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NEUMANN PROBLEM FOR A STOCHASTIC BENJAMIN–BONA–MAHONY EQUATION WITH RIESZ FRACTIONAL DERIVATIVE

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Abstract. In this work, we study an initial boundary-value problem for a stochastic Benjamin-Bona-Mahony equation with Riesz-fractional spatial derivative and white noise on the half-line. For the associated linear problem, we construct the Green's function adapting the main ideas of the Fokas method. Then, the main problem will be understood in the Walsh sense and the Picard scheme is used to prove existence and uniqueness of solutions. Moreover, an example is presented to show the results obtained.

1. Introduction

The waves are produced when the equilibrium state of a medium in a system is disturbed and the disturbance propagates from one region to another, its study is a broad and well known research topic, with numerous applications in several fields of science (e.g., see [1, 8, 10, 12] and related references). The propagation of unidirectional, one-dimensional, small-amplitude long waves has been studied through two classical models; the celebrated Korteweg-de Vries (KdV) equation [14]

$$u_t + u_x + uu_x + u_{xxx} = 0,$$

and the also famous Benjamin-Bona-Mahony (BBM) equation [4]

$$u_t + u_x + uu_x - u_{xxt} = 0,$$

among others. The KdV equation is suitable when the values of the wavenumber are small enough; however, if this does not hold the phase velocity can be negative, which contradicts the physical nature wave propagation. On the other hand, the BBM equation produces a more reasonable dispersion relation for any value of the wavenumber (see Benjamin et al. [4]). The description of the drift of waves in plasma physics, the propagation of wave in semi-conductors and optical devices and the behavior of Rossby

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waves in rotating fluids are some other phenomena that are modeled by this equation [7, 16].

The BBM equation has been studied by many authors using diverse methods; for example, in Besse [5] and Wang [25], the Crank Nicolson time discretization scheme is used to study continuous and discrete artificial boundary conditions for the linearized BBM equation. Moreover, two results of unique continuation on a bounded interval of the linearized BBM equation, using the eigenvector expansion and spectral analysis, appear in Zhang [26]. Also, Micu [17] shows the finite-domain controllability of the linearized BBM equation and its spectral non-controllability. Moreover, Vishal [23] use the Fokas method [11] to solve the linear BBM equation, considering a Robin problem on the half-line and a finite interval.

On the other hand, although most of the theoretical structure of fractional calculus has been realized, in recent years it has attracted the attention of scientists and engineers who have managed to rediscover and apply it in various fields. For example, the fractional Benjamin-Bona-Mahony equation is used to study the phenomena of propagation for small amplitude long unidirectional waves in a nonlocal elastic medium [19]. Also, this equation describes cold plasma for hydromagnetic and audio waves in harmonic crystals [21]. Recently, Pava [20] has considered the non-linear stability on a specific interval and the spectral instability of the solutions and Amaral [2] studied a Cauchy problem using the Petviashvili's iteration method, both considering a fractional BBM equation. This type of models can be generalized by including a random variable which leads to the study of fractional stochastic equations [3, 15, 18, 22]. Elmandouh and Fadhal, investigated the bifurcation of exact solutions with the influence of a multiplicative noise of the modified BBM equation [9].

In this work, we study a stochastic differential equation on the half-line with additive white noise and Riesz fractional derivative, the paper is organized as follows: Section 2 deals with the preliminaries and the statement of the problem. The Neumann's problem for a fractional BBM equation is analyzed in Section 3. The existence and uniqueness of the solution are proved in Section 4. In order to illustrate our results an explicit example is presented.

2. Preliminaries and statement of the problem

In this section we set up the notations and recall some basic definitions. Furthermore, the problem to be studied is presented.

DEFINITION 1. The *Laplace transform* of $u(x,t)$ is defined by

$$\widehat{u}(k,t) = \int_0^\infty e^{-ikx} u(x,t)dx, \quad \Im m k \leqslant 0,$$

and the *inverse Laplace transform* is given by

$$u(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx} \hat{u}(k,t)dk.$$

There exist numerous definitions of fractional integrals and fractional derivatives. This paper deals with the Riesz fractional derivative.

DEFINITION 2. The *Riesz fractional derivative* is defined by the following integral

$$\mathscr{R}_x^{\alpha} u(x,t) = -\frac{1}{\Gamma(2-\alpha)\cos(\frac{\pi}{2}\alpha)} \int_0^{\infty} \frac{\operatorname{sgn}(x-y)}{(x-y)^{\alpha-1}} \partial_y^2 u(y,t)dy,$$

where $u(x,t)$ is twice differentiable, $\alpha \in (1,2)$ and the integral has to be understood in the sense of Cauchy principal value.

Note that, the Laplace transform of the Riesz fractional derivative is

$$\widehat{\mathscr{R}_x^{\alpha}} u(k,t) = -|k|^{\alpha-2}\left[(ik)^2 \widehat{u}(k,t) - iku(0,t) - u_x(0,t)\right].$$

DEFINITION 3. The operator $\mathbf{F}$ is called *lipschitzian* if there exist a constant $M > 0$, such that

$$|\mathbf{F}v - \mathbf{F}w| \leqslant M|v-w|,$$

where v, w are real-valued functions.

LEMMA 1. (Gronwall's Lemma) (See Hirsch [13]) *Suppose* $\sigma_1, \sigma_2, \ldots : [0,T] \to \mathbb{R}^+$ *are measurable non-decreasing. Suppose also that here exist a constant M such that for all integers* $n \geqslant 1$, *and* $t \in [0,T]$,

$$\sigma_{n+1}(t) \leqslant M \int_0^t \sigma_n(s)ds.$$

Then,

$$\sigma_n(t) \leqslant \sigma_1(t)\frac{(Mt)^{n-1}}{(n-1)!}.$$

We study the following Neumann problem for a fractional BBM equation

$$\begin{cases} u_t - au_{xxt} + b\mathscr{R}_x^{\alpha} u = \mathbf{F}u + \dot{\mathscr{B}}, & x,t \geqslant 0, \\ u(x,0) = u_0(x), & x \geqslant 0, \\ u_x(0,t) = h_1(t), & t \geqslant 0, \end{cases} \tag{2.1}$$

where $\mathscr{R}_x^{\alpha}$ is the Riesz fractional derivative with $\alpha \in (1,2)$, $\mathbf{F}$ is a lipschitzian operator and $\dot{\mathscr{B}}$ is the white noise defined on a complete probability space $(\Omega, \mathscr{A}, \mathscr{A}_t, P)$. Here, $\mathscr{A}$ is a $\sigma-$algebra and $\mathscr{A}_{t\{t\geqslant 0\}}$ is a right-continuous filtration on $(\Omega, \mathscr{A})$ such that $\mathscr{A}_0$ contains all $P-$negligible subsets, being P a probability measure. Moreover, suppose that $\mathscr{B}$ generates a $(\mathscr{A}_t, t \geqslant 0)-$martingale measure in the Walsh sense [24], where

$$\mathscr{B} = \{\mathscr{B}(x,t) \mid \quad x \geqslant 0, \quad t \geqslant 0\}$$

be a center Gaussian field.

3. Neumann's problem for a fractional BBM equation

In this section, we construct the Green function for the linear Neumann problem (2.1), where the BBM equation with a Riesz fractional derivative is considered. In order to state precisely the main result of this paper, we define the following Green's operators:

$$\begin{aligned}
\mathscr{G}^I(t)u_0 &= \int_0^\infty \left[G^I(x+y,t)+G^I(x-y,t)\right]u_0(y)dy,\\
\mathscr{G}^B(x)h_1 &= \int_0^t G^B(x,t-s)h_1(s)ds,
\end{aligned}$$

and the Green's functions are

$$\begin{aligned}
G^I(x,t) &= \tfrac{1}{\pi}\int_0^\infty e^{-\frac{b|k|^\alpha}{1+ak^2}t}\cos(kx)dk,\\
G^B(x,t) &= \tfrac{2}{\pi}\int_0^\infty \tfrac{-b|k|^\alpha}{k^2(1+ak^2)^2}e^{-\frac{b|k|^\alpha}{1+ak^2}t}\cos(kx)dk,\\
G^{B^*}(x,t) &= \tfrac{2}{\pi}\int_0^\infty \tfrac{a}{1+ak^2}e^{-\frac{b|k|^\alpha}{1+ak^2}t}\cos(kx)dk.
\end{aligned} \tag{3.1}$$

THEOREM 1. *Let be the initial-value* $u_0(x)\in\mathbf{L}^1(\mathbb{R}^+)$ *and the boundary-value* $h_1(t)\in\mathbf{C}(\mathbb{R}^+)$. *Assume that there is some function* $u(x,t)$ *satisfying* (2.1)*, then* $u(x,t)$ *has the following integral representation*

$$u(x,t)=\mathscr{G}^I(t)u_0+\mathscr{G}^B(x)h_1+h_1(0)G^{B^*}(x,t)-h_1(t)G^{B^*}(x,0),\quad x,t\geqslant 0.$$

Proof. First, applying the Laplace transform in (2.1), we arrive

$$\hat{u}_t(k,t)+\gamma(k)\,\hat{u}(k,t)=-\tfrac{b|k|^\alpha}{k^2(1+ak^2)}\Big[h_1(t)+ikh_0(t)\Big]-\tfrac{a}{1+ak^2}\Big[h_1'(t)+ikh_0'(t)\Big], \tag{3.2}$$

where $\gamma(k):=b|k|^\alpha/(1+ak^2)$, $h_1(t)=u_x(0,t)$, $h_0(t)=u(0,t)$, $h_1'(t)=\partial_t u_x(0,t)$ and $h_0'(t)=\partial_t u(0,t)$. Now, multiplying the equation (3.2) by $e^{\gamma(k)t}$ and integrating from 0 to t, we get the global relation

$$\begin{aligned}
e^{\gamma(k)t}\hat{u}(k,t)-\hat{u}_0(k) = &-\tfrac{b|k|^\alpha}{k^2(1+ak^2)}\Big[ikH_0^0\big(\gamma(k),t\big)+H_1^0\big(\gamma(k),t\big)\Big]\\
&-\tfrac{a}{1+ak^2}\Big[ikH_0^1\big(\gamma(k),t\big)+H_1^1\big(\gamma(k),t\big)\Big],\quad \Im m k<0,
\end{aligned} \tag{3.3}$$

with

$$H_j^i(w,t)=\int_0^t e^{ws}\partial_s^i\partial_x^j u(0,s)ds,\quad i=0,1,\quad j=0,1.$$

Thus, by means of the inverse Laplace transform, $u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx}\hat{u}(k,t)dk,$ we arrive to

$$\begin{aligned} u(x,t) = {} & \tfrac{1}{2\pi}\int_{\mathbb{R}} e^{ikx-\gamma(k)t}\hat{u}_0(k)dk \\ & - \tfrac{1}{2\pi}\int_{\mathbb{R}} \tfrac{b|k|^\alpha}{k^2(1+ak^2)} e^{ikx-\gamma(k)t}\left[ikH_0^0\big(\gamma(k),t\big) + H_1^0\big(\gamma(k),t\big)\right]dk \\ & - \tfrac{1}{2\pi}\int_{\mathbb{R}} \tfrac{a}{1+ak^2} e^{ikx-\gamma(k)t}\left[ikH_0^1\big(\gamma(k),t\big) + H_1^1\big(\gamma(k),t\big)\right]dk. \end{aligned}$$

Now, dividing $\mathbb{R} = (-\infty,0]\cup(0,\infty)$ in the last two integrals in the above equation, considering the analytic extension

$$|k|^\alpha = \begin{cases} k^\alpha, & \Re e(k) > 0, \\ (-k)^\alpha, & \Re e(k) < 0, \end{cases}$$

and since that $\Re e\left(b|k|^\alpha/(1+ak^2)\right) > 0$ in the region

$$C_\theta = (-\infty,0]e^{-i\theta}\cup(0,\infty)e^{i\theta}, \quad \theta\in[0,\pi/4],$$

then by the Cauchy Theorem and Jordan's Lemma we can deform the integration contour $\mathbb{R}$ into $C_{\pi/4}$.

Moreover, since $\gamma(k) = \gamma(-k)$, the transformation $k\mapsto -k$ in the global relation (3.3) gives

$$\begin{aligned} \tfrac{ikb|k|^\alpha}{k^2(1+ak^2)} H_0^0\big(\gamma(k),t\big) + \tfrac{ika}{1+ak^2} H_0^1\big(\gamma(k),t\big) = {} & e^{\gamma(k)-t}\hat{u}(-k,t) - \hat{u}_0(-k) \\ & + \tfrac{b|k|^\alpha}{k^2(1+ak^2)} H_1^0\big(\gamma(k),t\big) \\ & + \tfrac{a}{1+ak^2} H_1^1\big(\gamma(k),t\big), \quad \Im m\, k > 0. \end{aligned}$$

Therefore, we get

$$\begin{aligned} u(x,t) = {} & \tfrac{1}{2\pi}\int_{\mathbb{R}} e^{ikx-\gamma(k)t}\hat{u}_0(k)dk + \tfrac{1}{2\pi}\int_{C_{\pi/4}} e^{ikx-\gamma(k)t}\hat{u}_0(-k)dk \\ & - \tfrac{1}{\pi}\int_{C_{\pi/4}} \tfrac{b|k|^\alpha}{k^2(1+ak^2)} H_1^0\big(\gamma(k),t\big) + \tfrac{a}{1+ak^2} e^{ikx-\gamma(k)t} H_1^1\big(\gamma(k),t\big)dk \\ & - \tfrac{1}{2\pi}\int_{C_{\pi/4}} e^{ikx}\hat{u}(-k,t)dk. \end{aligned} \tag{3.4}$$

Now, lets notice that by Cauchy's Theorem and Jordan's Lemma, we obtain

$$\int_{C_{\pi/4}} e^{ikx}\hat{u}(-k,t)dk = 0,$$

then, using the above equation and deforming back the integrate contour to the real line, (3.4) can be expressed by

$$\begin{aligned} u(x,t) &= \tfrac{1}{2\pi}\int_{\mathbb{R}} e^{ikx-\gamma(k)t}\int_0^\infty e^{-iky}u_0(y)dydk \\ &\quad+\tfrac{1}{2\pi}\int_{\mathbb{R}} e^{ikx-\gamma(k)t}\int_0^\infty e^{iky}u_0(y)dydk \\ &\quad-\tfrac{1}{\pi}\int_{\mathbb{R}} \tfrac{b|k|^\alpha}{k^2(1+ak^2)}e^{ikx-\gamma(k)t}\int_0^t e^{\gamma(k)s}h_1(s)dsdk \\ &\quad-\tfrac{1}{\pi}\int_{\mathbb{R}} \tfrac{a}{1+ak^2}e^{ikx-\gamma(k)t}\int_0^t e^{\gamma(k)s}h_1'(s)dsdk. \end{aligned}$$

Furthermore, applying Fubini's Theorem we obtain

$$\begin{aligned} u(x,t) &= \tfrac{1}{2\pi}\int_0^\infty\int_{\mathbb{R}} e^{ik(x-y)-\gamma(k)t}u_0(y)dkdy \\ &\quad+\tfrac{1}{2\pi}\int_0^\infty\int_{\mathbb{R}} e^{ik(x+y)-\gamma(k)t}u_0(y)dkdy \\ &\quad-\tfrac{1}{\pi}\int_0^t\int_{\mathbb{R}} \tfrac{b|k|^\alpha}{k^2(1+ak^2)}e^{ikx-\gamma(k)(t-s)}h_1(s)dkds \\ &\quad-\tfrac{1}{\pi}\int_0^t\int_{\mathbb{R}} \tfrac{a}{1+ak^2}e^{ikx-\gamma(k)(t-s)}h_1'(s)dkds. \end{aligned}$$

Integrating by parts the last term on the right-hand side and using the Green's functions (3.1), we obtain the desired result. □

4. Main problem

In this section, we consider the following Neumann problem for a stochastic equation

$$\begin{cases} u_t - au_{xxt} + b\mathscr{R}_x^\alpha u = \mathbf{F}u + \dot{\mathscr{B}}, & x,t \geqslant 0, \\ u(x,0) = u_0(x), \quad x \geqslant 0, \\ u_x(0,t) = h_1(t), \quad t \geqslant 0, \end{cases} \tag{4.1}$$

where $\mathbf{F}$ is a lipschitzian operator an $\dot{\mathscr{B}}(x,t)$ is the white noise. The function u is called a solution of the problem, if for all $x > 0$ and $t \in [0,T]$ satisfies

$$\begin{aligned} u(x,t) = &\int_0^\infty G^I(x,y,t)u_0(y)dy \\ &+\int_0^t G^B(x,t-s)h_1(s)ds \\ &+h_1(0)G^{B^*}(x,t) - h_1(t)G^{B^*}(x,0) \\ &+\int_0^t\int_0^\infty G(x-y,t-s)\mathbf{F}u(y,s)dyds \\ &+\int_0^t\int_0^\infty G(x-y,t-s)d\mathscr{B}(y,s), \end{aligned}$$

where

$$G(\eta,\tau) = \tfrac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ik\eta+\gamma(k)\tau}dk.$$

In the following theorem the existence of a unique solution is proved.

THEOREM 2. *Suppose that* $\mathbf{F}$ *is an lipschitzian operator. If for some* $p \geqslant 1$,

$$\sup_{x\in\mathbb{R}^+} \mathbb{E}(|u_0(x)|^p) < \infty, \tag{4.2}$$

then $u(x,t)$ *exists and is the unique solution to the problem* (4.1)*. Moreover, for all* $T > 0$ *and* $p > 1$,

$$\sup_{\substack{x\in\mathbb{R}^+\\ t\in[0,T]}} \mathbb{E}(|u(x,t)|^p) < \infty.$$

Proof. First, we construct a Picard iteration

$$\begin{aligned} u^{n+1}(x,t) =& u^0(x,t) + \int_0^t G^B(x,t-s)h_1(s)ds \\ &+ h_1(0)G^{B^*}(x,t) - h_1(t)G^{B^*}(x,0) \\ &+ \int_0^t\int_0^\infty G(x-y,t-s)\mathbf{F}u^n(y,s)dyds \\ &+ \int_0^t\int_0^\infty G(x-y,t-s)d\mathscr{B}(y,s), \end{aligned} \tag{4.3}$$

where

$$u^0(x,t) = \int_0^\infty G^I(x,y,t)u_0(y)dy.$$

For the convergence in $L^p(\Omega)$ of $\{u^n(x,t)\}_{n\geqslant 0}$, let $n \geqslant 2$ then

$$\begin{aligned} &\mathbb{E}\left(\left|u^{n+1}(x,t)-u^n(x,t)\right|^p\right) \\ =& \mathbb{E}\left(\left|\int_0^t\int_0^\infty G(x-y,t-s)\left[\mathbf{F}u^n(y,s)-\mathbf{F}u^{n-1}(y,s)\right]dyds\right|^p\right) \\ \leqslant& C(p)\int_0^t\int_0^\infty G(x-y,t-s)\mathbb{E}\left(\left|u^n(y,s)-u^{n-1}(y,s)\right|^p\right)dyds \\ \leqslant& C(p)\int_0^t \sup_{y\in\mathbb{R}^+}\mathbb{E}\left(\left|u^n(y,s)-u^{n-1}(y,s)\right|^p\right)ds, \end{aligned}$$

by Burkholder's inequality [6] and (4.2) we obtain

$$\sup_{x\in\mathbb{R}^+}\mathbb{E}\big(|u^1(x,t)-u^0(x,t)|^p\big) \leqslant C(p)\left(\sup_{x\in\mathbb{R}^+}\mathbb{E}\big(|u^1(x,t)|^p\big) + \sup_{x\in\mathbb{R}^+}\mathbb{E}\big(|u^0(x,t)|^p\big)\right) < \infty.$$

Then, the Lemma (1) shows that

$$\sum_{n\geqslant 0}\sup_{(x,t)\in\mathbb{R}^+\times[0,T]}\mathbb{E}\left(|u^n(x,t)-u^{n-1}(x,t)|^p\right)<\infty.$$

Therefore, $\{u^n(x,t)\}_{n\geqslant 0}$ is a Cauchy sequence in $L^p(\Omega)$. Let

$$u(x,t)=\lim_{n\to\infty}u^n(x,t).$$

Then for each $(x,t)\in\mathbb{R}^+\times[0,T]$

$$\sup_{(x,t)\in\mathbb{R}^+\times[0,T]}\mathbb{E}\left(|u(x,t)|^p\right)<\infty.$$

Thus, we show that $u(x,t)$ satisfies the problem (4.1) at both sides of (4.3) when $n\to\infty$ in $L^p(\Omega)$. Let v and w be two solutions of the problem (4.1), then

$$\begin{aligned}
&\mathbb{E}\left(|v(x,t)-w(x,t)|^p\right)\\
&=\mathbb{E}\left(|\int_0^t\int_0^\infty G(x-y,t-s)\left[\mathbf{F}v(y,s)-\mathbf{F}w(y,s)\right]dyds|^p\right)\\
&\leqslant C(p)\int_0^t\int_{\mathbb{R}}G(x-y,t-s)\mathbb{E}\left(|v(y,s)-w(y,s)|^p\right)dyds\\
&\leqslant C(p)\int_0^t\sup_{y\in\mathbb{R}^+}\mathbb{E}\left(|v(y,s)-w(y,s)|^p\right)ds.
\end{aligned}$$

Applying Lemma (1), we get

$$\mathbb{E}\left(|v(y,s)-w(y,s)|^p\right)=0.$$

Then, the uniqueness of the solution is proved. □

4.1. Example

We give an example to verify the theoretical analysis. Here, we consider problem (2.1) with initial condition $u_0(x)=xe^{-0.2x}$, boundary value $h_1(t)=\cos(t)$ and positive constants $a=1$, $b=3$. In the Figures 1, 2 and 3, we present the plot of the solution $u(x,t)$ for different α values. The graphs show that the behavior of the system is more diffusive for alpha values close to 2. That is, the non local property of the fractional derivative look more suitable to describe the non homogeneity of the medium.

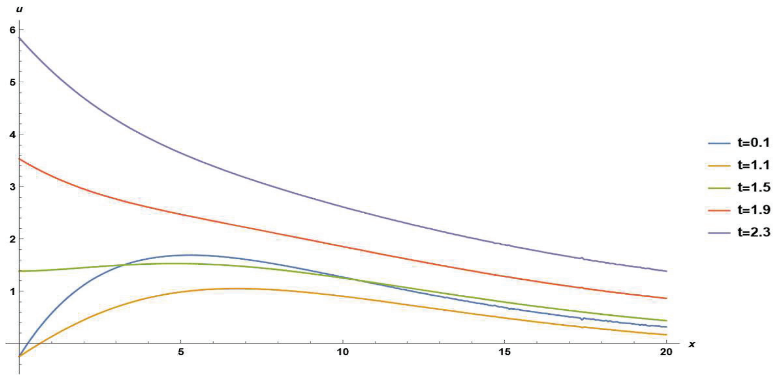

Figure 1: *Plot of the solution* $u(x,t)$ *for value* $\alpha = 1.5$.

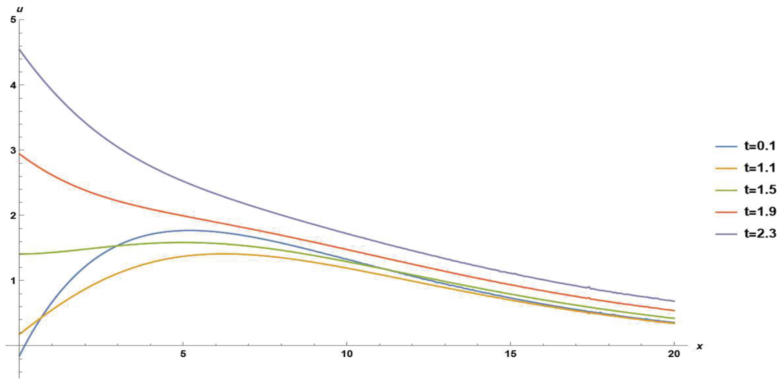

Figure 2: *Plot of the solution* $u(x,t)$ *for value* $\alpha = 1.7$.

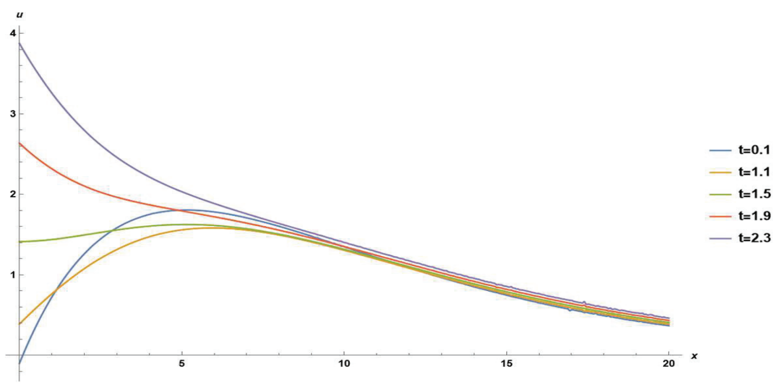

Figure 3: *Plot of the solution* $u(x,t)$ *for value* $\alpha = 1.9$.

REFERENCES

[1] M. Ablowitz, A. Fokas and Z. Musslimani, *On a New Nonlocal Formulation of Water Waves*, Journal of Fluid Mechanics **562**, (2006), 313–344.

[2] S. Amaral, H. Borluk, G. Muslu, F. Natali and G. Oruc, *On the existence, uniqueness, and stability of periodic waves for the fractional Benjamin-Bona-Mahony equation*, Studies in Applied Mathematics. (2021).

[3] M. P. Arciga-Alejandre, F. J. Ariza-Hernandez and J. Sanchez-Ortiz, *Stochastic evolution equation with Riesz-fractional derivative and white noise on the half-line*, Applied Numerical Mathematics. **104**, (2016), 103–109.

[4] T. Benjamin, J. Bona and J. Mahony, *Model Equations for Long Waves in Nonlinear Dispersive Systems*, Physical and Engineering Sciences. **272**, 1220 (1972), 47–78.

[5] C. Besse, B. Mesognon-Gireau and P. Noble, *Artificial boundary conditions for the linearized Benjamin-Bona-Mahony equation*, Numerische Mathematik. **139**, 2 (2018), 281–314.

[6] D. Burkholder, J. Burgess and R. Gundy, *Integral inequalities for convex functions of operators on martingales*, Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability. **2**, (1972), 223–240.

[7] Y. Chuntao, *Regularized long wave equation and inverse scattering transform*, Journal of mathematical physics. **34**, 6 (1993), 2618–2630.

[8] R. Davis and A. Acrivos, *Solitary internal waves in deep water*, Journal of Fluid Mechanics. **29**, 3 (1967), 593–607.

[9] A. Elmandouh and E. Fadhal, *Bifurcation of Exact Salutions for the Space-Fractional Stochastic Modified Benjamin-Bona-Mahony Equation*, Fractal and Fractional. **6**, 12 (2022).

[10] B. Engquist and A. Majda, *Absorbing Boundary Conditions for the Numerical Simulation of Waves*, Proceedings of the Royal Society of London Series A: Mathematical, Physical and Engineering Sciences. **31**, (1977), 629–651.

[11] A. Fokas, *A Unified Approach to Boundary Value Problems*, SIAM. (2022).

[12] M. Groves, *Steady Water Waves*, Journal of Nonlinear Mathematical Physics. **11**, 4 (2004), 435–460.

[13] M. Hirsch, S. Smale and R. Devaney, *Differential equations, dynamical systems and an introduction to chaos*, Academic press. (2012).

[14] D. Korteweg and G. De Vries, *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science. **39**, 240 (1895), 422–443.

[15] J. Liu, L. Yan and Y. Cang, *On a jump-type stochastic fractional partial differential equation with fractional noises*, Nonlinear Analysis: Theory, Methods and Applications. **75**, 16 (2012), 6060–6070.

[16] J. Meiss and W. Horton, *Solitary drift waves in the presence of magnetic shear*, The Physics of Fluids. **26**, 4 (1983), 990–997.

[17] S. Micu, *On the Controllability of the Linearized Benjamin-Bona-Mahony Equation*, SIAM Journal on Control and Optimization. **39**, 6 (2001), 1677–1696.

[18] W. Mohammed, N. Iqbal and J. Botmart, *Additive noise effects on the stabilization of fractional-space diffusion equation solutions*, Mathematics. **10**, 1 (2022), 130.

[19] G. Oruc, H. Borluk, and G. M. Muslu, *The generalized fractional Benjamin-Bona-Mahony equation: analytical and numerical results*, Physica D: Nonlinear Phenomena, (2020) 409, 132499.

[20] J. Pava, *Stability properties of solitary waves for fractional KdV and BBM equations*, Nonlinearity. **31**, 3 (2018), 920–956.

[21] K. Pavani, K. Raghavendar, and K. Aruna, *Solitary wave solutions of the time fractional Benjamin-Bona-Mahony-Burger equation*, Scientific Reports 14.1 (2024): 14596.

[22] K. Shi and Y. Wang, *On a stochastic fractional partial differential equation with a fractional noise*, Stochastics. **84**, 1 (2012), 21–36.

[23] V. Vishal and B. Deconinck, *Well-posedness of boundary-value problems for the linear Benjamin-Bona-Mahony equation*, Discrete and Continuous Dynamical Systems. **33**, 7 (2013), 3171–3188.

[24] J. Walsh, *An introduction to stochastic partial differential equations*, École d'Été de Probabilités de Saint Flour XIV – 1984. (1986), 265–439.

[25] L. WANG, X. LIAO AND H. YANG, *Unconditional superconvergence analysis of an energy-stable finite element scheme for nonlinear Benjamin-Bona-Mahony-Burgers equation*, Boundary Value Problems. **72**, (2022).

[26] X. ZHANG AND E. ZUAZUA, *Unique continuation for the linearized Benjamin-Bona-Mahony equation with space-dependent potential*, Mathematische Annalen. **325**, 3 (2003), 543–582.

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