Differential
Equations
& Applications
Volume 17, Number 1 (2025), 35–52
doi:10.7153/dea-2025-17-03

EXISTENCE AND GLOBAL BEHAVIOR OF POSITIVE SOLUTIONS OF SEMILINEAR FRACTIONAL DIRICHLET PROBLEMS IN EXTERIOR DOMAINS

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(*Communicated by D. Kang*)

Abstract. In this paper, we establish the existence and the global asymptotic behavior of positive solutions in an exterior domain $\Omega \subset \mathbb{R}^d$, $d \geqslant 3$,

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} x = f(t)x^p, & \text{in } \Omega, \\ x > 0, & \text{in } \Omega, \\ \lim\limits_{t \to \partial\Omega} \delta(t)^{1-\frac{\alpha}{2}} x(t) = 0, & \\ \lim\limits_{|t| \to \infty} x(t) = 0, & \end{cases}$$

where $(-\Delta)^{\frac{\alpha}{2}}$ is the infinitesimal generator of a killed symmetric α-stable process X^Ω on Ω, $0 < \alpha < 2$, $p < 1$ and the function f is positive and satisfies the suitable conditions related to the Karamata classes $\mathscr{K}_0$ and $\mathscr{K}_\infty$. Our approach relies on potential theory, Karamata regular variation theory, and the Schauder fixed point theorem.

1. Introduction

Numerous studies have focused on investigating the existence of a solution for the following specific type of fractional differential equation

$$(-\Delta)^{\frac{\alpha}{2}} x = \varphi(., x), \text{ in } \Omega \tag{1.1}$$

where Ω is a bounded or unbounded domain of $\mathbb{R}^d$. More details can be found in [1, 2, 3, 6, 8, 12, 14, 19, 20] and their references.

For a bounded $C^{1,1}$-domain $\Omega \subset \mathbb{R}^d$, $d \geqslant 2$, Chemmam et al., in [8], studied the equation (1.1). They well applied fixed point arguments and exploited properties of the Green function $G_\Omega^\alpha(t,s)$ associated to $(-\Delta)^{\frac{\alpha}{2}}$ in Ω as well as functions of a Kato class (see [8, Definition 2]) to establish the existence of a positive solution x of (1.1) that satisfies

$$x(t) = \int_\Omega G_\Omega^\alpha(t,s)\varphi(s,x(s))\, ds. \tag{1.2}$$

Mathematics subject classification (2020): 34A08, 31B05, 34B27, 34K37, 35R11, 60G22.
Keywords and phrases: Dirichlet problem, fractional Laplacian, Green function, Karamata class, Kato class, Schauder fixed point theorem.

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In [9], Chemmam et al. investigated equation (1.1) for $\varphi(t,x) = f(t)x^p$, where $p < 1$ and f is a function satisfying certain assumptions related to the class $\mathscr{K}_0$, the so-called Karamata class defined as in the Definition 3. The authors demonstrated that equation (1.1) possesses a positive continuous solution x in Ω that satisfies the condition $\lim_{t\to\partial\Omega} \delta(t)^{1-\frac{\alpha}{2}}x(t) = 0$. Here and always, $\delta(t)$ denotes the Euclidean distance between t and the boundary $\partial\Omega$.

Moreover, Chemmam et al., in [6], tackled the equation (1.1) in an exterior domain $\Omega \subset \mathbb{R}^d$, $d \geqslant 3$, with the following boundary conditions

$$\lim_{t\to\partial\Omega} \delta(t)^{1-\frac{\alpha}{2}}x(t) = 0 \text{ and } \lim_{|t|\to\infty} x(t) = 0.$$

The nonlinearity $\varphi(., x)$ satisfies some condition related to a new functional $\mathbf{K}^{\alpha}_{\infty}(\Omega)$, (see definition 2). Thanks to the Kelvin transform (see [5]), the authors in [6], gave precise estimates of a Green's function $G^{\alpha}_{\Omega}(t,s)$ associated to $(-\Delta)^{\frac{\alpha}{2}}$, which enabled them to introduce the class $\mathbf{K}^{\alpha}_{\infty}(\Omega)$. Then, using a fixed point theorem, they proved the existence, uniqueness and asymptotic behavior of a positive classical solution x in Ω, defined by

$$x(t) = \int_{\Omega} G^{\alpha}_{\Omega}(t,s)\varphi(s,v(s))\, ds.$$

Significant progress has been made in unbounded domains regarding the case $\alpha = 2$. For instance, Mâagli et al. have used in [15] the sub-super solution method and potential theory tools in their work [15] to investigate the existence of a positive solution on Ω, which is the outside of unit ball, for the differential equation $(-\Delta)x = f(t)x^p$, conditional on Dirichlet boundary conditions. Here, $p < 1$ and f satisfies certain assumptions associated with Karamata classes $\mathscr{K}_0$ and $\mathscr{K}_\infty$ (see Definition 4). Recently, in more general domains not necessarily radial, Mâagli et al. in [16] studied the following problem

$$\begin{cases} (-\Delta)x = f(t)x^p, & \text{in } \Omega, \\ x > 0, & \text{in } \Omega, \\ \lim\limits_{|t|\to\infty} x(t) = 0, & \\ \lim\limits_{t\to\partial\Omega} x(t) = 0, & \end{cases} \tag{1.3}$$

where $p < 1$ and Ω is an unbounded regular domain in $\mathbb{R}^d$ $(d \geqslant 3)$, with compact boundary. The function f needs to satisfy a specific condition that is relevant to classes $\mathscr{K}_0$ and $\mathscr{K}_\infty$. They used the sub-super method to establish the existence, uniqueness, and asymptotic behavior of a positive classical solution for the problem (1.3). More recently, in [20], the authors' attention was given to the following singular fractional problem

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}}x = f(t)x^p, & p < 1, \text{ in } \Omega\setminus\{0\}, \\ x > 0, & \text{in } \Omega\setminus\{0\}, \\ \lim\limits_{|t|\to 0} |t|^{\alpha-d}x(t) = 0, & \\ \lim\limits_{t\to\partial\Omega} \delta(t)^{1-\frac{\alpha}{2}}x(t) = 0, & \end{cases}$$

where Ω is a $C^{1,1}$-bounded domain in $\mathbb{R}^d$ containing zero with $d \geqslant 3$. The weight function $f(t)$ fulfills suitable conditions associated with the Karamata class $\mathscr{K}_0$. The authors employed Karamata's theory and Schauder's fixed point theorem to establish the existence of a continuous solution to the above problem.

Motivated by the works mentioned above, the main objective of this article is to investigate the existence and global asymptotic behavior of a positive classical solution for the following fractional Dirichlet problem in an exterior $C^{1,1}$-domain Ω, for $0 < \alpha < 2$,

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} x = f(t)x^p, & p < 1, \text{ in } \Omega, \\ x > 0, & \text{in } \Omega, \\ \lim\limits_{t \to \partial\Omega} \delta(t)^{1-\frac{\alpha}{2}} x(t) = 0, \\ \lim\limits_{|t| \to \infty} x(t) = 0. \end{cases} \tag{1.4}$$

This result extends the findings of [16] in the elliptic case $(\alpha = 2)$, revealing notable differences from the elliptic situation. It should be noted that the techniques of proofs provided by Chemmem et al. in [6] can be effectively applied to several proofs presented here. Furthermore, we remark that the special case of radial domains is treated in this work, particularly the outside of the ball. Here and always, a $C^{1,1}$-exterior domain Ω in $\mathbb{R}^d$ $(d \geqslant 3)$ means that $\bar{\Omega}^c = \cup_{1 \leqslant i \leqslant k} \Omega_i$ where Ω_i is a bounded $C^{1,1}$-domain of $\mathbb{R}^d$ and the intersection between any two domains, Ω_i and Ω_j, is empty when $i \neq j$. Then, the fractional power $(-\Delta)^{\frac{\alpha}{2}}$ is the infinitesimal generator of the following killed symmetric α-stable process

$$X_s^{\Omega}(w) = \begin{cases} X_s(w), & s < \tau_s \\ \delta, & s \geqslant \tau_s, \end{cases}$$

where $\tau_s := \inf\{s > 0;\ X_s \notin \Omega\}$ and δ is the cemetery point. We refer to [4, 6, 18] and the references therein for more details. For $t \in \Omega$, $g_1(t) \approx g_2(t)$ means that for two nonnegative functions g_1 and g_2 defined on a set S there exists $c > 0$ such that $\frac{1}{c} g_2(t) \leqslant g_1(t) \leqslant c g_2(t)$. The notation, $\mathscr{B}(\Omega)$ is the set of Borel measurable functions in Ω and $\mathscr{B}^+(\Omega)$ is the set of nonnegative ones. Furthermore, $\mathscr{C}(\Omega)$ is the set of continuous functions in Ω and $\mathscr{C}_0(\Omega) := \{x \in \mathscr{C}(\Omega), \lim\limits_{t \to \partial\Omega} x(t) = \lim\limits_{|t| \to \infty} x(t) = 0\}$ with its uniform norm $||x||_\infty := \sup_{t \in \Omega} |x(t)|$. Note that the letter c denotes a generic positive constant that can vary from one line to another.

Now, for $t_0 \in \bar{\Omega}^c$ and $\bar{B}(t_0,\ r) \subset \bar{\Omega}^c$, $r > 0$, we have, from [11],

$$G_{\Omega}^{\alpha}(t,s) = r^{\alpha-d} G_{\frac{\Omega - t_0}{r}}^{\alpha} \left(\frac{t - t_0}{r}, \frac{s - t_0}{r} \right),\ t,\ s \in \Omega.$$

Throughout this work, we assume, without loss of generality, that $t_0 = 0$ and $r = 1$.

Let $t^* = \frac{t}{|t|^2}$ be the Kelvin transformation from Ω onto $\Omega^* = \{t^* \in B(0,1) : t \in \Omega\}$ (as mentioned in [5]). Applying this transformation, we have the following relation

$$G_{\Omega}^{\alpha}(t,s) = |t|^{\alpha-d} |s|^{\alpha-d} G_{\Omega^*}^{\alpha}(t^*, s^*), \text{ for any } t,\ s \in \Omega, \tag{1.5}$$

where $G_{\Omega}^{\alpha}(t,s)$ is the Green's function associated to $(-\Delta)^{\frac{\alpha}{2}}$ in Ω and $G_{\Omega^*}^{\alpha}(t^*,s^*)$ is the one in the bounded domain Ω^* (see for instance [6, 8]). Furthermore, let $g \in \mathscr{B}^+(\Omega)$. We define the α-order Kelvin transform of g in Ω^* as g^*, given by the following

$$g^*(t^*) = |t|^{d-\alpha} g(t).$$

Then, for all $t \in \Omega$, we obtain

$$\rho(t) = \frac{\delta(t)}{1+\delta(t)} \text{ and } 1+\delta(t) \approx |t|, \tag{1.6}$$

and

$$\delta(t^*) \approx \rho(t) \approx \frac{\delta(t)}{|t|}, \tag{1.7}$$

where $\delta(t^*)$ be the Euclidean distance between $t^* \in \Omega^*$ and the boundary $\partial\Omega^*$.

Now, let us define the potential kernel Vg on $\mathscr{B}^+(\Omega)$ by

$$Vg(t) = \int_{\Omega} G_{\Omega}^{\alpha}(t,s) g(s)\, ds.$$

From [6], we have $Vg \neq \infty \Longleftrightarrow \int_{\Omega} \frac{\rho(t)^{\frac{\alpha}{2}}}{(1+|t|)^{d-\alpha}} g(t)\, dt < \infty$. Hence, for any $g \in \mathscr{B}^+(\Omega)$ and $\psi \in \mathscr{C}_c^{\infty}(\Omega)$, such that $Vg \neq \infty$, we obtain

$$\int_{\Omega} g(t)(-\Delta)^{\frac{\alpha}{2}} \psi(t)\, dt = \int_{\Omega} Vg(t)\psi(t)\, dt.$$

That is, in the distributional sense

$$(-\Delta)^{\frac{\alpha}{2}} Vg = g \text{ in } \Omega \tag{1.8}$$

Below, we review the definition of super-harmonic functions associated with the killed symmetric α-stable process (X^{Ω}), (see for instance [6]).

DEFINITION 1. Let g be a locally integrable function defined on Ω, taking values in $(-\infty,+\infty]$, and satisfying the condition $\int_{(|t|>1)\cap\Omega} |g(t)||t|^{\alpha-d} dt < \infty$ for $0 < \alpha < 2$. We say that g is α-superharmonic with respect to X^{Ω} if it is lower semicontinuous in Ω and for every open set S such that $\overline{S} \subset \Omega$, the following condition holds

$$E^t[|g(X_{\tau_r}^{\Omega})|] < \infty \text{ and } g(t) \geqslant E^t[|g(X_{\tau_r}^{\Omega})|], \text{ for } t \in S.$$

EXAMPLE 1. The functions $t \mapsto \int_{\Omega} G_{\Omega}^{\alpha}(t,s)g(s)ds$ for any $g \in \mathscr{B}^+(\Omega)$, $t \mapsto \rho(t)^{\frac{\alpha}{2}-1}$ and $t \mapsto G_{\Omega}^{\alpha}(t,s)$ are α-superharmonic with respect to X^{Ω}. From [13], if a function g is α-superharmonic with respect to X^{Ω}, it implies that its α-order Kelvin transform g^* is also α-superharmonic with respect to X^{Ω^*}.

Throughout this paper, we assume that the function f satisfies the following hypothesis.

$(\mathscr{H})$: f is a positive function in $\mathscr{C}^{\gamma}_{loc}(\Omega)$, $0<\gamma<1$ satisfying for $t\in\Omega$,

$$f(t)\approx(\rho(t))^{-\lambda}L(\rho(t))|t|^{-\xi}K(|t|), \tag{1.9}$$

where $\lambda<\alpha$, $\xi>\alpha+(2-\alpha)(1-p)$, $L\in\mathscr{K}_0$ defined on $(0,\eta]$, $\eta>1$ and $K\in\mathscr{K}_\infty$.

Here, we remark that we have

$$\int_0^\eta u^{\alpha-1-\lambda}L(u)\,du<\infty,\quad \int_1^\infty u^{\alpha-1-\xi+(2-\alpha)(1-p)}K(u)\,du<\infty. \tag{1.10}$$

We define the function Θ on Ω by

$$\Theta(t)=\rho(t)^{\min\left(\frac{\alpha}{2},\,\frac{\alpha-\lambda}{1-p}\right)}\varphi_{L,\lambda,p}(\rho(t))|t|^{\min\left(\alpha-d,\,\frac{\alpha-\xi}{1-p}\right)}\phi_{K,\xi,p}(|t|), \tag{1.11}$$

where $\varphi_{L,\lambda,p}$ defined on $(0,\eta]$, $\eta>1$, by

$$\varphi_{L,\lambda,p}(u)=\begin{cases}1, & \text{if } \lambda<\frac{\alpha}{2}(1+p),\\ \left(\int_u^\eta\frac{L(s)}{s}\,ds\right)^{\frac{1}{1-p}}, & \text{if } \lambda=\frac{\alpha}{2}(1+p),\\ (L(u))^{\frac{1}{1-p}}, & \text{if } \frac{\alpha}{2}(1+p)<\lambda<\alpha,\end{cases}$$

and $\phi_{K,\xi,p}$ is defined on $[1,\infty)$ as follows:

$$\phi_{K,\xi,p}(u)=\begin{cases}1, & \text{if } \xi>d-p(d-\alpha),\\ \left(\int_1^{1+u}\frac{K(s)}{s}\,ds\right)^{\frac{1}{1-p}}, & \text{if } \xi=d-p(d-\alpha),\\ (K(u))^{\frac{1}{1-p}}, & \text{if } \alpha+(2-\alpha)(1-p)<\xi<d-p(d-\alpha).\end{cases}$$

Let us, now, introduce our main result.

THEOREM 1. *Let $p<1$. Suppose that the function f satisfies the hypothesis $(\mathscr{H})$. Then the problem (1.4) has at least one positive continuous solution x on Ω satisfying for $c>0$,*

$$\frac{1}{c}\Theta(t)\leqslant x(t)\leqslant c\Theta(t),\ t\in\Omega. \tag{1.12}$$

We end this section with the outline of the paper. The next section will state already-known results for functions in the Karamata classes $\mathscr{K}_0$ and $\mathscr{K}_\infty$. Some results for the Kato class $\mathbf{K}^{\alpha}_{\infty}(\Omega)$ are also obtained. Then, in Section 3, we will focus on providing estimates on some potential functions. The last section will be devoted to the proof of the Theorem 1 and we end with illustrative examples.

2. Preliminaries and key tools

2.1. On the Kato class $\mathbf{K}_\infty^\alpha(\Omega)$

In this paragraph, we give some results concerning functions belonging to the Kato class $\mathbf{K}_\infty^\alpha(\Omega)$, for more details see [6].

DEFINITION 2. Let $q \in \mathscr{B}^+(\Omega)$. Then q is in the Kato class $\mathbf{K}_\infty^\alpha(\Omega)$ if q satisfies the following conditions

$$\lim_{r\to 0}\sup_{t\in\Omega}\int_{\Omega\cap B(t,r)}\left(\frac{\rho(s)}{\rho(t)}\right)^{\frac{\alpha}{2}} G_\Omega^\alpha(t,s)q(s)\,ds = 0,$$

and

$$\lim_{M\to\infty}\sup_{t\in\Omega}\int_{\Omega\cap(|s|\geqslant M)}\left(\frac{\rho(s)}{\rho(t)}\right)^{\frac{\alpha}{2}} G_\Omega^\alpha(t,s)q(s)\,ds = 0.$$

Let us recall the following result stated in [6, Proposition 4.5].

PROPOSITION 1. *Let $q \in \mathbf{K}_\infty^\alpha(\Omega)$. Then for any α-superharmonic function h and $t \in \Omega$, we have for $t_0 \in \overline{\Omega}$*

$$\lim_{r\to 0}\sup_{t\in\Omega}\frac{1}{h(t)}\int_{\Omega\cap B(t_0,r)} G_\Omega^\alpha(t,s)h(s)q(s)\,ds = 0,$$

and

$$\lim_{M\to\infty}\sup_{t\in\Omega}\frac{1}{h(t)}\int_{\Omega\cap(|s|\geqslant M)} G_\Omega^\alpha(t,s)h(s)q(s)\,ds = 0.$$

Next, we give the following theorem proven in [6, Theorem 4.9].

THEOREM 2. *Let $q \in \mathbf{K}_\infty^\alpha(\Omega)$ be a nonnegative function. Then the following family of functions defined in Ω by*

$$\Gamma = \{t \mapsto J(g)(t) := \int_\Omega\left(\frac{\rho(s)}{\rho(t)}\right)^{\frac{\alpha}{2}-1} G_\Omega^\alpha(t,s)g(s)ds:\ g \in \mathbf{K}_\infty^\alpha(\Omega),\ |g| \leqslant q\}$$

is equicontinuous and uniformly bounded in $\bar{\Omega}\cup\{\infty\}$. Furthermore, the set Γ is relatively compact in $\mathscr{C}_0(\Omega)$.

2.2. Properties of the Karamata classes $\mathscr{K}_0$ and $\mathscr{K}_\infty$

In this paragraph, we present several essential properties of Karamata functions that will be employed in subsequent sections.

2.2.1. On the Karamata class $\mathcal{K}_0$

DEFINITION 3. The function L defined on $(0,\eta]$, $\eta>0$, belongs to the Karamata class $\mathcal{K}_0$ if

$$L(u) := c\exp\left(\int_u^\eta \frac{y(s)}{s}\,ds\right),$$

where $c>0$ and y be the continuous function on $[0,\eta]$ with $y(0)=0$.

EXAMPLE 2. Let $u\in(0,\ \eta)$. Then

$$L(u)=\prod_{k=1}^{n}\left(\ln_k\left(\frac{2\eta}{u}\right)\right)^{\mu_k}\in\mathcal{K}_0,$$

where $\ln_k(t)=\ln o\ln o\ldots\ln(t)$ (k times), $\mu_k\in\mathbb{R}$ and $n\geqslant 1$.

LEMMA 1. [7, 21] *Let $L_1,\ L_2\in\mathcal{K}_0$ and $m\in\mathbb{R}$. Then we have the following:*
(i) *$L_1^m,\ L_1L_2$ and L_1+L_2 belongs to $\mathcal{K}_0$;*
(ii) *Let $L\in\mathcal{K}_0$ and $\varepsilon>0$, then $\lim\limits_{s\to0^+} s^{\varepsilon}L=0$ and $\lim\limits_{s\to0^+} s^{-\varepsilon}L=\infty$.*

LEMMA 2. [16] *If $L\in\mathcal{K}_0$ defined on $(0,\ \eta]$, $\eta>1$, and $m_1,\ m_2\in(0,\ 1)$, $c\geqslant 1$ such that $\frac{1}{c}m_2\leqslant m_1\leqslant cm_2$. Then, there exists $w\geqslant 0$ such that*

$$c^{-w}L(m_2)\leqslant L(m_1)\leqslant c^{w}L(m_2).$$

LEMMA 3. [7, 17, 21] *Let $\beta\in\mathbb{R}$ and $L_1\in\mathcal{K}_0$ defined on $(0,\ \eta]$, $\eta>1$. Then,*
(i) *If $\beta>-1$, then $\int_0^\eta t^\beta L_1(t)\,dt$ converges and $\int_0^t s^\beta L_1(s)\,ds\approx\frac{t^{\beta+1}L_1(t)}{\beta+1}$ as $t\to0^+$;*
(ii) *If $\beta<-1$, then $\int_0^\eta t^\beta L_1(t)\,dt$ diverges and $\int_t^\eta s^\beta L_1(s)\,ds\approx-\frac{t^{\beta+1}L_1(t)}{\beta+1}$ as $t\to0^+$.*

LEMMA 4. [7, 21] *Let $L\in\mathcal{K}_0$ defined on $(0,\ \eta]$, $\eta>1$. Then we have the following assertions:*
(i) *$\lim\limits_{t\to0^+}\frac{L(t)}{\int_t^\eta\frac{L(s)}{s}\,ds}=0$ and $t\mapsto\int_t^\eta\frac{L(s)}{s}\,ds\in\mathcal{K}_0$;*
(ii) *If $\int_0^\eta\frac{L(s)}{s}\,ds$ converges, then $\lim\limits_{t\to0^+}\frac{L(t)}{\int_0^t\frac{L(s)}{s}\,ds}=0$ and $t\mapsto\int_0^t\frac{L(s)}{s}\,ds\in\mathcal{K}_0$.*

Next, we state a key lemma for the proof of our result.

LEMMA 5. [20] *Let $\Omega^*\subset\mathbb{R}^d$, $d\geqslant3$, be a bounded $C^{1,1}$-domain. If $L_1,\ L_2\in\mathcal{K}_0$, and $\lambda_1,\ \lambda_2\in\mathbb{R}$. Then, for $t,\ s\in\Omega^*$, the following statements are equivalent.*
(i)

$$\lim_{r\to0}\sup_{t\in\Omega^*}\int_{\Omega^*\cap B(t,r)} G^{\alpha}_{\Omega^*}(t,s)|s|^{-\lambda_1}L_1(|s|)\delta_{\Omega^*}(s)^{-\lambda_2}L_2(\delta_{\Omega^*}(s))\,ds=0, \tag{2.1}$$

here $G^{\alpha}_{\Omega^*}(t,s)$ is the Green function of the fractional Laplacian with respect to X^{Ω^*}.
(ii) *$\lambda_i<\alpha$ or $\lambda_i=\alpha$ with $\int_0^\eta\frac{L_i(r)}{r}\,dr<\infty$, $i\in\{1,2\}$.*

2.2.2. On the Karamata class $\mathcal{K}_\infty$

DEFINITION 4. The function k, defined on $[1,\infty)$, belongs to the Karamata class $\mathcal{K}_\infty$ if

$$k(u) := c\exp\int_1^u \frac{y(s)}{s}\,ds,$$

where $c>0$ and y is a continuous function on $[1,\infty)$ such that $\lim_{s\to\infty} y(s)=0$.

EXAMPLE 3. Let $u\in[1,\infty)$. Then, we have

$$L(u)=\prod_{k=1}^{n}(\ln_k(wu))^{\mu_k}\in\mathcal{K}_\infty,$$

where $w>0$ sufficiently large number, $\mu_k\in\mathbb{R}$ and $n\geqslant 1$.

LEMMA 6. [16, 21] *Let L_1, $L_2\in\mathcal{K}_\infty$, $\eta>1$ and $m\in\mathbb{R}$. Then, we have*

(i) *L_1^m, L_1L_2 and L_1+L_2 belongs to $\mathcal{K}_\infty$.*

(ii) *Let $L\in\mathcal{K}_\infty$ and $\varepsilon>0$, then $\lim_{s\to\infty} s^{-\varepsilon}L(s)=0$ and $\lim_{s\to\infty} s^{\varepsilon}L(s)=\infty$.*

LEMMA 7. [16, 17, 21] *Let $\beta\in\mathbb{R}$ and $L_2\in\mathcal{K}_\infty$. Then we have the following.*

(i) *If $\beta<-1$, then $\int_1^\infty s^\beta L_2(s)\,ds$ converges and $\int_t^\infty s^\beta L_2(s)\,ds\approx-\frac{t^{\beta+1}L_2(t)}{\beta+1}$ as $t\to\infty$.*

(ii) *If $\beta>-1$, then $\int_1^\infty s^\beta L_2(s)\,ds$ diverges and $\int_1^t s^\beta L_2(s)\,dt\approx\frac{t^{\beta+1}L_2(t)}{\beta+1}$ as $t\to\infty$.*

LEMMA 8. [10] *Let $L\in\mathcal{K}_\infty$. Then*

(i) *$\lim_{t\to\infty}\frac{L(t)}{\int_1^t\frac{L(s)}{s}\,ds}=0$ and $t\mapsto\int_1^{t+1}\frac{L(s)}{s}\,ds\in\mathcal{K}_\infty$.*

(ii) *If $\int_1^\infty\frac{L(s)}{s}\,ds$ converges, then $\lim_{t\to\infty}\frac{L(t)}{\int_t^\infty\frac{L(s)}{s}\,ds}=0$ and $t\mapsto\int_t^\infty\frac{L(s)}{s}\,ds\in\mathcal{K}_\infty$.*

(iii) *There exists $m\geqslant 0$ such that for every $e>0$ and $u\geqslant 1$, we have*

$$(1+e)^{-m}L(u)\leqslant L(e+u)\leqslant(1+e)^{m}L(u).$$

As a relation between the Karamata classes $\mathcal{K}_0$ and $\mathcal{K}_\infty$, we have the following remark due to [16].

REMARK 1. The function $u\mapsto f(u)$ belongs to $\mathcal{K}_\infty$ if and only if $u\mapsto f(\frac{1}{u})\in\mathcal{K}_0$, $u\in(0,1]$.

3. Asymptotic behavior of potential functions

In this section, we give estimates on the potential function $V(f\Theta^p)$, where f is a function satisfying $(\mathscr{H})$ and Θ is the function given in (1.11). First, let us recall the following lemma due to Salah et al. [20].

LEMMA 9. *Let $\Omega^* \subset \mathbb{R}^d$ be a bounded $C^{1,1}$-domain containing 0, $d \geqslant 3$. Let $\gamma < d-2+\alpha$, $\mu < \alpha$ and L_1, $L_2 \in \mathscr{K}_0$ with $\eta > diam(\Omega^*)$. Suppose*

$$b(t) = \delta(t)^{-\mu} L_1(\delta(t))|t|^{-\gamma} L_2(|t|),\ t \in \Omega^* \setminus \{0\}.$$

Then, for $t \in \Omega^* \setminus \{0\}$, we have

$$Vb(t) \approx \delta(t)^{\min(\frac{\alpha}{2},\alpha-\mu)} \tilde{L_1}(\delta(t))|t|^{\min(0,\alpha-\gamma)} \tilde{L_2}(|t|),$$

here $\tilde{L_1}$ and $\tilde{L_2}$ are defined on $(0,\eta)$ by

$$\tilde{L_1}(u) := \begin{cases} 1, & \text{if } \mu < \frac{\alpha}{2}, \\ \int_u^\eta \frac{L_1(s)}{s}\,ds, & \text{if } \mu = \frac{\alpha}{2}, \\ L_1(u), & \text{if } \frac{\alpha}{2} < \mu < \alpha, \end{cases}$$

and

$$\tilde{L_2}(u) := \begin{cases} 1, & \text{if } \gamma < \alpha, \\ \int_u^\eta \frac{L_2(s)}{s}\,ds, & \text{if } \gamma = \alpha, \\ L_2(u), & \text{if } \alpha < \gamma < d-2+\alpha. \end{cases}$$

PROPOSITION 2. *Let g be a positive continuous function satisfies, for all $t \in \Omega$,*

$$g(t) \approx (\rho(t))^{-\beta_1} M_1(\rho(t))|t|^{-\beta_2} M_2(|t|), \tag{3.1}$$

where $\beta_1 < \alpha$, $\beta_2 > 2$, $M_1 \in \mathscr{K}_0$ and $M_2 \in \mathscr{K}_\infty$. Then, for all $t \in \Omega$, we have

$$Vg(t) \approx \rho(t)^{\min(\frac{\alpha}{2},\ \alpha-\beta_1)} \tilde{M}_1(\rho(t))|t|^{\min(\alpha-d,\alpha-\beta_2)} \tilde{M}_2(|t|),$$

where

$$\tilde{M}_1(u) := \begin{cases} 1, & \text{if } \beta_1 < \alpha - 1, \\ \int_u^\eta \frac{M_1(s)}{s}\,ds, & \text{if } \beta_1 = \alpha - 1, \\ M_1(u), & \text{if } \alpha - 1 < \beta_1 < \alpha, \end{cases}$$

and

$$\tilde{M}_2(u) := \begin{cases} 1, & \text{if } \beta_2 > d, \\ \int_1^{u+1} \frac{M_2(s)}{s}\,ds, & \text{if } \beta_2 = d, \\ M_2(u), & \text{if } 2 < \beta_2 < d. \end{cases}$$

Proof. By (3.1), we have, for $t \in \Omega$,

$$Vg(t) \approx \int_{\Omega} G^{\alpha}_{\Omega}(t,s)\rho(s)^{-\beta_1} M_1(\rho(s))|s|^{-\beta_2} M_2(|s|)\, ds.$$

From (1.7) and Lemma 2 we get, for $t \in \Omega$,

$$M_1(\rho(t)) \approx M_1(\delta_{\Omega^*}(t^*)). \tag{3.2}$$

Therefore, by (1.5), (1.7) and (3.2), we obtain

$$Vg(t) \approx |t^*|^{d-\alpha} \int_{\Omega^*} G^{\alpha}_{\Omega^*}(t^*,s^*)(\delta(s^*))^{-\beta_1} M_1(\delta(s^*))|s^*|^{\beta_2-d-\alpha} M_2\left(\frac{1}{|s^*|}\right) ds^*.$$

Let

$$\mu = \beta_1,\ \ \gamma = d+\alpha-\beta_2,\ \ L_1(t) = M_1(t) \ \text{ and } \ L_2(t) = M_2\left(\frac{1}{t}\right).$$

Since $\beta_2 > 2$ then $\gamma < d+\alpha-2$. By Remark 1, the function $t \mapsto M_2(\frac{1}{t})$ belongs to $\mathscr{K}_0$. So, from Lemma 9, we obtain

$$Vg(t) \approx \delta(t^*)^{\min(\frac{\alpha}{2},\ \alpha-\beta_1)} \tilde{M_1}(\delta(t^*))|t^*|^{d-\alpha+\min(0,\beta_2-d)} \tilde{L_2}(|t^*|),$$

where for all $u \in (0,1]$,

$$\tilde{M_1}(u) := \begin{cases} 1, & \text{if } \beta_1 < \frac{\alpha}{2}, \\ \int_u^{\eta} \frac{M_1(s)}{s}\, ds, & \text{if } \beta_1 = \frac{\alpha}{2}, \\ M_1(u), & \text{if } \frac{\alpha}{2} < \beta_1 < \alpha, \end{cases} \quad \text{and } \tilde{L_2}(u) := \begin{cases} 1, & \text{if } \beta_2 > d, \\ \int_u^{\eta} \frac{M_2(\frac{1}{s})}{s}\, ds, & \text{if } \beta_2 = d, \\ M_2(\frac{1}{u}), & \text{if } 2 < \beta_2 < d. \end{cases} \tag{3.3}$$

By (1.7), for $t \in \Omega$, we have

$$\delta(t^*)^{\min(\frac{\alpha}{2},\ \alpha-\beta_1)} \tilde{M_1}(\delta(t^*)) \approx \rho(t)^{\min(\frac{\alpha}{2},\ \alpha-\beta_1)} \tilde{M_1}(\rho(t)). \tag{3.4}$$

Since $\tilde{M_2}(\frac{1}{u}) = \tilde{L_2}(u)$ for $u \in (0,\ 1]$. Then, by (3.3), we obtain for $u \in [1,\infty)$

$$\tilde{M_2}(u) := \tilde{L_2}(\frac{1}{u}) = \begin{cases} 1, & \text{if } \beta_2 > d, \\ \int_{\frac{1}{u}}^{\eta} \frac{M_2(\frac{1}{s})}{s}\, ds, & \text{if } \beta_2 = d, \\ M_2(u), & \text{if } 2 < \beta_2 < d. \end{cases}$$

Therefore, using Lemma (8), we obtain

$$\tilde{M_2}(u) := \begin{cases} 1, & \text{if } \beta_2 > d, \\ \int_{\frac{1}{\eta}}^{u} \frac{M_2(r)}{r}\, dr, & \text{if } \beta_2 = d, \\ M_2(u), & \text{if } 2 < \beta_2 < d, \end{cases} \implies \tilde{M_2}(u) \approx \begin{cases} 1, & \text{if } \beta_2 > d, \\ \int_{1}^{u+1} \frac{M_2(r)}{r}\, dr, & \text{if } \beta_2 = d, \\ M_2(u), & \text{if } 2 < \beta_2 < d. \end{cases}$$

Moreover, since $|t^*| = \frac{1}{|t|}$, then

$$|t^*|^{d-\alpha+\min(0,\beta_2-d)}\tilde{M}_2\left(\frac{1}{|t^*|}\right) = |t|^{\alpha-d-\min(0,\beta_2-d)}\tilde{M}_2(|t|). \tag{3.5}$$

Using the fact that

$$d-\alpha+\min(0,\beta_2-d) = \min(\alpha-d,\alpha-\beta_2),$$

and by (3.4) and (3.5), we obtain

$$Vg(t) \approx \rho(t)^{\min(\frac{\alpha}{2},\ \alpha-\beta_1)}\tilde{M}_1(\rho(t))|t|^{\min(\alpha-d,\alpha-\beta_2)}\tilde{M}_2(|t|),\ t \in \Omega.$$

This finished the proof. $\square$

The proposition presented below holds a crucial role in this article.

PROPOSITION 3. *Assume that f is a function that satisfies $(\mathscr{H})$, and let Θ be the function given by equation* (1.11). *Then*

$$V(f\Theta^p)(t) \approx \Theta(t),\ t \in \Omega.$$

Proof. From the hypothesis $(\mathscr{H})$ and (1.11) we have, for $t \in \Omega$,

$$f(t)\Theta(t)^p \approx \rho(t)^{-\lambda+p\min(\frac{\alpha}{2},\ \frac{\alpha-\lambda}{1-p})}(L\varphi^p_{L,\lambda,p})(\rho(t))|t|^{-\xi-p\min\left(d-\alpha,\ \frac{\xi-\alpha}{1-p}\right)}(K\phi^p_{K,\xi,p})(|t|).$$

Assume that

$$\beta_1 = \lambda - p\min\left(\frac{\alpha}{2},\ \frac{\alpha-\lambda}{1-p}\right),\ \ \beta_2 = \xi + p\min\left(d-\alpha,\ \frac{\xi-\alpha}{1-p}\right),$$

$$M_1(t) = (L\varphi^p_{L,\lambda,p})(t) \ \text{ and } M_2(t) = (K\phi^p_{K,\xi,p})(t).$$

Then

$$f(t)\Theta(t)^p \approx \rho(t)^{-\beta_1}M_1(\rho(t))|t|^{-\beta_2}M_2(|t|).$$

By Lemma 1, Lemma 4 and the hypothesis $(\mathscr{H})$, we have $M_1 \in \mathscr{K}_0$. Moreover, using Lemma 6 and Lemma 8, we obtain $M_2 \in \mathscr{K}_\infty$. Since $\lambda < \alpha$ and $\xi > \alpha+(2-\alpha)(1-p)$ then, $\beta_1 < \alpha$ and $\beta_2 > 2$. So, by Lemma 3 and Lemma 7, we have

$$\int_0^\eta u^{\alpha-1-\beta_1}M_1(u)\,du < \infty \ \text{ and } \ \int_1^\infty u^{1-\beta_2}M_2(u)\,du < \infty.$$

Now, applying Proposition 2, we get, for $t \in \Omega$

$$G(f\Theta^p)(t) \approx \rho(t)^{\min(\frac{\alpha}{2},\alpha-\beta_1)}\tilde{M}_1(\rho(t))|t|^{\min(\alpha-d,\alpha-\beta_2)}\tilde{M}_2(|t|).$$

By computation, we have

$$\min\left(\frac{\alpha}{2},\ \alpha-\beta_1\right) = \min\left(\frac{\alpha}{2},\ \frac{\alpha-\lambda}{1-p}\right), \quad \min(\alpha-d,\ \alpha-\beta_2) = \min\left(\alpha-d,\ \frac{\alpha-\xi}{1-p}\right).$$

Also, by elementary calculus, we obtain, for $t \in \Omega$,

$$\tilde{M_1}(\rho(t)) = \varphi_{L,\,\lambda,\,p}(\rho(t)) \text{ and } \tilde{M_2}(|t|) = \phi_{K,\,\xi,\,p}(|t|).$$

Then, we conclude that

$$G(f\Theta^p)(t) \approx \Theta(t),\ t \in \Omega.$$

The proof is completed. $\square$

4. Proof of main result

In order to prove the existence result, we require the following lemma from [20].

LEMMA 10. *Let q be a function satisfying (2.1). Then the following function $v(t^*)$ defined by*

$$v(t^*) := |t^*|^{d-\alpha} \int_{\Omega^*} G^{\alpha}_{\Omega^*}(t^*,s^*)|s^*|^{\alpha-d} q(s^*)\, ds^*. \tag{4.1}$$

is in $\mathscr{C}_0(\overline{\Omega^*})$.

Now, we need to prove the following.

PROPOSITION 4. *Let $p < 0$. Assume that hypothesis $(\mathscr{H})$ is satisfied. Then, for $t \in \Omega$, we have*

$$h(t) = \rho(t)^{(1-\frac{\alpha}{2})(1-p)} f(t) \in \mathbf{K}^{\alpha}_{\infty}(\Omega). \tag{4.2}$$

Proof. Let $r > 0$ and $t \in \Omega$. By (1.5), (1.7) and (1.9), we have

$$\begin{aligned}
&\int_{\Omega\cap B(t,r)} \left(\frac{\rho(s)}{\rho(t)}\right)^{\frac{\alpha}{2}} G^{\alpha}_{\Omega}(t,s)\rho(s)^{(1-\frac{\alpha}{2})(1-p)} f(s)ds \\
&\leqslant c\int_{\Omega\cap B(t,r)} \left(\frac{\rho(s)}{\rho(t)}\right)^{\frac{\alpha}{2}} G^{\alpha}_{\Omega}(t,s)\rho(s)^{(1-\frac{\alpha}{2})(1-p)-\lambda} L(\rho(s))|s|^{-\xi} K(|s|)ds \\
&\leqslant c|t^*|^{d-\alpha} \int_{\Omega^*\cap B(t^*,r)} \left(\frac{\delta(s^*)}{\delta(t^*)}\right)^{\frac{\alpha}{2}} G^{\alpha}_{\Omega^*}(t^*,s^*)\delta(s^*)^{(1-\frac{\alpha}{2})(1-p)-\lambda} \\
&\quad \times L(\delta(s^*))|s^*|^{\xi-\alpha+d-2d} K\left(\frac{1}{|s^*|}\right) ds^* \\
&\leqslant c|t^*|^{d-\alpha}\delta(t^*)^{\frac{\alpha}{2}} \int_{\Omega^*\cap B(t^*,r)} G^{\alpha}_{\Omega^*}(t^*,s^*)\delta(s^*)^{(1-\frac{\alpha}{2})(1-p)+\frac{\alpha}{2}-\lambda} \\
&\quad \times L(\delta(t^*))|s^*|^{\xi-\alpha+d-2d} K\left(\frac{1}{|s^*|}\right) ds^* \\
&\leqslant c|t^*|^{d-\alpha} \int_{\Omega^*} G^{\alpha}_{\Omega^*}(t^*,s^*)|s^*|^{\alpha-d} f_1(s^*)ds^*,
\end{aligned}$$

where $f_1(s^*) := \delta(s^*)^{(1-\frac{\alpha}{2})(1-p)+\frac{\alpha}{2}-\lambda}L(\delta(s^*))|s^*|^{\xi-2\alpha}K(\frac{1}{|s^*|})$ belongs to the Kato class $\mathbf{K}^\alpha(\Omega^*)$, in a bounded domain Ω^* (see [8]). Indeed, since $p<0$, $\lambda<\alpha$ and $\xi>\alpha+(2-\alpha)(1-p)$ then, $\lambda_0=\lambda-(1-\frac{\alpha}{2})(1-p)-\frac{\alpha}{2}<\alpha$ and $\xi_0=2\alpha-\xi<\alpha$. Moreover, by Remark 1, we have $t\mapsto K(\frac{1}{t})\in\mathscr{K}_0$. Thus, by Lemma 5 and Lemma 10, we deduce that

$$\lim_{r\to 0}\sup_{t\in\Omega}\int_{\Omega\cap B(t,r)}\left(\frac{\rho(s)}{\rho(t)}\right)^{\frac{\alpha}{2}}G_\Omega^\alpha(t,s)\rho(s)^{(1-\frac{\alpha}{2})(1-p)}f(s)ds=0. \tag{4.3}$$

Now, let us prove that

$$\lim_{M\to\infty}\sup_{t\in\Omega}\int_{\Omega\cap(|s|\geqslant M)}\left(\frac{\rho(s)}{\rho(t)}\right)^{\frac{\alpha}{2}}G_\Omega^\alpha(t,s)\rho(s)^{(1-\frac{\alpha}{2})(1-p)}f(s)ds=0. \tag{4.4}$$

Let $M>1$. By (1.5) and [8, Proposition 1], we have

$$\begin{aligned}
&\int_{\Omega\cap(|s|\geqslant M)}\left(\frac{\rho(s)}{\rho(t)}\right)^{\frac{\alpha}{2}}G_\Omega^\alpha(t,s)\rho(s)^{(1-\frac{\alpha}{2})(1-p)}f(s)ds\\
&\leqslant c|t^*|^{d-\alpha}\int_{\Omega^*\cap(|s^*|\leqslant\frac{1}{M})}\left(\frac{\delta(s^*)}{\delta(t^*)}\right)^{\frac{\alpha}{2}}G_{\Omega^*}^\alpha(t^*,s^*)\delta(s^*)^{-\lambda_1}L(\delta(s^*))|s^*|^{-\xi_1}K\left(\frac{1}{|s^*|}\right)ds^*\\
&\leqslant c|t^*|^{d-\alpha}\int_{\Omega^*\cap B(t^*,r)}\left(\frac{\delta(s^*)}{\delta(t^*)}\right)^{\frac{\alpha}{2}}G_{\Omega^*}^\alpha(t^*,s^*)\delta(s^*)^{-\lambda_1}L(\delta(s^*))|s^*|^{-\xi_1}K\left(\frac{1}{|s^*|}\right)ds^*\\
&\quad+c|t^*|^{d-\alpha}\int_{\Omega^*\cap(|s^*|\leqslant\frac{1}{M})\cap(|t^*-s^*|\geqslant r)}\left(\frac{\delta(s^*)}{\delta(t^*)}\right)^{\frac{\alpha}{2}}G_{\Omega^*}^\alpha(t^*,s^*)\delta(s^*)^{-\lambda_1}\\
&\quad\times L(\delta(s^*))|s^*|^{-\xi_1}K\left(\frac{1}{|s^*|}\right)ds^*\\
&\leqslant c\varepsilon+c\int_{(|s^*|\leqslant\frac{1}{M})}\delta(s^*)^{\alpha-\lambda_1}L(\delta(s^*))|s^*|^{-\xi_1}K\left(\frac{1}{|s^*|}\right)ds^*\\
&\leqslant c\varepsilon+c\int_{(|s|>M)}\rho(s)^{\alpha-\lambda_1}L(\rho(s))|s|^{\xi_1-2d}K(|s|)ds\\
&\leqslant c\varepsilon+c\int_{(|s|>M)}|s|^{\alpha-d-\xi}K(|s|)ds\\
&\leqslant c\varepsilon+c\int_1^\infty u^{\alpha-\xi-1+(2-\alpha)(1-p)}K(u)du,
\end{aligned}$$

where $\lambda_1=\lambda-(1-\frac{\alpha}{2})(1-p)$ and $\xi_1=d+\alpha-\xi$. By (1.10), we have the limit value (4.4). Finally, the required result, (4.2), is obtained from (4.3) and (4.4). $\square$

Next, let us recall that the potential kernel V satisfies the complete maximum principle, we have the following.

LEMMA 11. [6] *Suppose $q\in\mathscr{B}^+(\Omega)$ and v is an α-superharmonic function. Let $w\in\mathscr{B}(\Omega)$ satisfy $V(q|w|)<\infty$ and $v=w+V(qw)$. Then w satisfies the following*

$$0\leqslant w\leqslant v.$$

Proof of Theorem 1. Let $p<1$ and f be a function satisfying $(\mathscr{H})$. By Proposition 3, there exists a constant $m \geqslant 1$ such that, for all $t \in \Omega$ and $q(t)=f(t)\Theta^{p}(t)$, we have

$$\frac{1}{m}\Theta(t) \leqslant Vq(t) \leqslant m\Theta(t), \tag{4.5}$$

where Θ is the function defined in (1.11).

The proof is divided into two cases, depending on the sign of p.

Case 1. If $p<0$. Let $t \in \Omega$. Let $\varphi(t,x)=\rho(t)^{(1-\frac{\alpha}{2})(1-p)}f(t)x^{p}(t)$. By Proposition 4, we deduce, from [6, Theorem 2.17], that the problem (1.4) has a positive continuous solution x in Ω such that

$$x(t)=\int_{\Omega} G_{\Omega}^{\alpha}(t,s)f(s)x^{p}(s)\,ds. \tag{4.6}$$

So it remains to prove that x satisfies (1.12).

From (4.5), we get

$$m^{p}(Vq)^{p}(t) \leqslant \Theta(t) \leqslant m^{-p}(Vq)^{p}(t),$$

Put $c=m^{\frac{p}{p-1}}$. Let $f \in \mathscr{B}^{+}(\Omega)$ be a function given by

$$h(t):=cf(t)[\Theta^{p}(t)-m^{p}(Vq)^{p}(t)].$$

Using an elementary calculus, we deduce the following.

$$cVq=V(f(cVq)^{p})+Vh. \tag{4.7}$$

By (4.6) and (4.7), we obtain

$$cVq-x+V(f(x^{p}-(cVq)^{p}))=Vh. \tag{4.8}$$

Let g be the function defined on D by

$$g(t)=\begin{cases} f(t)\frac{x^{p}(t)-(cVq)^{p}(t)}{cVq(t)-x(t)}, & \text{if } x(t)\neq (cVq)(t),\\ 0, & \text{if } x(t)=(cVq)(t).\end{cases}$$

This implies that $g \in \mathscr{B}^{+}(\Omega)$ and since $p<0$, we have

$$f(x^{p}-(cVq)^{p})=g(cVq-x). \tag{4.9}$$

Clearly, (4.8) becomes

$$cVq-x+V(g(cVq-x))=Vh.$$

Combining (4.9), (4.6), (4.7) and (4.5), we get

$$\begin{aligned} V(g|cVq-x|) &\leqslant V(fx^{p})+V(f(cVq)^{p})\\ &\leqslant x+cVq\\ &\leqslant x+cm\Theta<\infty. \end{aligned}$$

So, by Lemma 11, we have

$$x \leqslant cVq.$$

By the same manner, we find that

$$\frac{1}{c}Vq \leqslant x.$$

Hence (1.12) holds by (4.5).

Case 2. If $0 \leqslant p < 1$.

Let $\zeta(t) = \frac{1}{\rho(t)^{\frac{\alpha}{2}-1}}\Theta(t)$ for $t \in \Omega$. Then, by (4.5), we have

$$\frac{1}{m}\zeta(t) \leqslant \frac{1}{\rho(t)^{\frac{\alpha}{2}-1}}Vq(t) \leqslant m\zeta(t). \tag{4.10}$$

Let Z be the non-empty closed, convex set defined by

$$Z = \left\{w \in \mathscr{C}_0(\Omega);\ \frac{1}{c}\zeta \leqslant w \leqslant c\zeta\right\},\ \ c = m^{\frac{1}{1-p}}.$$

We define the operator Q on Z by

$$Qw(t) = \frac{1}{\rho(t)^{\frac{\alpha}{2}-1}}\int_\Omega G_\Omega^\alpha(t,s)f(s)\rho(s)^{(\frac{\alpha}{2}-1)p}w^p(s)\,ds.$$

It is clear from (4.10), that for all $w \in Z$, we have on Ω

$$\frac{1}{c}\zeta(t) \leqslant Qw(t) \leqslant c\zeta(t).$$

Since for all $w \in Z$ and $t \in \Omega$, we have

$$|w^p(t)| \leqslant c^p||\zeta^p||_\infty.$$

Then

$$|Qw(t)| \leqslant c\int_\Omega \left(\frac{\rho(s)}{\rho(t)}\right)^{\frac{\alpha}{2}-1} G_\Omega^\alpha(t,s)h(s)\,ds,$$

where $h(s)$ be the function given by (4.2). Therefore, using Proposition 4 and Theorem 2, we deduce that

$$Qw \in \mathscr{C}_0(\Omega),\ \text{for all } w \in Z.$$

So

$$QZ \subset Z.$$

Now, consider the sequence of functions $(w_k) \in \mathscr{C}_0(\Omega)$ defined by

$$w_0 = \frac{1}{c}\zeta \ \text{ and }\ w_{k+1} = Qw_k,\ \forall k \in \mathbb{N}.$$

Since $p \geqslant 0$, then the operator Q is nondecreasing on Z. Using the fact that $QZ \subset Z$, we obtain

$$\frac{1}{c}\zeta = w_0 \leqslant w_1 \leqslant w_2 \ldots \leqslant w_k \leqslant w_{k+1} \leqslant c\zeta.$$

Thus, from the monotone convergence theorem, we deduce that the sequence (w_k) converges to a function $w,$ such that for each $t \in \Omega,$

$$w(t) = \frac{1}{\rho(t)^{\frac{\alpha}{2}-1}} \int_\Omega G_\Omega^\alpha(t,s) f(y) \rho(s)^{(\frac{\alpha}{2}-1)p} w^p(s)\, ds,$$

and

$$\frac{1}{c}\zeta(t) \leqslant w(t) \leqslant c\zeta(t). \tag{4.11}$$

Using the same method as used above, we deduce, by (4.11) and Theorem (2), that

$$w \in \mathscr{C}_0(\Omega).$$

Let $x(t) = \rho(t)^{\frac{\alpha}{2}-1} w(t).$ Then $x \in \mathscr{C}(\Omega)$ and we have, for all $t \in \Omega,$

$$x(t) = \int_\Omega G_\Omega^\alpha(t,s) f(s) x^p(s)\, ds.$$

Finally, since $w \in \mathscr{C}_0(\Omega).$ Then

$$\lim_{t\to\partial\Omega} \delta(t)^{1-\frac{\alpha}{2}} x(t) = \lim_{|t|\to\infty} x(t) = 0.$$

By (1.8) and using the fact that $x \in \mathscr{C}(\Omega)$, we deduce that x is a solution of problem (1.4). The proof of Theorem 1 is finished. □

EXAMPLE 4. Let $p < 1.$ Suppose that f is a nonnegative function in $\mathscr{C}_{loc}^\gamma(\Omega),\ 0 < \gamma < 1,$ such that for $t \in \Omega,$

$$f(t) \approx \rho(t)^{-\lambda} \left(\ln\left(\frac{4}{\rho(t)}\right)\right)^{-\beta_1} (1+|t|)^{-\xi} (\ln(2(1+|t|)))^{-\beta_2},$$

where $\lambda < \alpha,\ \xi > \alpha + (2-\alpha)(1-p),\ \beta_1 > 1$ and $\beta_2 > 1.$ Then by Theorem 1, problem (1.4) has a positive solution x satisfying, for $t \in \Omega,$

$$x(t) \approx \varphi(\rho(t))\phi(|t|).$$

Where

$$\varphi(\rho(t)) = \begin{cases} \rho(t)^{\frac{\alpha}{2}}, & \text{if } \lambda \leqslant \frac{\alpha}{2}(1+p), \\ \rho(t)^{\frac{\alpha-\lambda}{1-p}} \left(\ln\left(\frac{4}{\rho(t)}\right)\right)^{\frac{-\beta_1}{1-p}}, & \text{if } \frac{\alpha}{2}(1+p) < \lambda < \alpha, \end{cases}$$

and

$$\phi(|t|) = \begin{cases} |t|^{\frac{\alpha-\xi}{1-p}}, & \text{if } \xi \geqslant d - p(d-\alpha), \\ |t|^{\alpha-d} (\ln(2|t|))^{\frac{-\beta_2}{1-p}}, & \text{if } \alpha + (2-\alpha)(1-p) < \xi < d - p(d-\alpha). \end{cases}$$

EXAMPLE 5. Let $\Omega = \{t \in \mathbb{R}^d,\ |t| > 1\}$ and f be a nonnegative measurable function such that, for all $t \in \Omega$

$$f(t) \approx \left(1 - \frac{1}{|t|}\right)^{-\lambda} \ln\left(\frac{4|t|}{|t|-1}\right) |t|^{-\xi} \ln(4|t|)^{-m},$$

where $\lambda < \alpha$, $\xi > \alpha + (2-\alpha)(1-p)$ and $m > 1$. Suppose that $L\left(1 - \frac{1}{|t|}\right) = \ln\left(\frac{4|t|}{|t|-1}\right)$ and $K(|t|) = \ln(4|t|)^{-m}$. Then by Theorem 1, problem (1.4) has a positive solution v satisfying, for $t \in \Omega$ and $p > 1$

$$x(t) \approx \varphi_{L,\lambda,p}\left(1 - \frac{1}{|t|}\right) \phi_{K,\xi,p}(|t|).$$

With

$$\varphi_{L,\lambda,p}\left(1 - \frac{1}{|t|}\right) = \begin{cases} \left(1 - \frac{1}{|t|}\right)^{\frac{\alpha}{2}}, & \text{if } \lambda \leqslant \frac{\alpha}{2}(1+p), \\ \left(1 - \frac{1}{|t|}\right)^{\frac{\alpha-\lambda}{1-p}} \left(\ln\left(\frac{4|t|}{|t|-1}\right)\right)^{\frac{1}{1-p}}, & \text{if } \frac{\alpha}{2}(1+p) < \lambda < \alpha \end{cases}$$

and

$$\phi_{K,\xi,p}(|t|) = \begin{cases} |t|^{\frac{\alpha-\xi}{1-p}}, & \text{if } \xi \geqslant d - p(d-\alpha), \\ |t|^{\alpha-d}(\ln(4|t|))^{\frac{-m}{1-p}}, & \text{if } \alpha + (2-\alpha)(1-p) < \xi < d - p(d-\alpha). \end{cases}$$

Declarations

Availability of data and materials. Data sharing not applicable to this paper as no data sets were generated or analyzed during the current study.

Conflict of interest. the authors declare that they have no conflict of interest.

Author's contributions. All authors read and approved the final manuscript.

Funding. Not available.

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(Received August 20, 2024)

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