x_0 \in P_h$ and construct successively a sequence $x_n = Ax_{n-1}$, $n = 1,2,\ldots$, we have $x_n \to x^*$ as $n \to \infty$.

3. Main results

For convenience, let $E = C[0,1]$, then E is a Banach space with the norm $\|u\| = \max\{|u(t)| : t \in [0,1]\}$. We will consider (1.1) in $E \times E$. For $(u,v) \in E \times E$, let

$$\|(u,v)\| = \max\{\|u\|, \|v\|\},$$

then $(E \times E, \|(\cdot,\cdot)\|)$ is a Banach space. Let $\overline{P} = \{(u,v) \in E \times E : u(t) \geqslant 0,\ v(t) \geqslant 0,\ t \in [0,1]\}$, $P = \{x \in E : x(t) \geqslant 0,\ t \in [0,1]\}$, then the cone $\overline{P} \subset E \times E$ and $\overline{P} = P \times P$ is normal, and the space $E \times E$ has a partial order:

$$(u_1, v_1) \leqslant (u_2, v_2) \Longleftrightarrow u_1(t) \leqslant u_2(t),\ \ v_1(t) \leqslant v_2(t),\ \ t \in [0,1].$$

From Lemma 2.1, we can obtain

LEMMA 3.1. *If $f(t,x), g(t,x)$ are continuous, then $(u,v) \in E \times E$ is a solution of (1.1) if and only if $(u,v) \in E \times E$ is a solution of the following equations:*

$$\begin{cases} u(t) = \int_0^1 \mathfrak{G}_1(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1} f(\zeta, v(\zeta)))d\zeta + \int_0^1 \mathfrak{G}_2(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2} g(\zeta, u(\zeta)))d\zeta,\ t \in [0,1], \\ v(t) = \int_0^1 \mathfrak{G}_3(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1} f(\zeta, v(\zeta)))d\zeta + \int_0^1 \mathfrak{G}_4(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2} g(\zeta, u(\zeta)))d\zeta,\ t \in [0,1]. \end{cases}$$

Let $\gamma_1, \gamma_2 \in (0,1]$, $\delta_1 \in (p-1, p]$, $\delta_2 \in (q-1, q]$, $p, q \in \mathbb{N}$, $p, q \geqslant 3$, $h_1(t) = t^{\delta_1 - 1}$, $h_2(t) = t^{\delta_2 - 1}$, $t \in [0,1]$. Assume that

(H_1) *$f, g \in C([0,1] \times [0,+\infty), [0,+\infty))$ and $f(t,0) \not\equiv 0$, $g(t,0) \not\equiv 0$, $t \in [0,1]$;*

(H_2) *f, g are increasing with respect to the second variable, i.e., $f(t,v_1) \leqslant f(t,v_2)$, $g(t,u_1) \leqslant g(t,u_2)$ for $t \in [0,1]$, $0 \leqslant v_1 \leqslant v_2$, $0 \leqslant u_1 \leqslant u_2$;*

(H_3) *there exist $\psi(\lambda) > \lambda^{1/(\rho_i - 1)}$, for any $\lambda \in (0,1)$, such that*

$$f(t, \lambda x) \geqslant \psi(\lambda) f(t,x),\ \ g(t, \lambda x) \geqslant \psi(\lambda) g(t,x),\ \ \forall t \in [0,1],\ \ x \in [0,+\infty).$$

We consider three operators $A_1, A_2 : P \to E$ and $T : \overline{P} \to E \times E$ defined by

$$\begin{aligned} A_1 u(t) = &\int_0^1 \mathfrak{G}_1(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1} f(\zeta, v(\zeta)))d\zeta \\ &+ \int_0^1 \mathfrak{G}_2(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2} g(\zeta, u(\zeta)))d\zeta,\ \ t \in [0,1], \end{aligned}$$

$$\begin{aligned}
A_2v(t) &= \int_0^1 \mathfrak{G}_3(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1} f(\zeta,v(\zeta)))d\zeta \\
&\quad + \int_0^1 \mathfrak{G}_4(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2} g(\zeta,u(\zeta)))d\zeta, \ \ t\in[0,1], \\
T(u,v)(t) &= (A_1u(t), A_2v(t)).
\end{aligned}$$

LEMMA 3.2. *Assume that* (H_1) *and* (H_2) *hold. Then* $T:\overline{P}\to\overline{P}$ *is increasing (i.e.* $A_1:P\to P$, $A_2:P\to P$ *are increasing).*

Proof. By Lemma 2.2 (1) and (H_1), we can easily get $A_1:P\to P$, $A_2:P\to P$, and thus $T:\overline{P}\to\overline{P}$.

Next, we need to prove that two operators A_1,A_2 are increasing. For $u_i,v_i\in E$, $i=1,2$ with $u_1\leqslant u_2$, $v_1\leqslant v_2$, then $u_1(t)\leqslant u_2(t)$, $v_1(t)\leqslant v_2(t)$, for all $t\in[0,1]$. Noting that $I_{0+}^{\gamma_i}$ $(i=1,2)$ are increasing in $\mathbb{R}_+$ by the definition of the Riemann-Liouville fractional integral. If $0\leqslant\zeta_1\leqslant\zeta_2$, then $\varphi_{\rho_1}(\zeta_1)\leqslant\varphi_{\rho_1}(\zeta_2)$ and $\varphi_{\rho_2}(\zeta_1)\leqslant\varphi_{\rho_2}(\zeta_2)$. Thus, φ_{ρ_i}, $(i=1,2)$ are increasing in $\mathbb{R}_+$. By (H_2), we get

$$\begin{aligned}
A_1u_1(t) &= \int_0^1 \mathfrak{G}_1(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1} f(\zeta,v_1(\zeta)))d\zeta + \int_0^1 \mathfrak{G}_2(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2} g(\zeta,u_1(\zeta)))d\zeta \\
&\leqslant \int_0^1 \mathfrak{G}_1(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1} f(\zeta,v_2(\zeta)))d\zeta + \int_0^1 \mathfrak{G}_2(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2} g(\zeta,u_2(\zeta)))d\zeta \\
&= A_1u_2(t),
\end{aligned}$$

$$\begin{aligned}
A_2v_1(t) &= \int_0^1 \mathfrak{G}_3(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1} f(\zeta,v_1(\zeta)))d\zeta + \int_0^1 \mathfrak{G}_4(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2} g(\zeta,u_1(\zeta)))d\zeta \\
&\leqslant \int_0^1 \mathfrak{G}_3(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1} f(\zeta,v_2(\zeta)))d\zeta + \int_0^1 \mathfrak{G}_4(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2} g(\zeta,u_2(\zeta)))d\zeta \\
&= A_2v_2(t).
\end{aligned}$$

In conclusion, A_1,A_2 are increasing. Therefore, $T:\overline{P}\to\overline{P}$ is increasing. □

LEMMA 3.3. *Assume that* (H_2), (H_3) *hold. Then for any* $(u,v)\in\overline{P}$ *and* $\lambda\in(0,1)$,

$$T(\lambda(u,v)) \geqslant \Psi(\lambda)T(u,v),$$

where $\Psi(\lambda)=\min\{\varphi_{\rho_i}(\psi(\lambda))\}$, $i=1,2$.

Proof. For any $(u,v)\in\overline{P}$ and $\lambda\in(0,1)$, we get

$$T(\lambda(u,v)) = (A_1(\lambda u), A_2(\lambda v)).$$

We next consider $A_1(\lambda u)$ and $A_2(\lambda v)$ respectively. For $\lambda\in(0,1)$, according to (H_3), we have

$$\Psi(\lambda)=\min\{(\psi(\lambda))^{\rho_1-1},(\psi(\lambda))^{\rho_2-1}\} > \min\{(\lambda^{1/(\rho_1-1)})^{\rho_1-1},(\lambda^{1/(\rho_2-1)})^{\rho_2-1}\}=\lambda.$$

Further, for $\lambda \in (0,1)$ and $u,v \in E$, by (H_3), we obtain

$$
\begin{aligned}
&A_1(\lambda u)(t)\\
&= \int_0^1 \mathfrak{G}_1(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1} f(\zeta,\lambda v(\zeta)))d\zeta + \int_0^1 \mathfrak{G}_2(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2} g(\zeta,\lambda u(\zeta)))d\zeta\\
&\geqslant \int_0^1 \mathfrak{G}_1(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1}\psi(\lambda) f(\zeta,v(\zeta)))d\zeta\\
&\quad + \int_0^1 \mathfrak{G}_2(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2}\psi(\lambda) g(\zeta,u(\zeta)))d\zeta\\
&= \varphi_{\rho_1}(\psi(\lambda))\int_0^1 \mathfrak{G}_1(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1} f(\zeta,v(\zeta)))d\zeta\\
&\quad + \varphi_{\rho_2}(\psi(\lambda))\int_0^1 \mathfrak{G}_2(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2} g(\zeta,u(\zeta)))d\zeta\\
&\geqslant \Psi(\lambda)\left(\int_0^1 \mathfrak{G}_1(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1} f(\zeta,v(\zeta)))d\zeta + \int_0^1 \mathfrak{G}_2(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2} g(\zeta,u(\zeta)))d\zeta\right)\\
&= \Psi(\lambda)A_1u(t),
\end{aligned}
$$

$$
\begin{aligned}
&A_2(\lambda v)(t)\\
&= \int_0^1 \mathfrak{G}_3(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1} f(\zeta,\lambda v(\zeta)))d\zeta + \int_0^1 \mathfrak{G}_4(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2} g(\zeta,\lambda u(\zeta)))d\zeta\\
&\geqslant \int_0^1 \mathfrak{G}_3(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1}\psi(\lambda) f(\zeta,v(\zeta)))d\zeta\\
&\quad + \int_0^1 \mathfrak{G}_4(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2}\psi(\lambda) g(\zeta,\lambda u(\zeta)))d\zeta\\
&= \varphi_{\rho_1}(\psi(\lambda))\int_0^1 \mathfrak{G}_3(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1} f(\zeta,v(\zeta)))d\zeta\\
&\quad + \varphi_{\rho_2}(\psi(\lambda))\int_0^1 \mathfrak{G}_4(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2} g(\zeta,u(\zeta)))d\zeta\\
&\geqslant \Psi(\lambda)\left(\int_0^1 \mathfrak{G}_3(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1} f(\zeta,v(\zeta)))d\zeta + \int_0^1 \mathfrak{G}_4(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2} g(\zeta,u(\zeta)))d\zeta\right)\\
&= \Psi(\lambda)A_2v(t).
\end{aligned}
$$

So we have

$$T(\lambda(u,v))(t) \geqslant (\Psi(\lambda)A_1u(t),\Psi(\lambda)A_2v(t)) = \Psi(\lambda)(A_1u(t),A_2v(t)) = \Psi(\lambda)T(u,v)(t).$$

That is $T(\lambda(u,v)) \geqslant \Psi(\lambda)T(u,v)$ for $\lambda \in (0,1)$, $(u,v) \in \overline{P}$. $\square$

LEMMA 3.4. *Assume that* (H_1) *and* (H_2) *hold. Then there exists* $h \in \overline{P_h}$ *such that* $Th \in \overline{P_h}$.

Proof. Set $h = (h_1,h_2)$, where $h_1(t) = t^{\delta_1 - 1}$, $h_2(t) = t^{\delta_2 - 1}$, for all $t \in [0,1]$. By

(H_1), (H_2), Lemma 2.2 and the Definitions of A_1, A_2, we obtain

$$
\begin{aligned}
A_1 h_1(t) &= \int_0^1 \mathfrak{G}_1(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1} f(\zeta,h_2(\zeta)))d\zeta + \int_0^1 \mathfrak{G}_2(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2} g(\zeta,h_1(\zeta)))d\zeta \\
&\geqslant \int_0^1 t^{\delta_1-1}\mathfrak{J}_1(\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1} f(\zeta,\zeta^{\delta_2-1}))d\zeta \\
&\quad + \int_0^1 t^{\delta_1-1}\mathfrak{J}_2(\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2} g(\zeta,\zeta^{\delta_1-1}))d\zeta \\
&\geqslant \int_0^1 t^{\delta_1-1}\mathfrak{J}_1(\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1} f(\zeta,0))d\zeta + \int_0^1 t^{\delta_1-1}\mathfrak{J}_2(\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2} g(\zeta,0))d\zeta \\
&= \left(\int_0^1 \mathfrak{J}_1(\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1} f(\zeta,0))d\zeta + \int_0^1 \mathfrak{J}_2(\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2} g(\zeta,0))d\zeta\right) h_1(t), \\
A_1 h_1(t) &= \int_0^1 \mathfrak{G}_1(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1} f(\zeta,h_2(\zeta)))d\zeta + \int_0^1 \mathfrak{G}_2(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2} g(\zeta,h_1(\zeta)))d\zeta \\
&\leqslant \int_0^1 \left(\frac{1}{\Gamma(\delta_1)} t^{\delta_1-1}(1-\zeta)^{\delta_1-\alpha_0-1} + \frac{t^{\delta_1-1}\Delta_1}{\Delta}\sum_{j=1}^m \int_0^1 \mathfrak{g}_{1j}(\tau,\zeta)d\mathfrak{K}_j(\tau)\right) \\
&\quad \times \varphi_{\rho_1}(I_{0+}^{\gamma_1} f(\zeta,1))d\zeta \\
&\quad + \int_0^1 \left(\frac{t^{\delta_1-1}\Gamma(\delta_2)}{\Delta\Gamma(\delta_2-\beta_0)}\sum_{j=1}^n \int_0^1 \mathfrak{g}_{2j}(\tau,\zeta)d\mathfrak{H}_j(\tau)\right)\varphi_{\rho_2}(I_{0+}^{\gamma_2} g(\zeta,1))d\zeta \\
&= \frac{1}{\Gamma(\delta_1)} t^{\delta_1-1}\int_0^1 (1-\zeta)^{\delta_1-\alpha_0-1}\varphi_{\rho_1}(I_{0+}^{\gamma_1} f(\zeta,1))d\zeta \\
&\quad + \frac{t^{\delta_1-1}\Delta_1}{\Delta}\int_0^1 \left(\sum_{j=1}^m \int_0^1 \mathfrak{g}_{1j}(\tau,\zeta)d\mathfrak{K}_j(\tau)\right)\varphi_{\rho_1}(I_{0+}^{\gamma_1} f(\zeta,1))d\zeta \\
&\quad + \frac{t^{\delta_1-1}\Gamma(\delta_2)}{\Delta\Gamma(\delta_2-\beta_0)}\int_0^1 \left(\sum_{j=1}^n \int_0^1 \mathfrak{g}_{2j}(\tau,\zeta)d\mathfrak{H}_j(\tau)\right)\varphi_{\rho_2}(I_{0+}^{\gamma_2} g(\zeta,1))d\zeta \\
&= \Bigg[\frac{1}{\Gamma(\delta_1)}\int_0^1 (1-\zeta)^{\delta_1-\alpha_0-1}\varphi_{\rho_1}(I_{0+}^{\gamma_1} f(\zeta,1))d\zeta \\
&\quad + \frac{\Delta_1}{\Delta}\int_0^1 \left(\sum_{j=1}^m \int_0^1 \mathfrak{g}_{1j}(\tau,\zeta)d\mathfrak{K}_j(\tau)\right)\varphi_{\rho_1}(I_{0+}^{\gamma_1} f(\zeta,1))d\zeta \\
&\quad + \frac{\Gamma(\delta_2)}{\Delta\Gamma(\delta_2-\beta_0)}\int_0^1 \left(\sum_{j=1}^n \int_0^1 \mathfrak{g}_{2j}(\tau,\zeta)d\mathfrak{H}_j(\tau)\right)\varphi_{\rho_2}(I_{0+}^{\gamma_2} g(\zeta,1))d\zeta\Bigg] h_1(t).
\end{aligned}
$$

Overall, we have

$$
\begin{aligned}
&\frac{1}{\Gamma(\delta_1)}\int_0^1 (1-\zeta)^{\delta_1-\alpha_0-1}\varphi_{\rho_1}(I_{0+}^{\gamma_1} f(\zeta,1))d\zeta \\
&+ \frac{\Delta_1}{\Delta}\int_0^1 \left(\sum_{j=1}^m \int_0^1 \mathfrak{g}_{1j}(\tau,\zeta)d\mathfrak{K}_j(\tau)\right)\varphi_{\rho_1}(I_{0+}^{\gamma_1} f(\zeta,1))d\zeta
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\Gamma(\delta_2)}{\Delta\Gamma(\delta_2-\beta_0)}\int_0^1\left(\sum_{j=1}^n\int_0^1 \mathfrak{g}_{2j}(\tau,\zeta)d\mathfrak{H}_j(\tau)\right)\varphi_{\rho_2}(I_{0+}^{\gamma_2}g(\zeta,1))d\zeta\\
\geqslant&\int_0^1\mathfrak{J}_1(\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1}f(\zeta,0))d\zeta+\int_0^1\mathfrak{J}_2(\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2}g(\zeta,0))d\zeta\\
=&\frac{1}{\Gamma(\delta_1)}\int_0^1(1-\zeta)^{\delta_1-\alpha_0-1}(1-(1-\zeta)^{\alpha_0})\varphi_{\rho_1}(I_{0+}^{\gamma_1}f(\zeta,0))d\zeta\\
&+\frac{\Delta_1}{\Delta}\int_0^1\left(\sum_{j=1}^m \mathfrak{g}_{1j}(\tau,\zeta)d\mathfrak{K}_j(\tau)\right)\varphi_{\rho_1}(I_{0+}^{\gamma_1}f(\zeta,0))d\zeta\\
&+\frac{\Gamma(\delta_2)}{\Delta\Gamma(\delta_2-\beta_0)}\int_0^1\left(\sum_{j=1}^n\int_0^1 \mathfrak{g}_{2j}(\tau,\zeta)d\mathfrak{H}_j(\tau)\right)\varphi_{\rho_2}(I_{0+}^{\gamma_2}g(\zeta,0))d\zeta.
\end{aligned}
$$

Let us denote

$$
\begin{aligned}
l_1=&\int_0^1\mathfrak{J}_1(\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1}f(\zeta,0))d\zeta+\int_0^1\mathfrak{J}_2(\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2}g(\zeta,0))d\zeta\\
=&\int_0^1\left(\frac{1}{\Gamma(\delta_1)}(1-\zeta)^{\delta_1-\alpha_0-1}(1-(1-\zeta)^{\alpha_0})+\frac{\Delta_1}{\Delta}\sum_{j=1}^m\int_0^1 \mathfrak{g}_{1j}(\tau,\zeta)d\mathfrak{K}_j(\tau)\right)\\
&\times\varphi_{\rho_1}(I_{0+}^{\gamma_1}f(\zeta,0))d\zeta\\
&+\int_0^1\left(\frac{\Gamma(\delta_2)}{\Delta\Gamma(\delta_2-\beta_0)}\sum_{j=1}^n\int_0^1 \mathfrak{g}_{2j}(\tau,\zeta)d\mathfrak{H}_j(\tau)\right)\varphi_{\rho_2}(I_{0+}^{\gamma_2}g(\zeta,0))d\zeta\\
=&\frac{1}{\Gamma(\delta_1)}\int_0^1(1-\zeta)^{\delta_1-\alpha_0-1}(1-(1-\zeta)^{\alpha_0})\varphi_{\rho_1}(I_{0+}^{\gamma_1}f(\zeta,0))d\zeta\\
&+\frac{\Delta_1}{\Delta}\int_0^1\left(\sum_{j=1}^m \mathfrak{g}_{1j}(\tau,\zeta)d\mathfrak{K}_j(\tau)\right)\varphi_{\rho_1}(I_{0+}^{\gamma_1}f(\zeta,0))d\zeta\\
&+\frac{\Gamma(\delta_2)}{\Delta\Gamma(\delta_2-\beta_0)}\int_0^1\left(\sum_{j=1}^n\int_0^1 \mathfrak{g}_{2j}(\tau,\zeta)d\mathfrak{H}_j(\tau)\right)\varphi_{\rho_2}(I_{0+}^{\gamma_2}g(\zeta,0))d\zeta,\\
l_2=&\frac{1}{\Gamma(\delta_1)}\int_0^1(1-\zeta)^{\delta_1-\alpha_0-1}\varphi_{\rho_1}(I_{0+}^{\gamma_1}f(\zeta,1)d\zeta\\
&+\frac{\Delta_1}{\Delta}\int_0^1\left(\sum_{j=1}^m\int_0^1 \mathfrak{g}_{1j}(\tau,\zeta)d\mathfrak{K}_j(\tau)\right)\varphi_{\rho_1}(I_{0+}^{\gamma_1}f(\zeta,1)d\zeta\\
&+\frac{\Gamma(\delta_2)}{\Delta\Gamma(\delta_2-\beta_0)}\int_0^1\left(\sum_{j=1}^n\int_0^1 \mathfrak{g}_{2j}(\tau,\zeta)d\mathfrak{H}_j(\tau)\right)\varphi_{\rho_2}(I_{0+}^{\gamma_2}g(\zeta,1))d\zeta.
\end{aligned}
$$

By (H_1), we have $\int_0^1 f(\zeta,0)d\zeta>0$, $\int_0^1 g(\zeta,0)d\zeta>0$. By (H_2), we can get $f(\zeta,1)\geqslant f(\zeta,0)$, $g(\zeta,1)\geqslant g(\zeta,0)$. Thus, $l_2\geqslant l_1>0$, and thus $l_1h_1(t)\leqslant A_1h_1(t)\leqslant l_2h_1(t)$. This shows $A_1h_1\in P_{h_1}$. Similarly, we can also get $A_2h_2\in P_{h_2}$. Consequently, by Lemma 2.3,

$$Th=(A_1h_1,A_2h_2)\in P_{h_1}\times P_{h_2}=\overline{P_h}.$$

The proof is completed. $\square$

THEOREM 3.1. *Assume that* (H_1)–(H_3) *are satisfied. Then the following conclusions hold:*

(1) *the system* (1.1) *has a unique solution* (u^*,v^*) *in* $\overline{P_h}$, *where*

$$h(t)=(h_1(t),h_2(t)),\ \ h_1(t)=t^{\delta_1-1},\ \ h_2(t)=t^{\delta_2-1},\ \ t\in[0,1];$$

(2) *for a given point* $(u_0,v_0)\in\overline{P_h}$, *construct the following sequences:*

$$u_{n+1}(t)=\int_0^1\mathfrak{G}_1(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1}f(\zeta,v_n(\zeta)))d\zeta+\int_0^1\mathfrak{G}_2(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2}g(\zeta,u_n(\zeta)))d\zeta,$$

$$v_{n+1}(t)=\int_0^1\mathfrak{G}_3(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1}f(\zeta,v_n(\zeta)))d\zeta+\int_0^1\mathfrak{G}_4(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2}g(\zeta,u_n(\zeta)))d\zeta,$$

$t\in[0,1]$, $n=0,1,2,\cdots$, *we have* $u_n(t)\to u^*(t)$, $v_n(t)\to v^*(t)$, *as* $n\to\infty$.

Proof. By Lemma 3.2, $T:\overline{P}\to\overline{P}$ is increasing. From Lemma 3.3, we get $T(\lambda(u,v))\geqslant\Psi(\lambda)T(u,v)$ for $\lambda\in(0,1)$, $(u,v)\in\overline{P}$, where $\Psi(\lambda)=\min\{(\psi(\lambda))^{\rho_1-1},(\psi(\lambda))^{\rho_2-1}\}$ and $\Psi(\lambda)>\lambda$, for all $\lambda\in(0,1)$. Further, by Lemma 3.4, we found $h=(h_1,h_2)\in\overline{P_h}$ and $Th\in\overline{P_h}$. Hence, all the conditions of Lemma 2.4 are satisfied, which implies: the system (1.1) has a unique solution (u^*,v^*) in $\overline{P_h}$, where

$$h(t)=(h_1(t),h_2(t)),\ \ h_1(t)=t^{\delta_1-1},\ \ h_2(t)=t^{\delta_2-1},\ \ t\in[0,1];$$

and for any given point $(u_0,v_0)\in\overline{P_h}$, construct the following sequences

$$u_{n+1}(t)=\int_0^1\mathfrak{G}_1(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1}f(\zeta,v_n(\zeta)))d\zeta+\int_0^1\mathfrak{G}_2(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2}g(\zeta,u_n(\zeta)))d\zeta,$$

$$v_{n+1}(t)=\int_0^1\mathfrak{G}_3(t,\zeta)\varphi_{\rho_1}(I_{0+}^{\gamma_1}f(\zeta,v_n(\zeta)))d\zeta+\int_0^1\mathfrak{G}_4(t,\zeta)\varphi_{\rho_2}(I_{0+}^{\gamma_2}g(\zeta,u_n(\zeta)))d\zeta,$$

$t\in[0,1]$, $n=0,1,2,\cdots$, we have $u_n(t)\to u^*(t)$, $v_n(t)\to v^*(t)$, as $n\to\infty$. $\square$

4. An example

Considering the following system:

$$\begin{cases} D_{0+}^{\frac12}(\varphi_{\frac32}(D_{0+}^{\frac52}u(t)))+t(v^{\frac13}+t)=0, & t\in(0,1),\\ D_{0+}^{\frac12}(\varphi_{\frac32}(D_{0+}^{\frac52}u(t)))+t(u^{\frac13}+3t)=0, & t\in(0,1),\end{cases}\tag{4.1}$$

subject to the coupled nonlocal boundary conditions

$$\begin{cases} u(0)=u'(0)=0,\ \ D_{0+}^{\frac52}u(0)=0,\ \ D_{0+}^{\frac65}u(1)=\int_0^1D_{0+}^{\frac25}v(\tau)d\tau+2\int_0^1D_{0+}^{\frac45}v(\tau)d\tau,\\ v(0)=v'(0)=0,\ \ D_{0+}^{\frac52}v(0)=0,\ \ D_{0+}^{\frac65}v(1)=\int_0^1D_{0+}^{\frac25}u(\tau)d\tau+3\int_0^1D_{0+}^{\frac45}u(\tau)d\tau\end{cases}\tag{4.2}$$

where $\gamma_1=\gamma_2=\frac{1}{2}$, $\delta_1=\delta_2=\frac{5}{2}$, $p=q=3$, $\rho_1=\rho_2=\frac{3}{2}$, $\alpha_0=\beta_0=\frac{6}{5}$, $\alpha_1=\beta_1=\frac{2}{5}$, $\alpha_2=\beta_2=\frac{4}{5}$, $\mathfrak{H}_1(t)=\mathfrak{K}_1(t)=t$, $\mathfrak{H}_2(t)=2t$, $\mathfrak{K}_2(t)=3t$, and

$$f(t,v)=t(v^{\frac{1}{3}}+t),\ \ g(t,u)=t(u^{\frac{1}{3}}+3t).$$

Obviously, $f,g\in C([0,1]\times[0,+\infty),[0,+\infty))$ and $f(t,0)=t^2\not\equiv 0$, $g(t,0)=3t^2\not\equiv 0$. Note that $x^{\frac{1}{3}}$ is increasing with respect to the second variable for $t\in[0,1]$. Moreover, set $\psi(\lambda)=\lambda^{\frac{1}{3}}$, $\lambda\in(0,1)$. Then, $\psi(\lambda)\in(0,1),\psi(\lambda)=\lambda^{\frac{1}{3}}>\lambda^{\frac{1}{2}}=\lambda^{\frac{1}{(\rho_1-1)}}=\lambda^{\frac{1}{(\rho_2-1)}}$,

$$f(t,\lambda v)=t[(\lambda v)^{\frac{1}{3}}+t]=\lambda^{\frac{1}{3}}v^{\frac{1}{3}}t+t^2\geqslant\lambda^{\frac{1}{3}}f(t,v)=\lambda^{\frac{1}{3}}(v^{\frac{1}{3}}t+t^2),$$

$$g(t,\lambda u)=t[(\lambda v)^{\frac{1}{3}}+3t]=\lambda^{\frac{1}{3}}u^{\frac{1}{3}}t+3t^2\geqslant\lambda^{\frac{1}{3}}g(t,u)=\lambda^{\frac{1}{3}}(u^{\frac{1}{3}}t+3t^2),$$

for any $t\in[0,1]$, $u,v\in[0,+\infty)$. Hence, all conditions of Theorem 3.1 are satisfied. Then, Theorem 3.1 shows that (4.1) and (4.2) has a unique positive solution (u^*,v^*) in $\overline{P_h}$, where $h_1(t)=t^{\frac{3}{2}}$, $h_2(t)=t^{\frac{3}{2}}$, $t\in[0,1]$ and taking any given point $(u_0,v_0)\in\overline{P_h}$, let

$$u_{n+1}(t)=\int_0^1\mathfrak{G}_1(t,\zeta)\varphi_{\frac{3}{2}}(I_{0+}^{\frac{1}{2}}\zeta(v_n(\zeta)^{\frac{1}{3}}+\zeta))d\zeta+\int_0^1\mathfrak{G}_2(t,\zeta)\varphi_{\frac{3}{2}}(I_{0+}^{\frac{1}{2}}\zeta(u_n(\zeta)^{\frac{1}{3}}+3\zeta))d\zeta,$$

$$v_{n+1}(t)=\int_0^1\mathfrak{G}_3(t,\zeta)\varphi_{\frac{3}{2}}(I_{0+}^{\frac{1}{2}}\zeta(v_n(\zeta)^{\frac{1}{3}}+\zeta))d\zeta+\int_0^1\mathfrak{G}_4(t,\zeta)\varphi_{\frac{3}{2}}(I_{0+}^{\frac{1}{2}}\zeta(u_n(\zeta)^{\frac{1}{3}}+3\zeta))d\zeta,$$

$n=1,2,\cdots$, then $u_n(t)\to u^*(t),v_n(t)\to v^*(t)$, as $n\to\infty$, where

$$\Delta_1=\frac{20}{17}\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{17}{10})}+\frac{10}{21}\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{21}{10})},\quad \Delta_2=\frac{30}{17}\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{17}{10})}+\frac{10}{21}\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{21}{10})},$$

$$\Delta=\left(\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{13}{10})}\right)^2-\left(\frac{20}{17}\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{17}{10})}+\frac{10}{21}\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{21}{10})}\right)\left(\frac{30}{17}\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{17}{10})}+\frac{10}{21}\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{21}{10})}\right),$$

$$\begin{aligned}
\mathfrak{G}_1(t,\zeta)&=\mathfrak{g}_1(t,\zeta)+\frac{t^{\frac{3}{2}}\Delta_1}{\Delta}(\int_0^1\mathfrak{g}_{11}(\tau,\zeta)d\tau+3\int_0^1\mathfrak{g}_{12}(\tau,\zeta)d\tau),\\
\mathfrak{G}_2(t,\zeta)&=\frac{t^{\frac{3}{2}}\Gamma(\frac{5}{2})}{\Delta\Gamma(\frac{13}{10})}\left(\int_0^1\mathfrak{g}_{21}(\tau,\zeta)d\tau+2\int_0^1\mathfrak{g}_{22}(\tau,\zeta)d\tau\right),\\
\mathfrak{G}_3(t,\zeta)&=\frac{t^{\frac{3}{2}}\Gamma(\frac{5}{2})}{\Delta\Gamma(\frac{13}{10})}\left(\int_0^1\mathfrak{g}_{11}(\tau,\zeta)d\tau+3\int_0^1\mathfrak{g}_{12}(\tau,\zeta)d\tau\right),\\
\mathfrak{G}_4(t,\zeta)&=\mathfrak{g}_2(t,\zeta)+\frac{t^{\frac{3}{2}}\Delta_1}{\Delta}(\int_0^1\mathfrak{g}_{21}(\tau,\zeta)d\tau+2\int_0^1\mathfrak{g}_{22}(\tau,\zeta)d\tau),
\end{aligned}$$

$$\mathfrak{g}_1(t,\zeta)=\frac{1}{\Gamma(\frac{5}{2})}\begin{cases} t^{\frac{3}{2}}(1-\zeta)^{\frac{3}{10}}-(t-\zeta)^{\frac{3}{2}}, & 0\leqslant\zeta\leqslant t\leqslant 1,\\ t^{\frac{3}{2}}(1-\zeta)^{\frac{3}{10}}, & 0\leqslant t\leqslant\zeta\leqslant 1,\end{cases}$$

$$\mathfrak{g}_{11}(\tau,\zeta)=\frac{1}{\Gamma(\frac{21}{10})}\begin{cases} \tau^{\frac{11}{10}}(1-\zeta)^{\frac{3}{10}}-(\tau-\zeta)^{\frac{11}{10}}, & 0\leqslant\zeta\leqslant\tau\leqslant 1,\\ \tau^{\frac{11}{10}}(1-\zeta)^{\frac{3}{10}}, & 0\leqslant\tau\leqslant\zeta\leqslant 1,\end{cases}$$

$$\mathfrak{g}_{12}(\tau,\zeta)=\frac{1}{\Gamma(\frac{17}{10})}\begin{cases} \tau^{\frac{7}{10}}(1-\zeta)^{\frac{3}{10}}-(\tau-\zeta)^{\frac{7}{10}}, & 0\leqslant\zeta\leqslant\tau\leqslant 1,\\ \tau^{\frac{7}{10}}(1-\zeta)^{\frac{3}{10}}, & 0\leqslant\tau\leqslant\zeta\leqslant 1,\end{cases}$$

$$\mathfrak{g}_2(t,\zeta)=\frac{1}{\Gamma(\frac{5}{2})}\begin{cases} t^{\frac{3}{2}}(1-\zeta)^{\frac{3}{10}}-(t-\zeta)^{\frac{3}{2}}, & 0\leqslant\zeta\leqslant t\leqslant 1,\\ t^{\frac{3}{2}}(1-\zeta)^{\frac{3}{10}}, & 0\leqslant t\leqslant\zeta\leqslant 1,\end{cases}$$

$$\mathfrak{g}_{21}(\tau,\zeta)=\frac{1}{\Gamma(\frac{21}{10})}\begin{cases} \tau^{\frac{11}{10}}(1-\zeta)^{\frac{3}{10}}-(\tau-\zeta)^{\frac{11}{10}}, & 0\leqslant\zeta\leqslant\tau\leqslant 1,\\ \tau^{\frac{11}{10}}(1-\zeta)^{\frac{3}{10}}, & 0\leqslant\tau\leqslant\zeta\leqslant 1,\end{cases}$$

$$\mathfrak{g}_{22}(\tau,\zeta)=\frac{1}{\Gamma(\frac{17}{10})}\begin{cases} \tau^{\frac{7}{10}}(1-\zeta)^{\frac{3}{10}}-(\tau-\zeta)^{\frac{7}{10}}, & 0\leqslant\zeta\leqslant\tau\leqslant 1,\\ \tau^{\frac{7}{10}}(1-\zeta)^{\frac{3}{10}}, & 0\leqslant\tau\leqslant\zeta\leqslant 1.\end{cases}$$

And we get

$$\mathfrak{J}_1(\zeta)=\mathfrak{h}_1(\zeta)+\frac{\Delta_1}{\Delta}\left(\int_0^1\mathfrak{g}_{11}(\tau,\zeta)d\tau+3\int_0^1\mathfrak{g}_{12}(\tau,\zeta)d\tau\right),\ \forall\zeta\in[0,1],$$

where $\mathfrak{h}_1(\zeta)=\frac{1}{\Gamma(\frac{5}{2})}(1-\zeta)^{\frac{3}{10}}(1-(1-\zeta)^{\frac{6}{5}}),\ \forall\zeta\in[0,1]$,

$$\mathfrak{J}_2(\zeta)=\frac{\Gamma(\frac{5}{2})}{\Delta\Gamma(\frac{13}{10})}\left(\int_0^1\mathfrak{g}_{21}(\tau,\zeta)d\tau+2\int_0^1\mathfrak{g}_{22}(\tau,\zeta)d\tau\right),\ \forall\zeta\in[0,1],$$

$$\mathfrak{J}_3(\zeta)=\frac{\Gamma(\frac{5}{2})}{\Delta\Gamma(\frac{13}{10})}\left(\int_0^1\mathfrak{g}_{11}(\tau,\zeta)d\tau+3\int_0^1\mathfrak{g}_{12}(\tau,\zeta)d\tau\right),\ \forall\zeta\in[0,1],$$

$$\mathfrak{J}_4(\zeta)=\mathfrak{h}_2(\zeta)+\frac{\Delta_2}{\Delta}\left(\int_0^1\mathfrak{g}_{21}(\tau,\zeta)d\tau+2\int_0^1\mathfrak{g}_{22}(\tau,\zeta)d\tau\right),\ \forall\zeta\in[0,1],$$

where $\mathfrak{h}_2(\zeta)=\frac{1}{\Gamma(\frac{5}{2})}(1-\zeta)^{\frac{3}{10}}(1-(1-\zeta)^{\frac{6}{5}}),\ \forall\zeta\in[0,1]$. $\square$

5. Conclusion

This paper is concerned with (1.1), a system of Riemann-Liouville fractional differential equations with ρ-Laplacian operators, which is coupled by nonlocal boundary conditions involving the Riemann-Stieltjes integrals. By means of an increasing operator fixed point theorem, we obtain the local existence and uniqueness of positive solutions to (1.1). This approach very cleverly solves the uniqueness of positive solutions for differential equations, providing a new way to solve some boundary value problems.

In addition, we can approximate the unique solution by constructing a convergent iterative sequence. In the end, we give a valid example to illustrate the main result, and the example also shows that the conditions of Theorem 3.1 are easy to be verified.

REFERENCES

[1] B. AHMAD, R. LUCA, *Existence of solutions for sequential fractional integro-differential equations and inclusions with nonlocal boundary conditions*, Appl. Math. Comput. **339** (2018) 516–534.

[2] B. AHMAD, S. K. NTOUYAS, *A fully Hadamard type integral boundary value problem of a coupled system of fractional differential equations*, Fract. Calc. Appl. Anal. **17** (2) (2014) 348–360.

[3] B. AHMAD, S. K. NTOUYAS, A. ALSAEDI, *A study of a coupled system of Hadamard fractional differential equations with nonlocal coupled initial-multipoint conditions*, Adv. Differ. Equ. 2021, **33** (2021).

[4] Z. BAITICHE, C. DERBAZI, G. WANG, *Monotone iterative method for nonlinear fractionalp-Laplacian differential equation in terms of ψ-Caputo fractional derivative equipped with a new class of nonlinear boundary conditions*, Math. Meth. Appl. Sci. **45** (2022) 967–976.

[5] L. CHEN, J. MA, *Solve Riemann-Liouville boundary value problems using collocation boundary value methods with the graded mesh*, J. Comput. Appl. Math. **443** (2024) 115762.

[6] F. HADDOUCHI, M. E. SAMEI, *Solvability of a ψ-Riemann-Liouville fractional boundary value problem with nonlocal boundary conditions*, Math. Comput. Simul. **219** (2024) 355–377.

[7] Z. HAN, H. LU, S. SUN, *Positive solutions to boundary-value problems of p-Laplacian fractional differential equations with a parameter in the boundary*, Electron. J. Differ. Equ. **213** (2012) 2012.

[8] Z. HAN, H. LU, C. ZHANG, *Positive solutions for eigenvalue problems of fractional differential equation with generalized p-Laplacian*, Appl. Math. Comput. **257** (2015) 526–536.

[9] J. HENDERSON, R. LUCA, *Nonexistence of positive solutions for a system of coupled fractional boundary value problems*, Bound. Value Probl. 2015, **138** (2015).

[10] J. HENDERSON, R. LUCA, A. TUDORACHE, *On a System of Riemann-Liouville Fractional Boundary Value Problems with ρ-Laplacian Operators and Positive Parameters*, Fractal Fract. **6** (6) (2022) 299.

[11] J. HENDERSON, R. LUCA, A. TUDORACHE, *Positive solutions for systems of coupled fractional boundary value problems*, Open J. Appl. Sci. **5** (10) (2015) 600–608.

[12] T. JESSADA, S. K. NTOUYAS, S. ASAWASAMRIT, *Positive solutions for Hadamard differential systems with fractional integral conditions on an unbounded domain*, Open Math. **15** (1) (2017) 645–666.

[13] M. LI, J. WANG, *Exploring delayed Mittag-Leffler type matrix functions to study finite time stability of fractional delay differential equations*, Appl. Math. Comput. **324** (2018) 254–265.

[14] S. LI, X. ZHANG, Y. WU, *Extremal solutions for p-Laplacian differential systems via iterative computation*, Appl. Math. Lett. **26** (12) (2013) 1151–1158.

[15] R. LUCA, *On a system of fractional differential equations with p-Laplacian operators and integral boundary conditions*, Revue. Roum. Math. Pures. Appl. **66** (2021) 749–766.

[16] D. LUO, T. ABDELJAWAD, Z. LUO, *Ulam-Hyers stability results for a novel nonlinear nabla Caputo fractional variable-order difference system*, Turk. J. Math. **45** (2021) 456–470.

[17] D. LUO, Q. ZHU, Z. LUO, *A novel result on averaging principle of stochastic Hilfer-type fractional system involving non-Lipschitz coefficients*, Appl. Math. Lett. **122** (2021), 107549.

[18] I. PODLUBNY, *Fractional Differential Equations, an Introduction to Fractional Derivatives*, Elsevier: Amsterdam, The Netherlands, 1998.

[19] L. REN, J. WANG, M. FECKAN, *Asymptotically periodic solutions for Caputo type fractional evolution equations*, Frac. Calc. Appl. Anal. **21** (2018) 1294–1312.

[20] C. SHEN, H. ZHOU, L. YANG, *On the existence of solution to a boundary value problem of fractional differential equation on the infinite interval*, Bound. Value Probl. 2015, **241** (2015).

[21] A. TUDORACHE, R. LUCA, *Positive solutions for a system of fractional boundary value problems with r-Laplacian operators, uncoupled nonlocal conditions and positive parameters*, Axioms **11** (4) (2022) 164.

[22] A. TUDORACHE, R. LUCA, *Positive solutions for a system of Riemann-Liouville fractional boundary value problems with p-Laplacian operators*, Adv. Differ. Equ. 2020, **292** (2020).

[23] G. WANG, J. QIN, L. ZHANG, D. BALEANU, *Monotone iterative method fora nonlinear fractional conformable p-Laplacian differential system*, Math. Meth. Appl. Sci. 2020, 1–11.

[24] G. WANG, H. YUAN, D. BALEANU, *Stability analysis and solutions of fractional boundary value problem on the cyclopentasilane graph*, Heliyon **10** (2024) 32411.

[25] J. WANG, M. FECKAN, Y. ZHOU, *Center stable manifold for planar fractional damped equations*, Appl. Math. Comput. **296** (2017) 257–269.

[26] J. WANG, X. LI, *A uniformed method to Ulam-Hyers stability for some linear fractional equations*, Mediterr. J. Math. **13** (2016) 625–635.

[27] J. XU, J. JIANG, D. O'REGAN, *Positive solutions for a class of p-Laplacian Hadamard fractionalorder three-point boundary value problems*, Mathematics **308** (2020) 2020.

[28] J. XU, L. LIU, S. BAI, Y. WU, *Solvability for a system of Hadamard fractional multi-point boundary value problems*, Nonlinear Anal. Model. **26** (3) (2021) 502–521.

[29] C. YANG, H. LI, *Existence of positive solutions for a system of nonlinear Caputo type fractional differential equations with two parameters*, Adv. Differ. Equ. 2021, **497** (2021).

[30] C. YANG, X. ZHU, *Positive solutions depending on parameters for a nonlinear fractional system with-Laplacian operators*, Adv. Math. Phys. 2020, 9563791.

[31] H. YANG, *Existence and approximate controllability of Riemann-Liouville fractional evolution equations of order with weighted time delay*, Bull. Sci. Math. **187** (2023) 103303.

[32] C. ZHAI, R. JIANG, *Unique solutions for a new coupled system of fractional differential equations*, Adv. Differ. Equ. 2018, **1** (2018).

[33] C. ZHAI, J. REN, *Some properties of sets, fixed point theorems in ordered product spaces and applications to a nonlinear system of fractional differential equations*, Topol. Methods Nonlinear Anal. **49** (2) (2017) 625–645.

[34] C. ZHAI, F. WANG, *Properties of positive solutions for the operator equation $Ax = \lambda x$ and applications to fractional differential equations with integral boundary conditions*, Adv. Differ. Equ. 2015, **366** (2015).

[35] C. ZHAI, L. WANG, *$\varphi-(h,e)$-concave operators and applications*, J. Math. Anal. Appl. **454** (2) (2017) 571–584.

[36] C. ZHAI, W. WANG, *Properties of positive solutions for m-point fractional differential equations on an infinite interval*, Rev. R. Acad. Cienc. Exactas. Fis. Nat. Ser. A. Mat. **113**(2) (2019) 1289–1298.

[37] C. ZHAI, W. WANG, *Solutions for a system of Hadamard fractional differential equations with integral conditions*, Numer. Funct. Anal. Optim. **41** (2) (2020) 209–229.

[38] C. ZHAI, X. ZHU, *Unique solution for a new system of fractional differential equations*, Adv. Differ. Equ. 2019, **394** (2019).

[39] K. ZHANG, Z. FU, *Solutions for a class of Hadamard fractional boundary value problems with sign-changing nonlinearity*, J. Funct. Spaces, 2019, 9046472.

[40] N. ZHANG, L. ZHANG, T. LIU, H. LIU, *Uniqueness solution and stability results for singular fractional Riemann-Stieltjes integral boundary problems*, Bull. Sci. math. **195** (2024) 103487.

[41] X. ZHANG, L. LIU, B. WIWATANAPATAPHEE, *The eigenvalue for a class of singular p-Laplacian fractional differential equations involving the Riemann-Stieltjes integral boundary condition*, Appl. Math. Comput. **235** (2014) 412–422.

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