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REMARKS ON LYAPUNOV–TYPE INEQUALITIES FOR (p,q)–LAPLACE EQUATIONS

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Abstract. For the (p,q)-Laplace equation: $-\Delta_p u-\Delta_q u=W(x)(\alpha|u|^{p-2}u+\beta|u|^{q-2}u)$ in Ω under the Dirichlet boundary condition, we provide Lyapunov-type inequalities using the Sobolev constants or the radius of the maximum inscribed ball. Moreover, we give an existence result for non-trivial and non-negative solutions, and show the optimality of the inequalities.

1. Introduction

It is known that Lyapunov ([23]) established the classical stability condition for solutions of the ordinary differential equation $u''+W(x)u=0$. The classical Lyapunov inequality introduced by Borg ([6]) is known to be a necessary condition

$$\int_a^b |W(x)|\,dx \geqslant \frac{4}{b-a}$$

for the existence of a non-trivial solution of the problem

$$u''+W(x)u=0 \quad \text{in } (a,b), \quad u(a)=u(b)=0.$$

This result is naturally extended to one-dimensional p-Laplace equations ([13], [29], [36]) and other ordinary problems ([5], [20], [34]). Refer to the books [1] and [30] also.

In [8] (and [9]), Canãda–Montero–Villegas extended the notion of Lyapunov inequality to the partial differential equations (Laplace equation) under Neumann (and Dirichlet) boundary condition. After that, many authors provide Lyapunov-type inequalities for p-Laplace equations ([17], [19], [35]). See [1] and [30] for other PDE problems. In particular, we mention that for the following p-Laplace equations

$$-\Delta_p u=W(x)|u|^{p-2}u \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega, \tag{1.1}$$

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Edward–Hudson–Leckband ([12]), and de Napoli–Pinasco ([11]) gave Lyapunov-type inequalities by using Sobolev constant $\lambda_{s,\sigma}$ (see (1.3)) or the inner radius r_Ω of Ω, respectively. In more detail, we find the Lyapunov-type inequalities

$$\|W_+\|_\gamma \geqslant \lambda_{p,p\gamma'} \quad \text{and} \quad \|W_+\|_\gamma \geqslant \frac{C}{r_\Omega^\sigma}$$

in [12, Theorem 2.2.] and in [11, Theorem 2.1, 2.4] with $\sigma = p-N$ if $N<p$ and $\sigma = p-N/\gamma$ if $\gamma > N/p > 1$, respectively. Here r_Ω is defined as follows:

$$r_\Omega := \max_{x\in\Omega} d_\Omega(x), \quad d_\Omega(x) := \mathrm{dist}\,(x,\partial\Omega) = \min_{y\in\partial\Omega} |x-y|. \tag{1.2}$$

The main purpose of this paper is to extend the results on the p-Laplace equation (1.1) in [12] and [11] to the (p,q)-Laplace equation (L), and the corresponding results are seen in Theorem 1 and Theorem 2:

$$\begin{cases} -\Delta_p u - \Delta_q u \ = \ W(x)\left(\alpha|u|^{p-2}u + \beta|u|^{q-2}u\right) & \text{in } \Omega, \\ \qquad\qquad u \ = \ 0 & \text{on } \partial\Omega, \end{cases} \tag{L}$$

where Ω is a bounded open set in $\mathbb{R}^N$ $(N \geqslant 1)$, $1<q<p<+\infty$, $\alpha,\beta \in \mathbb{R}$, and Δ_s with $s\in\{p,q\}$ stands for the standard s-Laplace operator defined as $\Delta_s u = \mathrm{div}\,(|\nabla u|^{s-2}\nabla u)$. Moreover, $W \in L^\gamma(\Omega)$ $(\gamma \in [1,\infty])$ is a weight function admitted to change sign.

DEFINITION 1. We say that $u \in W_0^{1,p}(\Omega)$ is a solution of (L) if it holds:

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u\right)\nabla v\,dx = \int_\Omega W(\alpha|u|^{p-2}u + \beta|u|^{q-2}u)v\,dx$$

for all $v \in W_0^{1,p}(\Omega)$.

The most difficulty is to show the optimality of our inequalities. It needs the results on eigenvalue problems for (p,q)-Laplacian, and so we modify the existence result in [26] (see Theorem 7).

The equation (L) is constructed from the nonlinear eigenvalue problems for p-Laplacian and q-Laplacian with weight W. We say that $\lambda\in\mathbb{R}$ is the eigenvalue of the s-Laplacian with weight W if the equation

$$-\Delta_s u = \lambda\, W(x)|u|^{s-2}u \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega,$$

has a non-trivial solution. It is well known that the first (positive) eigenvalue is described by minimizing the Rayleigh quotient $\int_\Omega |\nabla u|^s\,dx / \int_\Omega W|u|^s\,dx$. Moreover, the properties of the corresponding first eigenfunction are known (see [10] and (3.1) also). In viewpoint of the eigenvalue problems for (p,q)-Laplacian, for example, we study eigenvalue two parameters (α,β) such that (L) has a non-trivial solution, the second author has studied (p,q)-Laplace equation (L) with Motreanu ([26]) and Bobkov ([3]). Recently, many authors have studied (p,q)-Laplace eigenvalue problems including

Fučik–type spectrum, which is the generalization from the eigenvalue (see [15], [24], [31], [33]).

NOTATIONS. Throughout the paper, $\|\cdot\|_r$ stands for the standard Lebesgue norm of $L^r(\Omega)$ for $r\in[1,\infty]$. We set $s^*:=\infty$ (if $N\leqslant s$), $s^*:=sN/(N-s)$ (if $N>s$), and γ' stands for Hölder conjugate of $\gamma\in[1,\infty]$, namely, $\gamma':=1$ if $\gamma=\infty$, $\gamma':=\gamma/(\gamma-1)$ if $\gamma\in(1,\infty)$ and $\gamma':=\infty$ if $\gamma=1$. As usual, we consider $1/0$ and $1/\infty$ to be $+\infty$ and 0, respectively.

Here, we define $\lambda_{s,\sigma}$ by

$$\lambda_{s,\sigma}:=\inf\left\{\frac{\|\nabla u\|_s^s}{\|u\|_\sigma^s}\,:\,u\in W_0^{1,s}(\Omega)\setminus\{0\}\right\}>0 \tag{1.3}$$

for $s\in(1,\infty)$, and $\sigma\in[1,\infty]$ if $N<s$, $\sigma\in[1,\infty)$ if $N=s$ and $\sigma\in[1,s^*]$ if $N>s$. It is obvious that $\lambda_{s,\sigma}^{-s}$ is the Sobolev constant of the embedding from $W_0^{1,s}(\Omega)$ into $L^\sigma(\Omega)$. In case $1\leqslant\sigma<s^*$, thanks to the compactness of the embedding $W_0^{1,s}(\Omega)\hookrightarrow L^\sigma(\Omega)$, $\lambda_{s,\sigma}$ is attained by a non-negative function. In particular, it is easily seen that the minimizer $u\geqslant 0$ of $\lambda_{s,\sigma}$ ($\sigma\in(1,s^*)$) is a non-trivial and non-negative solution of the following equation with $\lambda=\lambda_{s,\sigma}$:

$$-\Delta_s u=\lambda\|u\|_\sigma^{s-\sigma}|u|^{\sigma-2}u \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega. \tag{1.4}$$

That is, in case $\sigma=s$, $\lambda_{s,s}$ is the first eigenvalue of s-Laplacian and the minimizer is the corresponding eigenfunction. Moreover, when Ω is additionally supposed to be connected, that is, a bounded domain, the first non-local eigenvalue $\lambda_{s,\sigma}$ is simple provided $\sigma\leqslant s$ and the corresponding first eigenfunction is *positive* (or negative) in Ω for $\sigma\in(1,s^*)$ (see [16] and [38] for the non-local eigenvalue problem and [18] in case $N=1$).

REMARK 1. Due to the standard Moser's iteration methods, any solution of (L) and (1.4) is bounded. In addition, under $C^{1,\kappa}$-regularity of Ω ($\kappa\in(0,1)$), any (weak) solution belongs to $C_0^{1,\mu}(\overline{\Omega})$ for some $\mu\in(0,1)$. This regularity result follows from [21, Theorem 1] (see [22, p. 320]). Moreover, we recall that if Ω is connected, that is, a bounded domain (without the regularity of Ω), then any non-negative minimizer of $\lambda_{s,\sigma}$ is positive in Ω. This is proved by Harnack inequality or maximum principle (see [32]). Finally, we remark that the positivity (and boundary point condition) of non-negative and non-trivial $C^1(\overline{\Omega})$-solutions for (L) follows from the strong maximum principle (refer to [32] and [27]) provided αW and βW are bounded from below, under C^2-regularity of Ω.

1.1. Main results on Lyapunov-type inequalities

To state main results corresponding to the sign of (α,β), we set

$$W_{\alpha,\beta}:=W_\pm \quad \text{if } \pm\alpha\cdot\beta\geqslant 0 \quad \text{and} \quad W_{\alpha,\beta}:=W \quad \text{if } \alpha\cdot\beta<0,$$

respectively, where $W_\pm:=\max\{\pm W,0\}$.

THEOREM 1. *Let $\gamma \in [1,\infty]$ if $N < q$ and $N/q \leqslant \gamma \in (1,\infty]$ if $N \geqslant q$. If (L) has a non-trivial solution, then*

$$\|W_{\alpha,\beta}\|_{\gamma} \geqslant \min\left\{ \frac{\lambda_{p,\sigma_p}}{|\alpha|}, \frac{\lambda_{q,\sigma_q}}{|\beta|} \right\}, \qquad \left(\sigma_s := s\gamma' \quad \text{for } s \in \{p,q\} \right) \tag{1.5}$$

holds, where $\lambda_{s,\sigma}$ is defined in (1.3). In particular, the equal sign above is not valid provided $1 < \gamma < \infty$.

In case $\gamma = \infty$, assuming an additional condition of (i) $\sim$ (iii) as in Proposition 1, we can see that the equal sign in (1.5) does not hold.

PROPOSITION 1. *Let $\gamma = \infty$. Assume that one of the following conditions:*

(i) $\alpha \cdot \beta \leqslant 0$*;*

(ii) $\lambda_{p,p}/|\alpha| \neq \lambda_{q,q}/|\beta|$*;*

(iii) *Ω is class of $C^{1,\kappa}$ (for some $\kappa \in (0,1)$) if $N \geqslant 2$.*

If (L) has a non-trivial solution, then the equal sign in (1.5) is not valid, namely,

$$\|W_{\alpha,\beta}\|_{\infty} > \min\left\{ \frac{\lambda_{p,p}}{|\alpha|}, \frac{\lambda_{q,q}}{|\beta|} \right\}$$

holds.

Here, we remark that we do not consider Lyapunov-type inequality using r_Ω in case $N \in \{p,q\}$ because we can not expect to get it for general sets Ω due to Osserman's results ([28]) in case $p = N = 2$.

The following result is proved for the case $N > s$ as in the argument in the proof of Theorem 2.4. in [11]. We provide the same result for the case $N < s$. Since λ_{s,s^*} is independent of Ω, we do not consider the case $\sigma = s^*$ and $s \leqslant N$ for the general open set Ω. See Remark 2 for convex sets and case $N = s$. It is shown in [7, Proposition 6.1] that we can not get an estimate of $\lambda_{s,\sigma}$ as in (1.6) in sublinear case $\sigma < s$.

PROPOSITION 2. *Let $s \in (1,\infty) \setminus \{N\}$, $\sigma \in [s,\infty]$ if $N < s$ and $\sigma \in [s,s^*)$ if $N > s$. In addition, we assume that Ω is a Lipschitz bounded domain if $N \geqslant s$. Then there exists a positive constant C depending only on N, s, σ, H_s such that*

$$\lambda_{s,\sigma} \geqslant C r_\Omega^{-s+N(1-s/\sigma)}, \tag{1.6}$$

where $\lambda_{s,\sigma}$ is defined in (1.3), and H_s is the constant as in Theorem 6 (Hardy inequality).

In particular, if $N < s$ and $\sigma = \infty$, then we can take $C = M_s^{-s}$ in (1.6), where $M_s = M_s(s,N)$ is the constant as in Morrey inequality (see Theorem 5).

REMARK 2. Although H_s depends on the capacity of $\mathbb{R}^N \setminus \Omega$ in general, it is known that the constant H_s can be taken independent of Ω for convex domains (refer to [25] and [2, Chapter 3]). In particular, for an open bounded "*convex*" set Ω, (1.6) is shown together with $N = s$ in [7, Corollary 5.1. and Proposition 6.3.].

According to Theorem 1 and Proposition 2, the following two results are proved. These results correspond to those in [11, Theorem 2.1 and Theorem 2.4] for the p-Laplace equation. Since we can not apply Proposition 2 to the case $s = N$ for the general domain Ω, we have to assume $N \notin \{p,q\}$.

THEOREM 2. *Let $\gamma = 1$ and $N < q$. If (L) has a non-trivial solution, then there exists a positive constant C such that*

$$\|W_{\alpha,\beta}\|_1 \geqslant \frac{C}{\max\left\{ |\alpha|\, r_\Omega^{p-N},\, |\beta|\, r_\Omega^{q-N} \right\}},$$

where C depends only on N, p and q.

THEOREM 3. *Let $N \notin \{p,q\}$, and $\gamma \in [1,\infty]$ if $N < q$ and $N/q < \gamma \in (1,\infty]$ if $N > q$. Assume that Ω is a Lipschitz bounded domain if $N \geqslant q$. If (L) has a non-trivial solution, then there exists a positive constant C such that*

$$\|W_{\alpha,\beta}\|_\gamma \geqslant \frac{C}{\max\left\{ |\alpha|\, r_\Omega^{p-N/\gamma},\, |\beta|\, r_\Omega^{q-N/\gamma} \right\}},$$

where C depends only on N, p, q, γ, and Hardy constants H_p and H_q (as in Theorem 6).

For convex sets, applying the results in [7, Corollary 5.1. and Proposition 6.3.] instead of Proposition 2 (refer to Remark 2), we get the following result including the cases $N = p$ and $N = q$.

THEOREM 4. *Assume that Ω is an open bounded convex set. Let $\gamma \in [1,\infty]$ if $N < q$ and $N/q < \gamma \in (1,\infty]$ if $N \geqslant q$. If (L) has a non-trivial solution, then there exists a positive constant C such that*

$$\|W_{\alpha,\beta}\|_\gamma \geqslant \frac{C}{\max\left\{ |\alpha|\, r_\Omega^{p-N/\gamma},\, |\beta|\, r_\Omega^{q-N/\gamma} \right\}},$$

where C depends only on N, p, q and γ.

1.2. Results on the optimality

First, we show the optimality of (1.5) except the case $\gamma = 1$.

PROPOSITION 3. *Let $N/q < \gamma \in (1,\infty]$. Assume that*

$$\min\left\{\frac{\lambda_{p,\sigma_p}}{|\alpha|}, \frac{\lambda_{q,\sigma_q}}{|\beta|}\right\} < \max\left\{\frac{\lambda_{p,\sigma_p}}{|\alpha|}, \frac{\lambda_{q,\sigma_q}}{|\beta|}\right\} \qquad (\sigma_s := s\gamma'). \tag{1.7}$$

Then, for any $\varepsilon > 0$ there exists $W \in L^\infty(\Omega)$ satisfying

$$\|W_{\alpha,\beta}\|_\gamma < \min\left\{\frac{\lambda_{p,\sigma_p}}{|\alpha|}, \frac{\lambda_{q,\sigma_q}}{|\beta|}\right\} + \varepsilon$$

such that (L) has a non-trivial and non-negative solution.

Finally, in case that Ω is a ball, we prove that the powers $\rho_s := s - N/\gamma$ of r_Ω in Theorem 4 are optimal. The same arguments for the p-Laplace equation are done in [11, Proposition 2.7.].

PROPOSITION 4. *Assume that $\Omega = B_R$, that is, Ω is the open ball of radius $R > 0$ centered at the origin. Let $N/q < \gamma \in [1,\infty]$ and $1 \leqslant \gamma < N/(N-1)$ if $N \geqslant 2$. For any $C > 0$ and $\varepsilon > 0$ satisfying $\varepsilon < \min\{1, \rho_p, \rho_q, p-q\}$, where $\rho_s := s - N/\gamma$, the following assertions hold:*

(i) *If $\alpha \neq 0$ and $\beta \in \mathbb{R}$, then for any sufficiently large $R \gg 1$ there exists $W \in L^\gamma(B_R)$ satisfying*

$$\|W_{\alpha,\beta}\|_\gamma < \frac{C}{|\alpha|R^{\rho_p-\varepsilon}} = \frac{C}{\max\{|\alpha|R^{\rho_p-\varepsilon}, |\beta|R^{\rho_q-\varepsilon}\}}$$

such that (L) has at least one non-trivial (and non-negative) solution.

(ii) *If $\beta \neq 0$ and $\alpha \in \mathbb{R}$, then for any sufficiently small $0 < R \ll 1$ there exists $W \in L^\gamma(B_R)$ satisfying*

$$\|W_{\alpha,\beta}\|_\gamma < \frac{C}{|\beta|R^{\rho_q+\varepsilon}} = \frac{C}{\max\{|\alpha|R^{\rho_p+\varepsilon}, |\beta|R^{\rho_q+\varepsilon}\}}$$

such that (L) has at least one non-trivial (and non-negative) solution.

The structure of the paper is as follows. In section 2, we prove Lyapunov-type inequalities: Theorem 1, Proposition 1 and Proposition 2. In section 3, we provide the proofs of Proposition 3 and Proposition 4. In Appendix, we give a sketch of the proof for the existence theorem stated in section 3.

2. Proofs for Lyapunov-type inequalities

First, we reall Morrey inequality and Hardy inequality. See [14] or [2, Theorem 3.2.1] for details.

THEOREM 5. *Let $N < s < \infty$. Then there exists a positive constant M_s depending only on s and N such that any $u \in W_0^{1,s}(\Omega)$ satisfies*

$$|u(x) - u(y)| \leqslant M_s \|\nabla u\|_s |x-y|^{1-N/s}$$

for all $x, y \in \overline{\Omega}$.

THEOREM 6. *Let $s \in (1,\infty)$. Assume that Ω be a Lipschitz bounded domain if $N \geqslant s$. Then, there exists a positive constant H_s such that*

$$\int_\Omega \left(\frac{|u|}{d_\Omega(x)} \right)^s dx \leqslant H_s \int_\Omega |\nabla u|^s \, dx$$

for any $u \in W_0^{1,s}(\Omega)$.

Let us start to prove Lyapunov-type inequality.

Proof of Theorem 1. Let u be a non-trivial solution of (L). Taking u as a test function, we have

$$\|\nabla u\|_p^p + \|\nabla u\|_q^q = \alpha \int_\Omega W|u|^p \, dx + \beta \int_\Omega W|u|^q \, dx \leqslant \|W_{\alpha,\beta}\|_\gamma \left(|\alpha| \|u\|_{\sigma_p}^p + |\beta| \|u\|_{\sigma_q}^q \right) \tag{2.1}$$

by Hölder inequality. This leads to Lyapunov-type inequality as follows:

$$\begin{aligned} \|W_{\alpha,\beta}\|_\gamma &\geqslant \frac{\|\nabla u\|_p^p + \|\nabla u\|_q^q}{|\alpha| \|u\|_{\sigma_p}^p + |\beta| \|u\|_{\sigma_q}^q} \geqslant \min \left\{ \frac{\|\nabla u\|_p^p}{|\alpha| \|u\|_{\sigma_p}^p}, \frac{\|\nabla u\|_q^q}{|\beta| \|u\|_{\sigma_q}^q} \right\} \\ &\geqslant \min \left\{ \frac{\lambda_{p,\sigma_p}}{|\alpha|}, \frac{\lambda_{q,\sigma_q}}{|\beta|} \right\}. \end{aligned} \tag{2.2}$$

Finally, we prove that Lyapunov-type inequality (1.5) is strict in case $1 < \gamma < \infty$ by contradiction. So, if the equality in (1.5) holds, then all equal signs in (2.1) and (2.2) hold. We easily see that the equality of (2.1) is impossible in the case $\alpha \cdot \beta < 0$. Moreover, if either α or β is zero, then the second inequality in (2.2) is strict. Let $\alpha, \beta > 0$. Then, the equality of (2.1) implies that $W \geqslant 0$ a.e. in Ω, and the equality condition of Hölder inequality guarantees that $(W/\|W\|_\gamma)^\gamma = (|u|/\|u\|_{\sigma_p})^{\sigma_p}$ and $(W/\|W\|_\gamma)^\gamma = (|u|/\|u\|_{\sigma_q})^{\sigma_q}$ a.e. in Ω. Since $\sigma_p > \sigma_q$, this gives that u is constant, that is, $u = 0$. This is a contradiction. In case $\alpha, \beta < 0$, we can get a contradiction in the same way. The proof is complete. □

Proof of Proposition 1. By contradiction, we suppose that the equality in (1.5), that is,

$$\|W_{\alpha,\beta}\|_\infty = \min\left\{ \frac{\lambda_{p,p}}{|\alpha|}, \frac{\lambda_{q,q}}{|\beta|} \right\}$$

holds for some weight function $W \in L^\infty(\Omega)$. Then, for some non-trivial solution u of (L) with such W, all equal sign holds in (2.1) and (2.2). Note that $\sigma_p = p$ and $\sigma_q = q$ since we are considering the case $\gamma = \infty$. The second equality in (2.2) shows that $\alpha \cdot \beta \neq 0$. Moreover, combining with the last equality in (2.2), we see that $\lambda_{p,p}/|\alpha| = \lambda_{q,q}/|\beta|$, and u is a minimizer of both $\lambda_{p,p}$ and $\lambda_{q,q}$. The first assertion is impossible provided the case (ii).

Next, let us consider case $\alpha \cdot \beta < 0$. The equality in (2.1) means that

$$0 = \int_\Omega (|\alpha|\|W\|_\infty - \alpha W)|u|^p\,dx = \int_\Omega (|\beta|\|W\|_\infty - \beta W)|u|^q\,dx. \tag{2.3}$$

If $\alpha > 0 > \beta$, then (2.3) is equivalent to $0 < \alpha\|W\|_\infty = \alpha W$ and $0 < (-\beta)\|W\|_\infty = \beta W$ a.e. in $\{x \in \Omega : u(x) \neq 0\} =: [u \neq 0]$. Hence $\|W\|_\infty = W = -W$ a.e. in $[u \neq 0]$, and so $W = 0$ a.e. in $[u \neq 0]$. This means that

$$\|\nabla u\|_p^p + \|\nabla u\|_q^q = \alpha \int_{[u=0]} W|u|^p\,dx + \beta \int_{[u=0]} W|u|^q\,dx = 0,$$

whence this contradicts to $u \neq 0$. Similarly, the case $\alpha < 0 < \beta$ is impossible.

Finally, let us consider the case (iii). Recalling that u is the minimizer of both $\lambda_{p,p}$ and $\lambda_{q,q}$, we may assume that $u \geqslant 0$ in Ω by considering $|u|$ if necessary. Since minimizers are solutions of corresponding eigenvalue equation, u is a non-trivial and non-negative solution of

$$-\Delta_s u = \lambda_{s,s}|u|^{s-2}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

for $s \in \{p,q\}$. Under the regularity of Ω as in (iii), it is known that $u \in C^{1,\theta}(\overline{\Omega})$ (see Remark 1). Take a component Ω' of Ω such that $u \neq 0$. Hence, $u > 0$ in Ω', that is, u is a positive solution of

$$-\Delta_s u = \lambda_{s,s}|u|^{s-2}u \quad \text{in } \Omega', \qquad u = 0 \quad \text{on } \partial\Omega'$$

for $s \in \{p,q\}$. Recall that only the first eigenfunction of s-Laplacian ($s \in (1,\infty)$) has a constant sign in Ω'. Thus, $\lambda_{s,s} = \lambda_{s,s}(\Omega')$ and u is the first eigenfunction corresponding to both $\lambda_{p,p}(\Omega')$ and $\lambda_{q,q}(\Omega')$. On the other hand, it is shown in [3, Proposition 13.] (note that the proof requires that positive eigenfunctions have a maximum point in Ω') that the first eigenfunctions of p-Laplacian and q-Laplacian are linearly independent. So, we have a contradiction.

As a result, we get a contradiction in all cases. The proof has been completed. □

Proof of Proposition 2. Let u be a minimizer of $\lambda_{s,\sigma}$ since it is attained by $\sigma < s^*$ (if $N \geqslant s$).

Case $N > s$ is shown by the same argument as in the proof of [11, Theorem 2.4]. For readers' convenience, we give only the sketch. Take $\tau \in (0,1]$ satisfying $\sigma = \tau s + (1-\tau)s^*$. Then, by Hölder inequality, Hardy inequality (Theorem 6) and Sobolev embedding, we get

$$\begin{aligned}\frac{\|u\|_\sigma^\sigma}{r_\Omega^{\tau s}} &\leqslant \int_\Omega \frac{|u|^{\tau s+(1-\tau)s^*}}{d_\Omega(x)^{\tau s}}\,dx \leqslant \left(\int_\Omega \left(\frac{|u|}{d_\Omega(x)}\right)^s dx\right)^\tau \|u\|_{s^*}^{(1-\tau)s^*} \\ &\leqslant \lambda_{s,s^*}^{-s^*(1-\tau)/s} H_s^\tau \|\nabla u\|_s^\sigma,\end{aligned}$$

where d_Ω is the distance function from the boundary $\partial\Omega$ defined in (1.2). This yields

$$\lambda_{s,\sigma} = \frac{\|\nabla u\|_s^s}{\|u\|_\sigma^s} \geqslant \lambda_{s,s^*}^{s^*(1-\tau)/\sigma} H_s^{-s\tau/\sigma} r_\Omega^{-s+N(1-s/\sigma)}.$$

Since λ_{s,s^*} depends only on s and N, our assertion is shown.

Case $N < s$*:* First, we recall that the argument as in [11, Theorem 2.1.] implies

$$\|u\|_\infty \leqslant M_s \|\nabla u\|_s r_\Omega^{1-N/s}, \tag{2.4}$$

where $M_s = M_s(s,N) > 0$ is the constant as in Morrey inequality (Theorem 5). In fact, this is easily shown to apply Morrey inequality with a maximum point $x \in \Omega$ of $|u|$ and $y \in \partial\Omega$ such that $|x-y| = \mathrm{dist}(x,\partial\Omega)\,(\leqslant r_\Omega)$.

In case $\sigma = \infty$, our assertion follows from (2.4). Now let $\sigma < \infty$ and we choose any $t \in (\sigma,\infty)$. Then, using (2.4), we have

$$\|u\|_t \leqslant \|u\|_\infty |\Omega|^{1/t} \leqslant M_s r_\Omega^{1-N/s} |\Omega|^{1/t} \|\nabla u\|_s, \tag{2.5}$$

where $|\Omega|$ is the Lebesgue measure of Ω. Let $(\tau_t =)\tau \in (0,1)$ satisfy $\sigma = \tau s + (1-\tau)t$. Then, by (2.5) instead of Sobolev embedding as in the above argument, we get

$$\frac{\|u\|_\sigma^\sigma}{r_\Omega^{\tau s}} \leqslant H_s^\tau \|\nabla u\|_s^{\tau s} \|u\|_t^{t(1-\tau)} \leqslant H_s^\tau M_s^{t(1-\tau)} |\Omega|^{1-\tau} r_\Omega^{t(1-\tau)(1-N/s)} \|\nabla u\|_s^\sigma,$$

and hence

$$r_\Omega^{-s+t(1-\tau)N/\sigma} H_s^{-s\tau/\sigma} M_s^{-t(1-\tau)s/\sigma} |\Omega|^{-s(1-\tau)/\sigma} \leqslant \frac{\|\nabla u\|_s^s}{\|u\|_\sigma^s} = \lambda_{s,\sigma}. \tag{2.6}$$

Letting $t \to \infty$ in (2.6), since

$$\tau = \frac{t-\sigma}{t-s} \to 1, \quad (1-\tau)t = \sigma - s\tau \to \sigma - s \qquad \text{as } t \to \infty,$$

we get

$$\lambda_{s,\sigma} \geqslant H_s^{-s/\sigma} M_s^{-(\sigma-s)s/\sigma} r_\Omega^{-s+N(1-s/\sigma)}.$$

The proof is complete. $\square$

3. Proofs of the optimality

3.1. Existence result

Here, we consider the following (p,q)-Laplace equation:

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda \left(V_p|u|^{p-2}u + V_q|u|^{q-2}u \right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\lambda \in \mathbb{R}$ and $V_s \in L^{\gamma_s}(\Omega)$ $(s \in \{p,q\})$.

To state the existence result, we define the first eigenvalue of s-Laplacian (for $s \in (1,\infty)$) with weight as follows:

$$\lambda_s(V) := \inf\left\{ \frac{\|\nabla u\|_s^s}{(\int_\Omega V|u|^s\,dx)^{1/s}} : u \in W_0^{1,s}(\Omega)\setminus\{0\},\ \int_\Omega V|u|^s\,dx > 0 \right\} \tag{3.1}$$

for $V \in L^\gamma(\Omega)$ satisfying $V_+ \not\equiv 0$ with $\gamma \geqslant N/s$ (if $N > s$), $\gamma \in (1,\infty]$ (if $N = s$), $\gamma \in [1,\infty]$ (if $N < s$). Moreover, we set $\lambda_s(V) = +\infty$ if $V \leqslant 0$ a.e. in Ω. Clearly, $\lambda_s(V) \geqslant \lambda_{s,\sigma}/\|V\|_\gamma$ holds, where $\sigma = s\gamma'$. In case $V_+ \not\equiv 0$ and $\gamma > N/s$ (if $N \geqslant s$), it is well known (see Remark 1 and [10]) that $\lambda_s(V)$ is attained by a non-negative solution belonging to $L^\infty(\Omega) \cap C^0_{\mathrm{loc}}(\Omega)$ for

$$-\Delta_s u = \lambda_s(V)\,V(x)|u|^{s-2}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

The following result is already proved in [26] provided that V_p and V_q are bounded.

THEOREM 7. *Assume that $\gamma_s \in [1,\infty]$ (if $N < s$) for $s \in \{p,q\}$, $\gamma_p \in (N/p,\infty]$ (if $N \geqslant p$), $\gamma_q \in (1,\infty]$ (if $N = q$) and $\gamma_q \in [N/q,\infty]$ (if $N > q$). If λ satisfies*

$$\min\{\lambda_p(V_p), \lambda_q(V_q)\} < \lambda < \max\{\lambda_p(V_p), \lambda_q(V_q)\} \ (\leqslant +\infty),$$

then (P) has at least one non-trivial and non-negative solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

In particular, under the additional condition that Ω is a bounded domain with C^2 boundary (if $N \geqslant 2$), if V_p and V_q are bounded from below a.e. in Ω, then $u \in C^{1,\theta}(\overline{\Omega})$ $(\theta \in (0,1))$, and it satisfies $u > 0$ in Ω and $\partial u/\partial \nu < 0$ on $\partial\Omega$, where ν denotes the outer normal vector on $\partial\Omega$.

The proof of Theorem 7 can be done in the same way as in [26, Theorem 1.3.]. So we give only the sketch of the proof in the Appendix.

3.2. Proofs of Propositions 3 and 4

For the proof of Proposition 3, we prepare the following calculation.

LEMMA 1. *Let $s,\gamma \in (1,\infty)$, $c>0$ and γ satisfy $\gamma > N/s$ if $N>s$. For the minimizer $\varphi \geqslant 0$ of $\lambda_{s,\sigma}$ with $\sigma = s\gamma'(<s^*)$, we set $V := \varphi^{s/(\gamma-1)} (= \varphi^{\sigma-s})$. Then $V \in L^\infty(\Omega)$, and*

$$\|V\|_\gamma = \|\varphi\|_\sigma^{\sigma-s} \quad \text{and} \quad \lambda_s(cV) = \frac{\lambda_s(V)}{c} = \frac{\lambda_{s,\sigma}}{c\|V\|_\gamma}$$

hold, where $\lambda_{s,\sigma}$ and $\lambda_s(V)$ are defined in (1.3) *and* (3.1)*, respectively.*

Proof. The boundedness of V follows from $\varphi \in L^\infty(\Omega)$ (see Remark 1). According to simple calculations, we have $\|V\|_\gamma = \|\varphi\|_\sigma^{\sigma-s}$ by $\sigma/\gamma = \sigma - s$ and

$$\begin{aligned}
\lambda_s(cV) &= \inf\left\{ \frac{\|\nabla u\|_s^s}{\int_\Omega cV|u|^s\,dx} : 0 \neq u \in W_0^{1,s}(\Omega) \right\} \\
&= \frac{1}{c} \inf\left\{ \frac{\|\nabla u\|_s^s}{\int_\Omega V|u|^s\,dx} : 0 \neq u \in W_0^{1,s}(\Omega) \right\} = \frac{\lambda_s(V)}{c} \\
&\geqslant \frac{1}{c\|V\|_\gamma} \inf\left\{ \frac{\|\nabla u\|_s^s}{\int_\Omega |u|^\sigma\,dx} : 0 \neq u \in W_0^{1,s}(\Omega) \right\} = \frac{\lambda_{s,\sigma}}{c\|V\|_\gamma},
\end{aligned} \tag{3.2}$$

where we used $\int_\Omega V|u|^s\,dx \leqslant \|V\|_\gamma \|u\|_\sigma^s$ by Hölder inequality. On the other hand, because it holds

$$\int_\Omega V|\varphi|^s\,dx = \int_\Omega \varphi^\sigma\,dx = \|\varphi\|_\sigma^s\,\|\varphi\|_\sigma^{\sigma-s} = \|\varphi\|_\sigma^s\,\|V\|_\gamma,$$

by taking the minimizer φ of $\lambda_{s,\sigma}$ as an admissible function, the definition of $\lambda_s(V)$ leads to

$$\lambda_s(V) \leqslant \frac{\|\nabla\varphi\|_s^s}{\int_\Omega V|\varphi|^s\,dx} = \frac{\|\nabla\varphi\|_s^s}{\|\varphi\|_\sigma^s\,\|V\|_\gamma} = \frac{\lambda_{s,\sigma}}{\|V\|_\gamma}.$$

Thus the opposite inequality in (3.2) is shown, whence our assertion is complete. □

Proof of Proposition 3. Our assumption (1.7) is divided into the following cases (i) is (ii):

$$\text{(i) } \beta \neq 0 \quad \text{and} \quad \frac{\lambda_{q,\sigma_q}}{|\beta|} < \frac{\lambda_{p,\sigma_p}}{|\alpha|}\,(\leqslant \infty) \quad \text{(ii) } \alpha \neq 0 \quad \text{and} \quad \frac{\lambda_{p,\sigma_p}}{|\alpha|} < \frac{\lambda_{q,\sigma_q}}{|\beta|}\,(\leqslant \infty). \tag{3.3}$$

Corresponding to case (i) or (ii), we shall set suitable λ and V_s ($s \in \{p,q\}$) and provide a non-trivial and non-negative solution of (L) applying Theorem 7.

First, we consider the case $\gamma = \infty$. Then $\sigma_p = p$ and $\sigma_q = q$. Define

$$\text{(i) } V_p := \operatorname{sign}(\beta)\,\alpha, \quad V_q := |\beta| \qquad \text{and} \qquad \text{(ii) } V_p := |\alpha|, \quad V_q := \operatorname{sign}(\alpha)\,\beta,$$

where $\operatorname{sign}(t) = t/|t|$ for $t \neq 0$, and put

$$\lambda_\delta := \min\{\lambda_p(V_p), \lambda_q(V_q)\} + \delta \quad \text{for } \delta > 0.$$

Since it holds

$$\lambda_s(\pm C) \geqslant \lambda_s(|C|) = \frac{\lambda_{s,s}}{|C|} \qquad \text{for } C \in \mathbb{R},\ s \in \{p,q\},$$

we have

$$\begin{aligned}\min\{\lambda_p(V_p), \lambda_q(V_q)\} = \min\left\{\frac{\lambda_{p,p}}{|\alpha|}, \frac{\lambda_{q,q}}{|\beta|}\right\} < \max\left\{\frac{\lambda_{p,p}}{|\alpha|}, \frac{\lambda_{q,q}}{|\beta|}\right\} \\ \leqslant \max\{\lambda_p(V_p), \lambda_q(V_q)\},\end{aligned}$$

and so $\lambda_\delta < \max\{\lambda_p(V_p), \lambda_q(V_q)\}$ for small $\delta > 0$. Therefore, by setting $W_\delta = \operatorname{sign}(\beta)\lambda_\delta$ in case (i) or $W_\delta = \operatorname{sign}(\alpha)\lambda_\delta$ in case (ii) with small $\delta > 0$, Theorem 7 guarantees that our equation (L) (with W_δ) has a non-trivial solution, whence the proof is done in case $\gamma = \infty$.

Next, let $1 < \gamma < \infty$. For $s \in \{p,q\}$ we let $\varphi_s \geqslant 0$ be the minimizer of λ_{s,σ_s} ($\sigma_s := s\gamma'$), and set $V_s^* = \varphi_s^{\sigma_s - s}$ because our assumption $N/q < \gamma \in (1,\infty]$ gives $\sigma_s < s^*$ if $N \geqslant s$. Then it follows from Lemma 1 that

$$\lambda_s(cV_s^*) = \frac{\lambda_s(V_s^*)}{c} = \frac{\lambda_{s,\sigma_s}}{c\|V_s^*\|_\gamma} \quad \text{for } c > 0. \tag{3.4}$$

Moreover, because $\int_\Omega V_t^*|u|^s\,dx \leqslant \|V_t^*\|_\gamma \|u\|_{\sigma_s}^s$ by Hölder inequality, we have

$$\lambda_s(cV_t^*) = \frac{\lambda_s(V_t^*)}{c} \geqslant \frac{\lambda_{s,\sigma_s}}{c\|V_t^*\|_\gamma} \quad \text{for } c > 0 \tag{3.5}$$

if $s \neq t \in \{p,q\}$. Define

$$\text{(i) } V_p := \operatorname{sign}(\beta)\,\alpha V_q^*, \quad V_q := |\beta| V_q^* \quad \text{and} \quad \text{(ii) } V_p := |\alpha| V_p^*, \quad V_q := \operatorname{sign}(\alpha)\,\beta V_p^*.$$

Hereafter, we put $s = q$ or $s = p$ in case (i) or (ii), respectively. We claim that for any $\varepsilon > 0$ we can take a small $\delta > 0$ satisfying

$$\lambda_\delta := \min\{\lambda_p(V_p), \lambda_q(V_q)\} + \delta < \max\{\lambda_p(V_p), \lambda_q(V_q)\} \tag{3.6}$$

and

$$\|\lambda_\delta V_s^*\|_\gamma < \min\left\{\frac{\lambda_{p,\sigma_p}}{|\alpha|}, \frac{\lambda_{q,\sigma_q}}{|\beta|}\right\} + \varepsilon. \tag{3.7}$$

If these claims are shown, applying Theorem 7, we can get a non-trivial and non-negative solution of

$$-\Delta_p u - \Delta_q u = \lambda_\delta\left(V_p|u|^{p-2}u + V_q|u|^{q-2}u\right) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

So, the proof is done with $W = \operatorname{sign}(\beta)\,\lambda_\delta V_q^*$ in case (i) and $W = \operatorname{sign}(\alpha)\,\lambda_\delta V_p^*$ in case (ii).

Now, let us show our claim. We shall consider only the case (i) with $\beta > 0$ because other cases can be shown in the same way. First claim (3.6) follows from (3.4), (3.3) and (3.5) that

$$\lambda_q(V_q) = \frac{\lambda_{q,\sigma_q}}{\beta\,\|V_q^*\|_\gamma} < \frac{\lambda_{p,\sigma_p}}{|\alpha|\,\|V_q^*\|_\gamma} \leqslant \lambda_p(|\alpha|V_q^*) \leqslant \lambda_p(\alpha V_q^*) = \lambda_p(V_p)(\leqslant \infty).$$

Hence $\lambda_\delta = \lambda_q(V_q) + \delta$ and

$$\|\lambda_\delta V_q^*\|_\gamma = \left(\frac{\lambda_{q,\sigma_q}}{\beta\,\|V_q^*\|_\gamma} + \delta\right)\|V_q^*\|_\gamma = \frac{\lambda_{q,\sigma_q}}{\beta} + \delta\|V_q^*\|_\gamma$$
$$= \min\left\{\frac{\lambda_{p,\sigma_p}}{|\alpha|}, \frac{\lambda_{q,\sigma_q}}{|\beta|}\right\} + \delta\|V_q^*\|_\gamma.$$

Thus, (3.7) holds if $0 < \delta < \varepsilon/\|V_q^*\|_\gamma$. □

For the proof of Proposition 4, we prepare two simple calculations, which are also argued in [11, Proposition 2.7.]. The first result is easily shown by the direct calculation. We omit the proof.

LEMMA 2. *Let $\gamma \in [1,\infty)$, $\delta > 0$ and $\rho > -N/\gamma$. Set*

$$W_*(x) := \chi_{[0,\delta]}(|x|)\,|x|^\rho \quad \text{for } x \in \mathbb{R}^N, \tag{3.8}$$

where χ_I denotes the characteristic function of an interval I. Then, it holds

$$\|W_*\|_\gamma = \omega_N^{1/\gamma}(\rho\gamma + N)^{-1/\gamma}\,\delta^{\rho + N/\gamma},$$

where ω_N denotes the surface measure of the unit ball in $\mathbb{R}^N$.

LEMMA 3. *Let $s \in (1,\infty)$, $-\min\{N,s\} < \rho \leqslant 1-N$, and assume that Ω includes the open ball B_δ centered at the origin with radius $\delta > 0$. Then $\lambda_s(W_*)$ with W_* defined by (3.8) satisfies*

$$\lambda_s(W_*) \leqslant \delta^{-s-\rho}\,(s-1)\left(\frac{\pi}{s\,\sin(\pi/s)}\right)^s.$$

Proof. By considering the zero extension, we may suppose that $W_0^{1,s}(B_\delta) \subset W_0^{1,s}(\Omega)$. First, we note that we can see $W \in L^\gamma(\Omega)$ with $\min\{1, N/s\} < \gamma < N/|\rho|$ by the assumption $-\min\{N,s\} < \rho(\leqslant 0)$. Thus, $\int_\Omega W_*|u|^s\,ds$ is well defined for any $u \in$

$W_0^{1,s}(\Omega)$. By $\rho + N - 1 \leqslant 0$, the simple calculations guarantee that

$$\begin{aligned}
\lambda_s(W_*) &= \inf\left\{ \frac{\|\nabla u\|_s^s}{\int_\Omega W_*|u|^s\,dx} : u \in W_0^{1,s}(\Omega),\ \int_\Omega W_*|u|^s\,dx > 0 \right\} \\
&\leqslant \inf\left\{ \frac{\|\nabla u\|_s^s}{\int_{B_\delta} W_*|u|^s\,dx} : u \in W_0^{1,s}(B_\delta),\ \int_{B_\delta} W_*|u|^s\,dx > 0 \right\} \\
&\leqslant \inf\left\{ \frac{\int_0^\delta r^{N-1}|u'|^s\,dt}{\int_0^\delta r^{\rho+N-1}|u|^s\,dt} : u \in W^{1,s}(0,\delta),\ u'(0) = 0 = u(\delta) \right\} \\
&\leqslant \frac{\delta^{N-1}}{\delta^{\rho+N-1}}\,\lambda_s(0,\delta) = \delta^{-s-\rho}\,(s-1)\left(\frac{\pi}{s\sin(\pi/s)}\right)^s,
\end{aligned}$$

where $\lambda_s(0,\delta) = \delta^{-s}(s-1)(\pi/s\sin(\pi/s))^s$ is the first eigenvalue of one dimensional s-Laplacian in the interval $(-\delta,\delta)$. □

Now let us prove Proposition 4.

Proof of Proposition 4. Let $\Omega = B_R$ and $\varepsilon > 0$ and $C > 0$ been given in Proposition 4. Here we note that our assumption $\gamma < N/(N-1)$ if $N \geqslant 2$ gurantees $-N/\gamma < 1-N$. So, we set $\rho = 0$ if $N = 1$ and $\gamma = \infty$, $-N/\gamma < \rho \leqslant 1-N (\leqslant 0)$ otherwise. Note that $-N/\gamma \geqslant -\min\{N,s\}$ follows from the assumptions $N/s < \gamma$ and $\gamma \geqslant 1$. Put $\rho_s := s - N/\gamma$.

As the same argument in the proof of Proposition 3, applying Theorem 7 to V_p and V_q (defined later in corresponding to the case), we shall find λ and R such that the following equation

$$-\Delta_p u - \Delta_q u = \lambda\left(V_p|u|^{p-2}u + V_q|u|^{q-2}u\right) \quad \text{in } B_R, \quad u = 0 \quad \text{on } \partial B_R, \tag{3.9}$$

has a non-trivial (and non-negative) solution.

(i) $\alpha \neq 0$ and $\beta \in \mathbb{R}$: First, we assume that $\gamma \neq \infty$ if $N = 1$. Since $\rho_p > \rho_q$ and $\alpha \neq 0$, we may assume that $|\alpha|R^{\rho_p} > |\beta|R^{\rho_q}$ for large $R \gg 1$. Take $\delta = R^a (< R)$ with $a \in (0,1)$ and $R \gg 1$. By using the function W_* defined by (3.8), we set

$$V_p := |\alpha|W_* \quad \text{and} \quad V_q := \operatorname{sign}(\alpha)\,\beta W_*.$$

Here, in case $\Omega = B_R$, we recall that the constant in (1.6) is independent of R and (1.6) holds in case $N \in \{p,q\}$ too (see Remark 2). According to Lemma 2, Lemma 3 and (1.6) with $\Omega = B_R$ and $\delta = R^a$, as $R \to \infty$ we have

$$\|W_*\|_\gamma = O\left(R^{a(\rho+N/\gamma)}\right) \qquad \lambda_p(V_p) = \frac{\lambda_p(W_*)}{|\alpha|} \leqslant O\left(R^{-a(p+\rho)}\right) =: I$$

and

$$\lambda_q(V_q) \geqslant \lambda_q(|\beta|W_*) \geqslant \frac{\lambda_{q,q\gamma'}}{|\beta|\|W_*\|_\gamma} \geqslant O\left(R^{-a(\rho+N/\gamma)-\rho_q}\right) =: II$$

where $O(R^t)$ denotes the term such that $\lim_{R\to\infty} O(R^t)/R^t > 0$. Therefore, to obtain λ satisying $\lambda_p(V_p) < \lambda < \lambda_q(V_q)$ and

$$\|\lambda W_*\|_\gamma < CR^{-\rho_p+\varepsilon}/|\alpha|, \text{ equivalently, } 0 < \lambda < O\left(R^{-\rho_p+\varepsilon-a(\rho+N/\gamma)}\right) =: III$$

for large $R \gg 1$, it sufficient to show that $I < III < II$ for large $R \gg 1$, that is, $a(p+\rho) > \rho_p - \varepsilon + a(\rho+N/\gamma) > a(\rho+N/\gamma)+\rho_q$. The last inequality follows from $\rho_p - \varepsilon > \rho_q$. Moreover, the first inequality is obtained by taking a such that $1 > a > 1-\varepsilon/\rho_p$. Consequently, for such large $R \gg 1$, choosing λ above, equation (3.9) has a non-trivial and non-negative solution. Therefore, our assertion is proved with $W := \mathrm{sign}(\alpha)\lambda W_*$.

Now let us consider the case $\gamma = \infty$ and $N = 1$. Recall that it is known that $\lambda_{q,q} = (q-1)(\pi/q\sin(\pi/q))^q R^{-q}$ in case $\Omega = (-R,R)$. Because of $\rho = 1-N = 0$, $W_*(x) = \chi_{[-\delta,\delta]}(x)$ and $\|W_*\|_\infty = 1$. Take $\delta = R^a$ with $a \in (0,1)$, and then we observe that

$$\lambda_p(|\alpha|W_*) \leqslant O\left(R^{-ap}\right) =: I, \qquad \lambda_q(|\beta|W_*) \geqslant \frac{\lambda_{q,q}}{|\beta|\|W_*\|_\infty} \geqslant O\left(R^{-q}\right) =: II$$

and

$$\|\lambda W_*\|_\infty = |\lambda| < \frac{C}{|\alpha|}R^{-p+\varepsilon} =: III.$$

Choosing $a \in (0,1)$ such that $ap > p-\varepsilon$, we see that $I < III < II$ for large $R \gg 1$ because of $\varepsilon < \rho_p - \rho_q = p-q$. Hence, by the same argument above, this case can be shown.

(ii) $\beta \neq 0$ and $\alpha \in \mathbb{R}$: The proof can be done in the same way as above. So, we give a sketch of the proof. Since $\rho_p > \rho_q$ and $\beta \neq 0$, we may assume that $|\alpha|R^{\rho_p} < |\beta|R^{\rho_q}$ for small $0 < R \ll 1$. Take $\delta = R^b (< R < 1)$ with $b > 1$ and $0 < R \ll 1$. Set

$$V_p := \mathrm{sign}(\beta)\,\alpha W_* \quad \text{and} \quad V_q := |\beta|W_*.$$

By taking suitable $\lambda > 0$ and small $0 < R \ll 1$. Theorem 7 guarantees the existence of a non-trivial (and non-negative) solution of (3.9), whence the proof is done by setting $W := \mathrm{sign}(\beta)\,\lambda\, W_*$. Now let us see the existence of λ. As $R \to +0$, we have

$$\|W_*\|_\gamma = O\left(R^{b(\rho+N/\gamma)}\right), \quad \lambda_q(V_q) \leqslant O\left(R^{-b(q+\rho)}\right) =: I$$

and

$$\lambda_p(V_p) \geqslant \lambda_p(|\alpha|W_*) \geqslant \frac{\lambda_{p,p\gamma'}}{|\alpha|\|W_*\|_\gamma} \geqslant O\left(R^{-b(\rho+N/\gamma)-\rho_p}\right) =: II$$

Moreover, we see that

$$\|\lambda W_*\|_\gamma < CR^{-\rho_q-\varepsilon}/|\beta|, \text{ equivalently, } 0 < \lambda < O\left(R^{-\rho_q-\varepsilon-b(\rho+N/\gamma)}\right) =: III. \tag{3.10}$$

So, if we choose $b > 1$ such that $b < 1+\varepsilon/\rho_q$, then $I < III < II$ as $R \to +0$. Therefore, for such small $R \ll 1$ we can get λ satisying (3.10) and $\lambda_q(V_q) < \lambda < \lambda_p(V_p)$. The proof has been completed. □

4. Appendix: Proof of Theorem 7

Here, we give the sketch of the proof of Theorem 7.

First, we note that $\lambda_s(V_s)$ is attained if $\gamma_s > N/s$ if $N \geqslant s$. We choose one (non-negative) minimizer of $\lambda_s(V_s)$ and denote it by $0 \leqslant \psi_s \in W_0^{1,s}(\Omega) \cap L^\infty(\Omega) \cap C^0_{\mathrm{loc}}(\Omega)$ (see [10] for the details about the minimizer). Remark that we don't get the positivity of ψ_s in general because Ω is not supposed to be connected.

Since $\lambda_q(V_q)$ is not attained in case $\gamma_q = N/q$ if $N > q$, and the minimizer ψ_q may not belong to $W_0^{1,p}(\Omega)$, we need to prepare the following result.

LEMMA 4. *Let $\gamma_q \in [1,\infty]$ $(N<q)$ and $\gamma_q \geqslant N/q$ if $N \geqslant q$. Assume that $\lambda_q(V_q) < \infty$. Then, for any $\varepsilon > 0$ there exists $\psi_{q,\varepsilon} \in W_0^{1,p}(\Omega)$ such that*

$$\psi_{q,\varepsilon} \geqslant 0, \quad \int_\Omega V_q \psi_{q,\varepsilon}^q\, dx = 1 \quad \text{and} \quad \int_\Omega |\nabla \psi_{q,\varepsilon}|^q\, dx < \lambda_q(V_q) + \varepsilon. \tag{4.1}$$

Proof. Take any $\varepsilon > 0$. First, due to $\lambda_q(V_q) < \infty$, we can choose $\psi \in W_0^{1,q}(\Omega)$ such that

$$\int_\Omega V_q |\psi|^q\, dx = 1 \quad \text{and} \quad \int_\Omega |\nabla \psi|^q\, dx < \lambda_q(V_q) + \varepsilon/2.$$

Since $C_c^\infty(\Omega)$ is dense in $W_0^{1,q}(\Omega)$, thanks to the continuous embedding from $W_0^{1,q}(\Omega)$ into $L^{q\gamma_q'}(\Omega)$, there exists a sequence $\{\psi_n\}_n \subset C_c^\infty(\Omega)$ such that

$$\lim_{n\to\infty} \int_\Omega V_q |\psi_n|^q\, dx = \int_\Omega V_q |\psi|^q\, dx = 1 \quad \text{and} \quad \lim_{n\to\infty} \int_\Omega |\nabla \psi_n|^q\, dx = \int_\Omega |\nabla \psi|^q\, dx.$$

So, $|\psi_n|/(\int_\Omega V_q |\psi_n|^q\, dx)^{1/q}$ satisfies (4.1) for large n. $\square$

Next, we set an energy functional E_λ^+ on $W_0^{1,p}(\Omega)$ as follows:

$$E_\lambda^+(u) := \frac{1}{p} H_\lambda^+(u) + \frac{1}{q} G_\lambda^+(u) \quad \text{for } u \in W_0^{1,p}(\Omega),$$
$$H_\lambda^+(u) := \|\nabla u\|_p^p - \lambda \int_\Omega V_p u_+^p\, dx \quad \text{and} \quad G_\lambda^+(u) := \|\nabla u\|_q^q - \lambda \int_\Omega V_q u_+^q\, dx,$$

where $u_\pm := \max\{\pm u, 0\}$. We see that any critical point u of E_λ^+ satisfies $u \geqslant 0$ by taking u_- as a test function. It is easily shown that any critical point of E_λ^+ corresponds to a non-negative solution of (P) (refer to [26, Remark 3.1.] or Remark 1 for the regularity of solutions).

The proof is done with the same argument in [26, Theorem 1.3.] using

$$I := \left| \int_\Omega V_s u_+^s\, dx \right| \leqslant \|V_s\|_{\gamma_s} \|u_+\|_{\sigma_s}^s \leqslant \frac{\|V_s\|_{\gamma_s} \|\nabla u\|_s^s}{\lambda_{s,\sigma_s}}, \ \frac{\|V_s\|_{\gamma_s} \|\nabla u\|_p^s}{\lambda_{p,\sigma_s}^{s/p}}$$

for $s \in \{p,q\}$ with $\sigma_s := s\gamma_s'$ instead of $I \leqslant \|V_s\|_\infty \|u_+\|_s^s \leqslant \|V_s\|_\infty \|\nabla u\|_s^s / \lambda_{s,s}$ in [26].

Moreover, we use the following result instead of [26, Lemma 3.3]. The proof is done by the same argument as in [37, Lemma 9.] according to the compactness of the embedding from $W_0^{1,p}(\Omega)$ into $L^q(\Omega)$ and $L^{\sigma_p}(\Omega)$. For readers' convenience, we give a sketch of the proof.

LEMMA 5. *Let* $q,\sigma_p \in [1,\infty]$ *if* $N < p$ *and* $1 \leqslant q,\sigma_p < p^*$ *if* $N \geqslant p$. *Set*

$$X(d) := \left\{ u \in W_0^{1,p}(\Omega) \,:\, \|\nabla u\|_p^p \leqslant d\,\|u\|_{\sigma_p}^p \right\}$$

for $d > 0$. *Then there exists* $C_d > 0$ *such that*

$$\|\nabla u\|_p \leqslant C_d \|u\|_q \quad \text{for all } u \in X(d).$$

Proof. Suppose, by contradiction, that there exist $d > 0$ and a sequence $\{u_n\}_n \subset X(d)$ such that $\|\nabla u_n\|_p > n\|u_n\|_q$ for all $n \in \mathbb{N}$. Then, since a normalized sequence $v_n := u_n/\|u_n\|_{\sigma_p}$ is bounded in $W_0^{1,p}(\Omega)$ by $u_n \in X(d)$, we may assume, up to a subsequence, that v_n converges some v_0 weakly in $W_0^{1,p}(\Omega)$ and strongly in $L^{\sigma_p}(\Omega)$ and $L^q(\Omega)$. Because $\|v_n\|_{\sigma_p} = 1$ for all n, we have $v_0 \neq 0$. On the other hand, our contradictional assumption leads to that $d^{1/p} \geqslant \limsup_{n\to\infty} \|\nabla v_n\|_p \geqslant \lim_{n\to\infty}(n\|v_n\|_q) = \infty$, from which we get the desired conclusion. $\square$

Proof of Theorem 7. Case $\lambda_q(V_q) < \lambda < \lambda_p(V_p)(\leqslant \infty)$: In this case, we shall show that E_λ^+ has a global minimum point with negative energy. In fact, by the argument as in [26, page 11.], we can show that

$$E_\lambda^+(u) \geqslant \frac{\varepsilon}{p}\|\nabla u\|_p^p - \frac{\lambda\|V_q\|_{\gamma_q}}{q\,\lambda_{p,\sigma_q}^{q/p}}\|\nabla u\|_p^q \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

where $\varepsilon \in (0,1]$ satisfying $(1-\varepsilon)\lambda_p(V_p) > \lambda$ if $\lambda_p(V_p) < \infty$. Hence, E_λ^+ is coercive and bounded from below, because $\sigma_p, \sigma_q < p^*$ (if $N \geqslant p$) guarantees the weakly lower semi-continuity of E_λ^+, and so E_λ^+ has a global minimizer. Since from $\lambda_q(V_q) < \lambda$ and $q < p$, for small $\delta \in (0, \lambda - \lambda_q(V_q))$, the nonnegative function $\psi_{q,\delta} \in W_0^{1,p}(\Omega)$ obtained in Lemma 4 satisfies that $E_\lambda^+(t\psi_{q,\delta}) < 0$ for small $t > 0$. Thus, the minimum value of E_λ^+ is negative, and so E_λ^+ has a non-trivial critical point.

Case $\lambda_p(V_p) < \lambda < \lambda_q(V_q)(\leqslant \infty)$: First, we note that by the standard argument (refer to [37, Lemma 12.] or see [3, Lemma 3.2.] for the boundedness of the Palais–Smale sequence), it is proved that the functional E_λ^+ satisfies the Palais–Smale condition provided $\lambda \neq \lambda_p(V_p)$.

In this case, we shall see that E_λ^+ has the mountain pass geometry. In Lemma 5 we take d satisfying

$$d > \max\left\{1, \lambda\|V_p\|_{\gamma_p}, \frac{\lambda\|V_p\|_{\gamma_p}}{\lambda_{p,\sigma_p}}\right\}.$$

Using above d in [26, (19)], the arguments as in [26, p. 12–13] leads to

$$E_\lambda^+(u) \geqslant -\frac{(d-1)C_d^p}{p}\|u\|_q^p + \frac{\varepsilon\lambda_{q,q}}{q}\|u\|_q^q \qquad \text{for any } u \in W_0^{1,p}(\Omega), \tag{4.2}$$

where C_d is the constant obtained by Lemma 5 and $\varepsilon \in (0,1]$ such that $(1-\varepsilon)\lambda_q(V_q) > \lambda$ if $\lambda(V_q) < \infty$. Moreover, it is easily shown that $E_\lambda^+(R\psi_p) \to -\infty$ as $R \to \infty$ by $\lambda < \lambda_p(V_p)$. Thus, we define a mountain pass value as follows:

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} E_\lambda^+(\gamma(t)),$$
$$\Gamma := \left\{\gamma \in C\left([0,1], W_0^{1,p}(\Omega)\right) : \gamma(0) = 0 \text{ and } \gamma(1) = R\psi_p\right\},$$

where $R > 0$ is a large number satisfying $E_\lambda^+(R\psi_p) < 0$. Thanks to (4.2), $c > 0$ holds, whence c is a positive critical value of E_λ^+.

Consequently, the proof has finished. □

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