Differential
Equations
& Applications
Volume 17, Number 1 (2025), 21–34
doi:10.7153/dea-2025-17-02

EXPLORING MULTIPLE SOLUTIONS AND NUMERICAL APPROACHES FOR A SIXTH–ORDER BOUNDARY VALUE PROBLEM

ANDRÉ LUÍS M. MARTINEZ, EMERSON V. CASTELANI*,
CRISTIANE A. PENDEZA MARTINEZ AND
CARLOS ALEXANDRE RIBEIRO MARTINS

(*Communicated by S. K. Ntouyas*)

Abstract. We analyze the existence of multiple solutions for a sixth-order boundary value problem. Firstly, we introduce an operator that transforms the problem into a fixed-point problem and delineate its key properties. Subsequently, we investigate the existence of solutions in the functional space $C^1[0,1]$, employing the fixed-point theorem of Avery-Peterson. We then provide non-trivial examples and establish a theorem based on the Banach-Piccard theorem, motivating the definition of a numerical method based on the compression principle for the problem. Additionally, we discuss the utilization of nonlinear optimization methods for the problem and compare them with the classical method based on the contraction principle.

1. Introduction

In this article, we undertake a study concerning the necessary conditions for the existence of multiple solutions to a sixth-order boundary value problem (BVP). Higher-order differential equations, particularly sixth-order BVPs, arise in various physical and engineering contexts. Equations of the form

$$u^{(6)}(t) = f(t,u,u'',u^{(4)}), \quad 0 < t < 1, \tag{1}$$

are widely used in elasticity theory to model the behavior of circular ring beams [17], as well as in the study of sandwich beams in structural mechanics [2]. Furthermore, sixth-order differential equations appear in vibration analysis in the automotive industry [4] and are relevant in astrophysics and hydrodynamics [12], [7], [24], and [25]. The boundary conditions considered in this study reflect physically significant constraints, such as clamped, simply supported, or sliding-supported structures. The mathematical modeling of such problems plays a crucial role in understanding the stability and deformation of materials under external forces. By analyzing this equation under specific conditions, we aim to provide insights into the existence of multiple solutions and

Mathematics subject classification (2020): 34B15, 34B18.
Keywords and phrases: Numerical solutions, sixth-order, boundary value problem and multiple solutions.

* Corresponding author.

assess the numerical efficiency of computational methods used for higher-order differential models.

The problem under consideration is explored in [21], with its equation being more generalized compared to those addressed in [26] and [20]. Hence, we delve into an investigation regarding the existence of multiple solutions to the sixth-order boundary value problem:

$$u^{(6)}(t)+f(t,u,u')=0, \quad 0<t<1, \tag{2}$$

with the boundary conditions:

$$u(0)=u'(0)=u''(0)=0,\ u'(1)=u'''(1)=u^{(5)}(1)=0. \tag{3}$$

Where, $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

The problem defined in (2)–(3) has been the subject of study by various authors, who often consider simpler versions of the problem in their works. Typically, the problem is addressed without the dependency of f on the term u'. This simplification allows for obtaining qualitative and quantitative results regarding the existence of solutions. In this regard, we recommend consulting [1], [3], [2], [9], [10], [11], [13], [14], [23], [8], and the references therein.

Works such as [9], [14], [21], [20], and [26] have presented conditions for the existence of solutions to the problem (2)–(3). In the paper [26], the dependence of f on u and u' is not considered, and the existence of solution is obtained by applying Krasnoselskii's fixed-point theorem. In [20], a more general version of the problem is considered, the dependence of f with respect to the term u is included, and results regarding the existence of multiple solutions are presented by applying the Avery-Peterson theorem. In the work [21], the authors conduct a study based on Krasnoselskii's theorem for the problem (2)–(3), aiming to determine sufficient conditions for the existence of a positive solution.

Most articles addressing the sixth-order problem do not delve into numerical solution determination. However, in the work [20], the authors provide a study for numerical solutions based on Banach's Contraction Principle. Conversely, in [21], numerical optimization techniques are employed to determine numerical solutions. In this context, we propose a comparative approach in this work between the techniques proposed by [20] and [21]. In other words, we present a comparison between the use of nonlinear programming methods and methods based on numerical solutions using Banach's Contraction Principle.

A recent study [5] investigates the existence of positive solutions for a sixth-order boundary value problem, considering a function f in (2) that depends up to the fifth derivative of u. Their approach is based on the Leray-Schauder fixed-point theorem, focusing primarily on establishing sufficient conditions for the existence of solutions. However, their analysis is restricted to verifying the compression of the integral operator. In contrast, our work explores not only the compression of the operator but also its expansion, ensuring the existence of at least three positive solutions. This broader approach requires us to work within the C^1 norm rather than higher-order norms, such as C^5, as the latter would significantly complicate the analysis and potentially render it infeasible.

We consider the function f as dependent only on t, u, and u', excluding its dependence on higher-order derivatives. This choice is motivated by the fact that incorporating u'' would require working in the C^2 norm, which significantly complicates the functional framework and the mathematical treatment of the problem. Several authors have adopted similar assumptions [21], [6], and [15], allowing for a more tractable analysis while still capturing relevant nonlinear behaviors. We highlight that in [15], the authors investigate the existence and uniqueness of solutions for (2), assuming that f depends only on t and u, considering nonlocal and integral boundary conditions. The authors analyze the properties of Green's functions and employ the Krasnoselskii-Zabreiko fixed point theorem to ensure the existence of at least one nontrivial solution. Additionally, they present two numerical examples to illustrate the obtained results.

The main contributions of this work are as follows:

- A new result demonstrating the existence of multiple solutions to the problem (2)–(3), obtained through the application of the Avery-Peterson Theorem (Section 2).
- A comparative analysis between discretization and the use of optimization techniques versus numerical solutions using Banach's Contraction Principle for determining numerical solutions to the problem (2)–(3) (Section 3).
- Presentation of examples to validate the theoretical and numerical results (Sections 2 and 3).

2. Positive solutions

Analogously to the approach presented in [26] and [21], we can represent problem (2)–(3) as a fixed-point problem for the operator $T : C^1[0,1] \to C^1[0,1]$. In this manner, we can explore properties of the operator T to obtain results regarding the existence of solutions within the domain of the operator, namely, in the functional space $C^1[0,1]$, and according to [21] its formulation is presented below:

$$Tu(t) = \int_0^1 G(t,s) f(s,u,u')ds \tag{4}$$

where G is the Green's function:

$$G(t,s) = \left(\frac{t^3}{2} - \frac{t^4}{8}\right)\frac{(1-s)^4}{24} - \left(\frac{t^3}{12} - \frac{t^4}{16}\right)\frac{(1-s)^2}{2} + \frac{t^3}{48} - \frac{5t^4}{192} + \frac{t^5}{120} - \frac{(t-s)^5}{120}H(t-s), \tag{5}$$

$$\frac{\partial}{\partial t}G(t,s) = \left(\frac{3t^2}{2} - \frac{t^3}{2}\right)\frac{(1-s)^4}{24} + \left(-\frac{t^2}{4} + \frac{t^3}{4}\right)\frac{(1-s)^2}{2} + \frac{t^2}{16} - \frac{5t^3}{48} + \frac{t^4}{24} - \frac{(t-s)^4}{24}H(t-s), \tag{6}$$

where is the Heaviside function

$$H(\zeta) = \begin{cases} 1, & \zeta \geqslant 0 \\ 0, & \zeta < 0 \end{cases} . \tag{7}$$

Next, we outline the key properties of the Green's function G and its derivative $\frac{\partial}{\partial t}G$, as presented in [26] and [21], which will be essential for demonstrating multiple solutions:

- How $G(1,s) = \frac{s^3}{960}(20-25s+8s^2) \geqslant 0$ there exist increasing polynomials $x(t)$ and $y(t)$ such that:

$$0 \leqslant x(t) \leqslant y(t) \leqslant 1, \tag{8}$$

$$x(t)G(1,s) \leqslant G(t,s) \leqslant y(t)G(1,s), \tag{9}$$

  where $x(t) = \frac{t^3}{3}(20-25t+8t^2)$, $y(t) = 4t^2-4t^3+t^4$.

- As $\frac{\partial}{\partial t}G$ is restricted to the interval $[0,1]\times[0,1]$, we can conclude that:

$$\frac{\partial}{\partial t}G(0,t) \leqslant \frac{\partial}{\partial t}G(t,s) \leqslant \max_{t\in[0,1]}\left[\frac{\partial}{\partial t}G(t,s)\right], \tag{10}$$

- $t \in [x,y]$

$$\min_{t\in[x,y]}\left[\frac{\partial}{\partial t}G(t,s)\right] \leqslant \frac{\partial}{\partial t}G(t,s) \leqslant \max_{t\in[x,y]}\left[\frac{\partial}{\partial t}G(t,s)\right], \tag{11}$$

To ascertain the existence of multiple solutions, we shall examine the cone defined as

$$E = \{u \in C^1[0,1] : u(0) = u'(0) = 0,\ u(t) \geqslant 0, \forall t \in [0,1]\},$$

where $C^1[0,1]$ represents the Banach space of continuously differentiable functions on $[0,1]$, equipped with the norm

$$\|u\|_{C^1} = \max\{\|u\|_\infty, \|u'\|_\infty\}$$

By applying the Arzela-Ascoli theorem, it is established that T is completely continuous as an integral operator, as elucidated in [21]. To illustrate the main result of this study, which establishes the existence of multiple solutions, we need to introduce the principal tool to be employed.

AVERY-PETERSON THEOREM. *Now, we need to consider the convex sets*

$$P(\gamma,d) = \{x \in P | \gamma(x) < d\}$$

$$P(\gamma,\alpha,b,d) = \{x \in P | b \leqslant \alpha(x) \text{ and } \gamma(x) \leqslant d\}$$

$$P(\gamma,\theta,\alpha,b,c,d) = \{x \in P | b \leqslant \alpha(x), \theta(x) \leqslant c \text{ and } \gamma(x) \leqslant d\}$$

and the closed set

$$R(\gamma,\psi,a,d) = \{x \in P | a \leqslant \psi(x) \text{ and } \gamma(x) \leqslant d\}.$$

THEOREM 1. *Let P be a cone in a real Banach space X. Let γ and θ nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x) \leqslant \lambda\psi(x)$ for $0 \leqslant \lambda \leqslant 1$, such that for some positive numbers μ and d,*

$$\alpha(x) \leqslant \psi(x) \text{ and } \|x\| \leqslant \mu\gamma(x),$$

for all $x \in \overline{P(\gamma,d)}$. Suppose

$$T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$$

is completely continuous and there exist positive numbers a, b, c with $a < b$, such that

$$\{u \in P(\gamma,\theta,\alpha,b,c,d) | \alpha(u) > b\} \neq \emptyset \text{ and}$$

$$u \in P(\gamma,\theta,\alpha,b,c,d) \Rightarrow \alpha(Tu) > b, \tag{12}$$

$$\alpha(Tu) > b \text{ for } u \in P(\gamma,\alpha,b,d) \text{ with } \theta(Tu) > c, \tag{13}$$

$$0 \notin R(\gamma,\psi,a,d) \text{ and } \psi(Tu) < a \text{ for} \tag{14}$$

$$u \in R(\gamma,\psi,a,d) \text{ with } \psi(u) = a.$$

Then T has at least three distinct fixed points $x_1, x_2, x_3 \in \overline{P(\gamma,d)}$ such that $\gamma(x_i) \leqslant d$, $i = 1,2,3$; $b < \alpha(x_1)$; $a < \psi(x_2)$; $\alpha(x_2) < b$; $\psi(x_3) < a$.

To apply the Theorem 1 it will be necessary to consider some hypotheses about the functions that make up the problem (2)–(3).

(A1) Then, for the problem (2)–(3), there exists a positive constant d so that:

- For all $(s,u,v) \in [0,1] \times [0,d] \times [0,d]$ then $0 \leqslant f(s,u,v) \leqslant \frac{d}{r_1}$
- $r_1 = \displaystyle\int_0^1 \max_{t\in[0,1]} \left[\frac{\partial}{\partial t} G(t,s)\right] ds;$

The lemma presented will be fundamental for demonstrating our main result.

LEMMA 1. *Suppose that* **(A1)** *holds, $P = E$, and $\gamma(.) = \|.\|_{C^1}$. Then, T defined in (4) satisfies: $T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$.*

Proof. Then, based on (A1) and Remark 1 in [21]. Let's take an element u belonging to E, with a norm $|u|_E$ not exceeding d. Then, based on **(A1)**, we can deduce:

$$\begin{aligned}
\|Tu\|_E &= \max_{t\in[0,1]} \left|\frac{\partial}{\partial t}(Tu)(t)\right|, \\
&\leqslant \max_{t\in[0,1]} \int_0^1 \left|\frac{\partial}{\partial t} G(t,s)\right| |f(s,u(s),u'(s))| ds \\
&\leqslant \int_0^1 \max_{t\in[0,1]} \left[\frac{\partial}{\partial t} G(t,s)\right] |f(s,u(s),u'(s))| ds \\
&\leqslant \frac{d}{r_1} \int_0^1 \max_{t\in[0,1]} \left[\frac{\partial}{\partial t} G(t,s)\right] ds \\
&\leqslant d.
\end{aligned}$$

Therefore $T:\overline{P(\gamma,d)}\to\overline{P(\gamma,d)}$. $\square$

Theorem 2 outlines the circumstances in which the problem described in (2)–(3) must possess a minimum of three distinct positive solutions.

THEOREM 2. *Assuming hypothesis* **(A1)** *is satisfied, and further, let's assume the existence of* a *such that* $0<a<d$*, with* f *satisfying the following conditions:*

(A2) $f(s,u,v)>\dfrac{2a}{r_2}$, $\forall(s,u,v)\in[0,1]\times[2a,12a]\times[0,d]$, *where* $r_2=\dfrac{19}{256}\displaystyle\int_{\frac{1}{4}}^{\frac{3}{4}}G(1,s)ds$.

(A3) $f(s,u,v)<\dfrac{a}{r_3}$, $\forall(s,u,v)\in[0,1]\times[0,a]\times[0,d]$, *where* $r_3=\displaystyle\int_0^1 G(1,s)ds$.

Consequently, Problem (2)–(3) *possesses a minimum of three positive solutions.*

Proof. We shall utilize the Avery-Peterson theorem, where we define T and P as previously stated. Additionally, it is necessary to introduce the following functionals:

$$\begin{aligned}
\gamma(u) &= \|u\|_E = \max_{t\in[0,1]}\{|u'(t)|\},\\
\psi(u) &= \max_{t\in[0,1]}\{|u(t)|\},\\
\theta(u) &= \max_{t\in[\frac{1}{4},\frac{3}{4}]}\{|u(t)|\},\\
\alpha(u) &= \min_{t\in[\frac{1}{4},\frac{3}{4}]}\{|u(t)|\}.
\end{aligned}$$

Hence, according to Lemma 1, we derive the mapping

$$T:\overline{P(\gamma,d)}\to\overline{P(\gamma,d)}$$

where T is completely continuous. Moreover, there exist positive numbers b and c, with $a<b$, such that

$$\{u\in P(\gamma,\theta,\alpha,b,c,d)|\alpha(u)>b\}\neq\emptyset \quad \text{and}$$

$$u\in P(\gamma,\theta,\alpha,b,c,d)\Rightarrow\alpha(Tu)>b \tag{15}$$

$$\alpha(Tu)>b \text{ for } u\in P(\gamma,\alpha,b,d) \text{ with } \theta(Tu)>c, \tag{16}$$

$$0\notin R(\gamma,\psi,a,d) \text{ and } \psi(Tu)<a \text{ for } u\in C^1, \tag{17}$$

$$\text{for } u\in R(\gamma,\psi,a,d) \text{ with } \psi(u)=a.$$

We can define the constants b and c as follows:

$$b=2a$$

and

$$c=12a.$$

Clearly, we have $u \in P(\gamma,\theta,\alpha,b,c,d)|\alpha(u) > b \neq \emptyset$. Now, let's establish (15). By employing **(A2)**, we acquire:

$$\begin{aligned}
\alpha(Tu) &= \min_{t\in[\frac{1}{4},\frac{3}{4}]} |(Tu)(t)| \\
&= \min_{t\in[\frac{1}{4},\frac{3}{4}]} \left(\int_0^1 G(t,s)f(s,u(s),u'(s))ds\right) \\
&\geqslant \min_{t\in[\frac{1}{4},\frac{3}{4}]} \left(\int_0^1 x(t)G(1,s)f(s,u(s),u'(s))ds\right) \\
&\geqslant x\left(\frac{1}{4}\right)\int_0^1 G(1,s)f(s,u(s),u'(s))ds \\
&\geqslant \frac{19}{256}\int_0^1 G(1,s)f(s,u(s),u'(s))ds \\
&\geqslant \frac{19}{256}\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)f(s,u(s),u'(s))ds \\
&\geqslant \frac{19}{256}\frac{2a}{r_2}\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds \\
&\geqslant 2a = b.
\end{aligned}$$

Let's prove (13). Suppose $u \in P(\gamma,\alpha,b,d)$ with $\theta(Tu) > c$. Then,

$$\begin{aligned}
\alpha(Tu) &= \min_{t\in[\frac{1}{4},\frac{3}{4}]} (Tu)(t) \\
&= \min_{t\in[\frac{1}{4},\frac{3}{4}]} \left(\int_0^1 G(t,s)f(s,u(s),u'(s))ds\right) \\
&\geqslant \min_{t\in[\frac{1}{4},\frac{3}{4}]} \left(\int_0^1 x(t)G(1,s)f(s,u(s),u'(s))ds\right) \\
&\geqslant x\left(\frac{1}{4}\right)\left(\int_0^1 G(1,s)f(s,u(s),u'(s))ds\right) \\
&\geqslant y\left(\frac{3}{4}\right)\frac{x\left(\frac{1}{4}\right)}{y\left(\frac{3}{4}\right)}\left(\int_0^1 G(1,s)f(s,u(s),u'(s))ds\right) \\
&\geqslant \frac{x\left(\frac{1}{4}\right)}{y\left(\frac{3}{4}\right)}\max_{t\in[\frac{1}{4},\frac{3}{4}]}\left(\int_0^1 y(t)G(1,s)f(s,u(s),u'(s))ds\right) \\
&\geqslant \frac{19}{207}\max_{t\in[\frac{1}{4},\frac{19}{207}]}\left(\int_0^1 G(t,s)f(s,u(s),u'(s))ds\right) \\
&\geqslant \frac{19}{207}\theta(Tu) \\
&> \frac{19}{207}c = b.
\end{aligned}$$

To establish (14), let's take $u \in R(\gamma,\psi,a,d)$ such that $\psi(u) = a$. Based on **(A3)**, we have:

$$\begin{aligned}\psi(Tu) &= \max_{t\in[0,1]} |(Tu)(t)| \\ &\leqslant \max_{t\in[0,1]} \int_0^1 |G(t,s)||f(s,u(s),u'(s))|ds \\ &\leqslant \max_{t\in[0,1]} \int_0^1 y(t)G(1,s)|f(s,u(s),u'(s))|ds \\ &\leqslant \frac{a}{r_3}\left[\int_0^1 G(1,s)ds\right]\max_{t\in[0,1]} y(t) \\ &\leqslant a.\end{aligned}$$

Utilizing the Avery-Peterson theorem, we ascertain that the problem yields a minimum of three distinct solutions within the set $\overline{P(\gamma,d)}$, ensuring their non-negativity. Moreover, these solutions are required to meet hypothesis **(A2)** and thus cannot be null. Consequently, Problem (2)–(3) exhibits a minimum of three positive solutions. □

We present an example satisfying hypotheses **(A1)**, **(A2)**, and **(A3)** to illustrate the consistency of these assumptions. The following example illustrates the conditions specified in Theorem 2.

EXAMPLE 1. Let the component function f of the problem (2)–(3) be represented as

$$f(s,u,v) = \begin{cases} s + \left(\frac{109}{8}\right)^4 + \frac{(u-2a)^2}{10} + \left(\frac{v}{10}\right)^2, & u \geqslant 2a \\ s + \left(\sqrt{\frac{109}{8}}u\right)^8 + \left(\frac{v}{10}\right)^2, & u < 2a \end{cases}$$

To verify that the problem defined with this function f satisfies hypotheses **(A1)**, **(A2)**, and **(A3)**, we need to choose the constants:

$$d = 100, \ \ a = \frac{1}{2},$$

We can readily verify that under these conditions, indeed with these choices, we calculate the constants:

$$\begin{aligned} b &= 2a = 1; \\ c &= 12a = 6; \\ r_1 &= \int_0^1 \max_{t\in[0,1]}\left[\frac{\partial}{\partial t}G(t,s)\right]ds = 0.00239740 \\ r_2 &= \frac{19}{256}\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds = \frac{18031}{377487360} \\ r_3 &= \int_0^1 G(1,s)ds = \frac{1}{720}\end{aligned}$$

It becomes straightforward to verify that hypotheses **(A1)**, **(A2)**, and **(A3)** are satisfied, in fact,

- **(A1)** $0 \leqslant f(s,u,v) \leqslant \max f(s,u,v) = 3.5544 \times 10^4 < \dfrac{d}{r_1} = 4.1712 \times 10^4$.
- **(A2)** $\dfrac{377487360}{18031} = \dfrac{2a}{r_2} < 3.4462 \times 10^4 \leqslant f(s,u,v), \forall (s,u,v) \in [0,1] \times [2a,12a] \times [0,d]$.
- **(A3)** $f(s,u,v) \leqslant 235.6189 < \dfrac{a}{r_3} = 360$, $\forall (s,u,v) \in [0,1] \times [0,a] \times [0,d]$.

3. Numerical testing and method performance analysis

In this section, we establish the existence and uniqueness of solutions for (2)–(3). Drawing inspiration from previous works such as [20], [19], we will introduce a method based on Banach's fixed point theorem. Furthermore, we will compare this classical approach with the utilization of nonlinear programming methods, as proposed in [21], [22], [18].

3.1. Numerical solution methods

Although classical, this approach is crucial for defining numerical methods for our problem. We begin by considering the iterative sequence

$$\begin{aligned} u^{k+1}(t) &= (Tu^k)(t) \\ &= \int_0^1 G(t,s) f(s,u^k(s),v^k(s))ds. \end{aligned}$$

and the basic hypothesis

(A4) $|f(s,u,v) - f(s,\overline{u},\overline{v})| \leqslant \dfrac{\xi}{r_1}|u(s) - \overline{u}(s)|$; $\forall u, v, \overline{u}, \overline{v} \in [0,d]$, $s \in [0,1]$ and $\xi \in (0,1)$.

THEOREM 3. *Assume that* **(A1)** *and* **(A4)** *hold. Then,* (2)–(3) *possesses a unique solution* u *with* $|u|_E \leqslant d$. *Furthermore,* $u^{k+1} = T(u^k) \to u^*$.

Proof. We will demonstrate that the operator T is a contraction. To begin, let's consider $u, \overline{u} \in E$ with $\|u\|_E \leqslant d$ and $\|\overline{u}\|_E \leqslant d$. Then

$$\begin{aligned} \|Tu - T\overline{u}\|_E &= \|(Tu - T\overline{u})'\|_\infty \\ &= \max_{t\in[0,1]} \left| \int_0^1 \frac{\partial}{\partial t} G(t,s)[f(s,u(s),v(s)) - f(s,\overline{u}(s),\overline{v}(s))]ds \right| \\ &\leqslant \max_{t\in[0,1]} \int_0^1 \frac{\partial}{\partial t} G(t,s)|f(s,u(s),v(s)) - f(s,\overline{u}(s),\overline{v}(s))|ds \end{aligned}$$

$$
\begin{aligned}
&\leqslant \int_0^1 \max_{t\in[0,1]}\left[\frac{\partial}{\partial t}G(t,s)\right]|f(s,u(s),v(s))-f(s,\overline{u}(s),\overline{v}(s))|ds\\
&\leqslant \left(\frac{\xi}{r_1}\max_s|u(s)-\overline{u}(s)|\right)\int_0^1 \max_{t\in[0,1]}\left[\frac{\partial}{\partial t}G(t,s)\right]ds\\
&\leqslant \xi\max_s|u(s)-\overline{u}(s)|\\
&\leqslant \xi\|u-\overline{u}\|_E.
\end{aligned}
$$

Consequently, by the contraction mapping principle, there exists only one solution, which can be obtained as the limit of the sequence $u^{k+1}=T(u^k)\to u^*$. □

Inspired by the previous result, we can introduce Algorithm 1.

Algorithm 1 *Fixed-Point (FP)*

```
Define a uniformly distributed mesh s_j in the interval [0,1];
Establish an initial approximation u_j^0 = u^0(s_j) and a tolerance ε > 0;
k=0;
while ||u^{k+1} - u^k||_∞ > ε or k = 0 do
    Compute u_j^{k+1} using

        u^{k+1} = T(u^k) and Trapezoidal Rule

    k = k + 1;
end while
output: u^k.
```

We will also test a method based on nonlinear programming for determining multiple solutions to the problem (2)–(3), briefly describing the method presented in [21].

To grasp the functionality of our numerical approach, it's imperative to comprehend the discretized problem model in terms of optimization. To this end, let's delineate $t_j, j=0,1,\ldots,n$ as a discretization of $[0,1]$ using an equally spaced mesh, where $h=t_{j+1}-t_j,\ j=0,1,\ldots,n-1$, and $u_j\approx u(t_j),\ j=0,1,\ldots,n$. We define the vector $\mathbf{u}=(u_0,u_1,\ldots,u_n)$. By substituting the classical finite difference schemes into (2) and (3), we derive the nonlinear system $R(\mathbf{u})=0$, where $R:\mathbb{R}^{n+1}\to\mathbb{R}^{n+7}$ is defined as:

$$
\begin{aligned}
&R_i(\mathbf{u})=\overline{u}_+^{(6)}(t_i,h)+f(t_i,\mathbf{u},\overline{u}_+^{(1)}(t_i,h))=0,\, i=0,1,2\\
&R_i(\mathbf{u})=\overline{u}^{(6)}(t_i,h)+f(t_i,\mathbf{u},\overline{u}^{(1)}(t_i,h))=0,\, 3\leqslant i\leqslant n-3\\
&R_i(\mathbf{u})=\overline{u}_-^{(6)}(t_i,h)+f(t_i,\mathbf{u},\overline{u}_-^{(1)}(t_i,h))=0,\, n-2\leqslant i\leqslant n\\
&R_{n+1}(\mathbf{u})=u_0=0\\
&R_{n+2}(\mathbf{u})=\overline{u}_+^{(1)}(t_0,h)=0\\
&R_{n+3}(\mathbf{u})=\overline{u}_+^{(2)}(t_0,h)=0\\
&R_{n+4}(\mathbf{u})=\overline{u}_-^{(1)}(t_n,h)=0
\end{aligned}
$$

$$R_{n+5}(\mathbf{u}) = \overline{u}_-^{(3)}(t_n,h) = 0$$
$$R_{n+6}(\mathbf{u}) = \overline{u}_-^{(5)}(t_n,h) = 0$$

The nonlinear system $R(\mathbf{u}) = 0$ consists of $n+7$ equations. Assuming $u_0 = 0$, our objective is to solve for $u_1, u_2, \ldots, u_n$, thus aiming to determine $n+1$ variables. Conventionally, numerical solutions utilize fixed-point methods. In this instance, the method involves an iterative sequence based on operator (3). In this paper, we use a method centered on the Gauss-Newton approach [16]. Here is the Algorithm 2 for solving the nonlinear system $R(\mathbf{u}) = 0$.

Algorithm 2 *Gauss-Newton (GN)*

1: Define a uniformly distributed mesh s_j in $[0,1]$;
2: Set an initial approximation $\mathbf{u}^0$ (where $u_j^0 \approx u(s_j)$) and tolerance $\varepsilon > 0$;
3: $k = 0$;
4: **while** $\|\mathbf{u}^{k+1} - \mathbf{u}^k\|_\infty > \varepsilon$ or $k = 0$ **do**
5: Compute vector $\mathbb{R}_k = R(\mathbf{u}^k)$ and matrix

$$\mathbf{A}_k = \begin{bmatrix} \nabla R_0(\mathbf{u}^k) \\ \nabla R_1(\mathbf{u}^k) \\ \vdots \\ \nabla R_{n+6}(\mathbf{u}^k) \end{bmatrix}$$

6: Find Δ_k such that:

$$(\mathbf{A}_k^T \mathbf{A}_k)\Delta_k = -\mathbf{A}_k^T \mathbb{R}_k$$

7: Determine α_k such that the Armijo's condition is satisfied
8: Compute $\mathbf{u}^{k+1} = \mathbf{u}^k + \alpha_k \Delta_k$
9: **end while**
10: output: $\mathbf{u}^k$

3.2. Numerical experiments and method performance comparison

Following this, examples are provided to demonstrate the effectiveness of Algorithm 1 and Algorithm 2. We implemented the algorithms using MATLAB 8.0, and conducted the tests on a computer running Windows 10 Home Language, version 1803, equipped with an Intel(R) Core processor i7-7700U @CPU 2.80GHz, 16.00GB of RAM, and a 64-bit operating system.

For conducting the tests, we consider Example 1, along with the following problems:

EXAMPLE 2. We consider the example provided in [21], considering (2)–(3) with

$$f(s,u,v) = s + 20u^4 + 30v^4,$$

has at least one positive solution.

EXAMPLE 3. We consider the example provided in [21], considering (2)–(3):

$$f(s,u,v) = u^2 + v^2$$

The analytical solution of (2)–(3) is $u^*(s) = 0$.

EXAMPLE 4. We consider the example provided in [20], considering (2)–(3):

$$f(s,u,v) = -(32400s(s-1)^2 + 14400(s-1)^3 + 6480s^2(2s-2) + 720s^3);$$

The analytical solution of (2)–(3) is $u^*(s) = s^3(1-s)^6$.

EXAMPLE 5. We consider the example provided in [20], considering (2)–(3):

$$f(t,u,v) = \begin{cases} 6e^s + 6561 + 5\frac{(u-2a)^2}{a}, & u \geqslant 2a \\ 6e^s + \left(\frac{9u}{2a}\right)^4, & u < 2a \end{cases}$$

Choosing the constants $d = 10$, $a = 1$, has at least three positive solution.

We applied Algorithms 1 and 2 to Examples 1, 2, 3, 4, and 5, considering uniformly distributed mesh s_j in $[0,1]$, $j = 1,\ldots,n$, $n = 20$, and maximum number of iteration for algorithms is 50. In Table 1, we summarize the results obtained, ε^k denotes $\|u^{k+1} - u^k\|_\infty$. Additionally, "It" denotes "iteration" and "Time" denotes the processing time to achieve convergence in seconds.

Example	Algortithm	Convergence	It	Time	ε^k
1	1 (FP)	Yes	11	0.131045	0.90758×10^{-6}
1	2 (GN)	No	50	2.158712	0.426053
2	1 (FP)	Yes	4	0.070846	5.60207×10^{-16}
2	2 (GN)	Yes	8	1.510303	2.76105×10^{-8}
3	1 (FP)	Yes	3	0.062347	9.59504×10^{-9}
3	2 (GN)	Yes	5	1.471715	9.77773×10^{-9}
4	1 (FP)	Yes	4	0.057821	3.44169×10^{-8}
4	2 (GN)	Yes	1	0.450391	4.14672×10^{-7}
5	1 (FP)	Yes	4	0.066197	6.72933×10^{-11}
5	2 (GN)	No	50	1.765164	0.352691

Table 1: *Algorithms 1 and 2 considering Examples 1, 2, 3, 4 and 5.*

In Table 1, it can be observed that the method based on the contraction principle, Algorithm 1, achieved better performance in terms of processing time (this performance was expected due to the lower complexity of the method, which has lower computational cost per iteration compared to nonlinear programming methods). Additionally, it is noteworthy that the method demonstrated greater robustness, achieving convergence in all tested problems. In contrast, the method of Algorithm 2 failed in Examples 1 and 5, unable to approach convergence. In Examples 2, 3, and 4, both methods converged and achieved comparable levels of accuracy.

4. Final remarks

In summary, this study addressed problem (2), (3) under broader conditions than most literature references. We explored the existence of multiple solutions by analyzing whether the function f satisfies specific conditions using the Avery-Peterson theorem. Additionally, we introduced two numerical solution approaches: one employing the contraction principle (Algorithm 1) and the other based on discretizing the problem with the Gauss-Newton method (Algorithm 2). Moreover, we demonstrated the convergence conditions for the iterative sequence $u^{k+1} = Tu^k$ in Algorithm 1 through the contraction principle. Through a comparative analysis using five non-trivial examples, we illustrated the feasibility of the proposed methods. Notably, Algorithm 1 displayed superior robustness and significantly faster convergence compared to Algorithm 2. Therefore, for the problem at hand, the results suggest that Algorithm 1 is more suitable.

REFERENCES

[1] M. ADJUSTOVS AND A. LEPINS, *Extremal solutions of a boundary value problem for a sixth-order equation*, Differential Equations, **50** (2), 141–146, 2014.

[2] R. P. AGARWAL, B. KOVACS AND D. O'REGAN, *Existence of positive solution for a sixth–order differential system with variable parameters*, Journal of Applied Mathematics and Computing, **44**, 437–454, 2014.

[3] R. P. AGARWAL, B. KOVACS AND D. O'REGAN, *Positive solutions for a sixth-order boundary value problem with four parameters*, Boundary Value Problems, **2013** (1), 1–22, 2013.

[4] MAGNUS ALVELID, *Sixth order differential equation for sandwich beam deflection including transverse shear*, Composite Structures, **102**, 29–37, 2013.

[5] ZOUAOUI BEKRI AND SLIMANE BENAICHA, *Positive solutions for boundary value problem of sixth-order elastic beam equation*, Open Journal of Mathematical Sciences, **4** (1), 9–17, 2020.

[6] GABRIELE BONANNO, PASQUALE CANDITO AND DONAL O'REGAN, *Existence of Nontrivial Solutions for Sixth-Order Differential Equations*, Mathematics, **9** (16), 2021.

[7] SUBRAHMANYAN CHANDRASEKHAR, *Hydrodynamic and hydromagnetic stability*, Courier Corporation, 2013.

[8] J. V. CHAPAROVA, L. A. PELETIER, S. A. TERSIAN FAZHAN GENG AND YONGGANG YE, *Existence and nonexistence of nontrivial solutions of semilinear sixth-order ordinary differential equations*, Appl. Math. Lett., **17** (10), 1207–1212, 2004.

[9] T. GARBUZA, *On solutions of one 6-th order nonlinear boundary value problem*, Mathematical Modelling and Analysis, **13** (3), 349–355, 2008.

[10] SUQUIN GE, WANYI WANG AND QIUXIA YANG, *Dependence of eigenvalues of sixth-order boundary value problems on the boundary*, Bull. Aust. Math. Soc., **90** (3), 457–468, 2014.

[11] K. GHANBARI AND H. MIRZAEI, *On the isospectral sixth order Sturm-Liouville equation*, J. Lie Theory, **23** (4), 921–935, 2013.

[12] GARY A. GLATZMAIER, *Numerical simulations of stellar convective dynamos III. At the base of the convection zone*, Geophysical & Astrophysical Fluid Dynamics, **31** (1–2), 137–150, 1985.

[13] J. R. GRAEF AND B. YANG, *Boundary value problems for sixth order nonlinear ordinary differential equations*, Dynam. Systems Appl., **10** (4), 465–475, 2001.

[14] T. GYULOV, *Trivial and nontrivial solutions of a boundary value problem for a sixth-order ordinary differential equation*, Comptes Rendus-Academie Bulgarie Des Science, **58** (9), 1013–1018, 2005.

[15] FAOUZI HADDOUCHI AND NOURREDINE HOUARI, *Solvability of a sixth-order boundary value problem with multi-point and multi-term integral boundary conditions*, Mathematical Methods in the Applied Sciences, **48** (3), 3400–3413, 2025.

[16] C. T. KELLEY, *Iterative methods for optimization*, SIAM, 1999.

[17] AMMAR KHANFER, LAZHAR BOUGOFFA AND NAWAL ALHELALI, *On the Sixth-Order Beam Equation of Small Deflection with Variable Parameters*, Mathematics, **13** (5), 2025.

[18] A. L. M. MARTINEZ, E. V. CASTELANI, C. A. P. MARTINEZ, G. M. BRESSAN AND R. M. SOUZA, *Multiple Solutions To A Third-Order Three-Point Nonhomogeneous Boundary Value Problem Aided By Nonlinear Programming Methods*, Differential Equations & Applications, **13** (1), 35–49, 2021.

[19] A. L. M. MARTINEZ, M. R. A. FERREIRA AND E. V. CASTELANI, *Theoretical and Numerical Aspects of a Third-order Three-point Nonhomogeneous Boundary Value Problem*, Trends in Applied and Computational Mathematics, **20** (3), 417–427, 2019.

[20] A. L. M. MARTINEZ, C. A. P. MARTINEZ, G. M. BRESSAN, R. MOLINA SOUZA AND E. W. STIEGELMEIER, *Multiple Solutions for a Sixth Order Boundary Value Problem*, Trends in Computational and Applied Mathematics, **22**, 2–12, 2021.

[21] A. L. M. MARTINEZ, C. A. PENDEZA MARTINEZ, T. S. PINTO AND E. V. CASTELANI, *Analysis of Solutions for a Sixth-Order Boundary Value Problem and Numerical Search using Nonlinear Programming Techniques*, Engineering Letters, **32** (3), 488–492, 2024.

[22] C. A. P. MARTINEZ, A. L. M. MARTINEZ, G. M. BRESSAN, E. V. CASTELANI AND R. M. SOUZA, *Multiple solutions for a fourth order equation with nonlinear boundary conditions: theoretical and numerical aspects*, Differential Equations & Applications, **11** (3), 335–348, 2019.

[23] M. MÖLLER AND B. ZINSOU, *Sixth order differential operators with eigenvalue dependent boundary conditions*, Applicable Analysis and Discrete Mathematics, 378–389, 2013.

[24] SAEID SHOKOOH, *Existence of solutions for a sixth-order nonlinear equation*, Rendiconti del Circolo Matematico di Palermo Series 2, **72** (8), 4251–4271, 2023.

[25] JUKI TOOMRE, J.-P. ZAHN, JEAN LATOUR AND EDWARD A. SPIEGEL, *Stellar convection theory. II. Single-mode study of the second convection zone in an A-type star*, Astrophysical Journal, **207**, 545–563, 1976.

[26] B. YANG, *Positive solutions to a nonlinear sixth order boundary value problem*, Differential Equations & Applications, **11** (2), 307–317, 2019.

(Received June 12, 2024)

André Luís M. Martinez
Academic Department of Mathematics
Universidade Tecnológica Federal do Paraná
Cornélio Procópio, Paraná, Brazil
e-mail: martinez@utfpr.edu.br

Emerson V. Castelani
Department of Mathematics
Universidade Estadual de Maringá
Maringá, Paraná, Brazil
e-mail: evcastelani@uem.br

Cristiane A. Pendeza Martinez
Academic Department of Mathematics
Universidade Tecnológica Federal do Paraná
Cornélio Procópio, Paraná, Brazil
e-mail: crismartinez@utfpr.edu.br

Carlos Alexandre Ribeiro Martins
Academic Department of Mathematics
Universidade Tecnológica Federal do Paraná
Pato Branco, Paraná, Brazil
e-mail: carlos@utfpr.edu.br