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ON CONTROLLABILITY OF SEMILINEAR GENERALIZED IMPULSIVE SYSTEMS ON FINITE DIMENSIONAL SPACE

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Abstract. This article presents sufficient conditions for the complete controllability of generalized semilinear impulsive systems in a finite-dimensional space. The analysis focuses on cases where the nonlinear perturbation functions satisfy the Lipschitz continuity condition. We establish these conditions by leveraging functional analysis techniques and various fixed-point theorems. Furthermore, a numerical example is included to demonstrate the effectiveness of the proposed results.

1. Introduction

Shah et al. [1,2] have explored the existence and uniqueness of solutions for generalized impulsive evolution equations, where nonlinear perturbations exhibit abrupt changes at impulse instants. Such integro-differential models frequently arise in real-world scenarios, such as vehicle motion dynamics within urban environments and other physical systems influenced by sudden state transitions. Given the significance of these models, this work investigates the controllability of generalized impulsive systems governed by

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + f_k(t,x(t)) + B_k(t)u(t), \quad t \in \cup_{k=1}^{p}[t_{k-1},t_k) \cup [t_p,T] \\ x(t_0) &= x_0, \\ x(t_k^+) &= [I_n + D^k u(t_k)]x(t_k^-), \;\; k-1,2,\cdots p \end{aligned} \tag{1.1}$$

over the time interval $[t_0,T]$. $x(t)$ represents the system state as an n-dimensional vector, while $u(t)$ is an m-dimensional control input. The function $f_k(\cdot,\cdot):\mathbb{R}^+\times\mathbb{R}^n \to \mathbb{R}^n$ adheres to the Carathéodory conditions-ensuring measurability in t for fixed x and continuity in x for almost all $t \in [t_0,T]$. Additionally, the system matrices $A(\cdot) \in C([t_0,T],\mathbb{R}^{n\times n})$, $B_k(\cdot) \in C([t_0,T],\mathbb{R}^{n\times m})$, and the impulsive effects are characterized by $D^k u(t_k) = \sum_{i=1}^{m} d_i^k u(t_k) I_n$ with $d_i^k \in \mathbb{R}$. The impulse times $t_0 < t_1 < \ldots < t_p < T$ represent instances where abrupt changes occur in the system state.

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Impulsive differential equations have been widely used to model complex systems across physics, engineering, and biological sciences, where sudden perturbations influence the system's evolution. Examples include the impact of Earth's oblateness on satellite trajectories, ecological systems undergoing abrupt harvesting, and population dynamics affected by sudden resource fluctuations [3–12]. These equations capture the instantaneous effects of external forces, making them a fundamental tool in modeling real-world discontinuities.

Controllability, a cornerstone of mathematical control theory, is concerned with determining whether a control function can drive the system from an initial state to a desired final state [13, 14]. Research on impulsive control systems has evolved significantly since Leela et al. [15] introduced controllability conditions for time-invariant impulsive linear systems. Benzaid and Sznaier [16] later established necessary and sufficient conditions for global controllability under impulse constraints at discontinuity points. Subsequent studies have expanded the theory, including functional analytic approaches for semilinear systems [17], controllability analyses of time-varying linear impulsive models [18–20], and sufficient conditions for semilinear systems under Lipschitz continuity assumptions [21]. Dubey and George [22] demonstrated that applying control in earlier intervals is preferable to later stages, refining controllability conditions for both linear and semilinear impulsive equations.

However, existing research has primarily focused on specific cases, with limited exploration of more generalized impulsive systems where discontinuous effects are integrated with broader control strategies. Furthermore, while numerous studies address applications of impulsive systems, fundamental theoretical aspects, such as solution definitions and necessary optimality conditions, are often overlooked. Foundational work by Bressan and Rampazzo [25] introduced the notion of graph completion solutions, while Karamzin [26] provided an alternative framework through discontinuous time transformations. Recent developments have extended these ideas, incorporating time delays and deriving maximum principles for optimal control [27, 28].

Similarly, the controllability of impulsive systems has been closely linked to asymptotic controllability, feedback stabilization, and Lyapunov function approaches. Seminal works such as those by Sontag [29] and Clarke et al. [30] established key connections between these properties, which were further refined in later studies [31]. More recent advances have incorporated higher-order Lie bracket conditions and cost regulation constraints, providing a deeper understanding of feedback stabilizability [32–34].

Building upon these theoretical advancements, Shah et al. [35,37] sufficient conditions for the existence and trajectory controllability of conformable fractional evolution equations using nonlinear functional analysis, Banach's fixed point principle, and Gronwall's inequality. Their study validated the theoretical results with examples in both finite and infinite-dimensional Banach spaces. Similarly, Ghansh et al. [36, 38, 39]investigated the controllability of fractional-order nonlinear dynamical systems, proposing a novel theorem that defines sufficient conditions for controllability while considering constraints on the steering operator and nonlinear perturbations. Their work also analyzed the controllability of a coal mill system modeled as a nonlinear differential system, demonstrating practical implications for industrial applications.

This paper aims to bridge these theoretical gaps by extending the controllability analysis of generalized impulsive systems. Unlike previous studies that focus on specific subclasses, our approach provides a more comprehensive framework by incorporating both classical and modern advancements in impulsive control theory. In particular, we analyze the interplay between system controllability and impulse-driven perturbations while integrating contemporary theoretical developments. By leveraging recent insights into asymptotic controllability and feedback stabilization, we offer novel conditions that enhance the applicability of impulsive control strategies across various domains.

2. Preliminaries

This section is devoted to mathematical preliminaries related to linear control systems and some concepts from nonlinear functional analysis.

2.1. Controllability of linear systems

In this section, we are going to discuss some basic definitions and facts related to the controllability of linear systems. Consider the linear system without impulses

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ x(t_0) &= x_0 \end{aligned} \tag{2.1}$$

over the interval $[t_0, T]$, $t_0 < T$. Where for each $t \in [t_0, T]$, $x(t) \in \mathbb{R}^n$ is state of the system, $u(t) \in \mathbb{R}^m$ is control, $A(t)$ and $B(t)$ are $n \times n$ and $n \times m$ matrices, respectively.

DEFINITION 1. [22] (Controllability) System (2.1) is said to controllable to a state $x_1 \in \mathbb{R}^n$ if for any time $T (> t_0)$ if there exists a control $u(\cdot) \in L^2([t_0, T]; \mathbb{R}^m)$ such that the solution of the system (2.1) satisfies $x(T) = x_1$. That is,

$$x_1 = \Phi(T, t_0)x_0 + \int_{t_0}^{T} \Phi(T, s)B(s)u(s)ds,$$

where $\Phi(t, s)$ is the transition matrix for the linear system $\dot{x}(t) = A(t)x(t)$. If x_1 is arbitrary, then the system is completely controllable.

The following theorem gives a characterization of complete controllability of the system (2.1).

THEOREM 1. [22] *The following statements are equivalent:*

(i) System (2.1) is completely controllable.

(ii) The operator $C : L^2([t_0, T]; \mathbb{R}^m) \to \mathbb{R}^n$ defined by $Cu(t) = \int_{t_0}^{T} \Phi(T, s)B(s)u(s)ds$ is onto.

(iii) The operator C^* (adjoint of C) defined by $(C^*x)(t) = B^*(t)\Phi^*(T, t)x$ is one-one.

(iv) CC^* is onto.

Note that the operator CC^* is called the controllability grammian of the system (2.1), denoted by $W(t_0,T)$. If this controllability grammian is invertible then the control for the system $u_0(t) = B^*(t)\Phi^*(T,t)W^{-1}(t_0,T)(x_1 - \Phi(T,t_0)x_0)$ steers the system (2.1) from x_0 at time t_0 to a desired state x_1 at $t = T$.

If $A(t)$ and $B(t)$ are time-invariant real matrices, then we have the following simple criterion for controllability.

THEOREM 2. [22] *The following statements are equivalent:*

(i) The time invarying system (2.1) *is completely controllable.*

(ii) $Rank([B|AB|A^2B|\cdots|A^{n-1}B]) = n$ *(Kalmann Condition).*

(iii) No eigen vector of A^T *lies in kernel of* B^T.

(iv) $Rank(A - \lambda IB) = n$ *for every eigen value* λ *of* A.

2.2. Some concepts from nonlinear functional analysis

In this section, we introduce some fundamental concepts from nonlinear functional analysis.

DEFINITION 2. [23] Let X be a real Banach space. Let $Lip(X)$ be the set of all operators $N : X \to X$ which satisfy Lipschitz condition, that is there exist $\alpha > 0$ such that $||Nx - Ny|| \leqslant \alpha||x - y||$, for all $x, y \in X$.

For $N \in Lip(X)$, define norm $||N||^* = sup\frac{||Nx-Ny||}{||x-y||}$ for all $x, y \in X$, $x \neq y$.

DEFINITION 3. [23] Let H be the real Hilbert space. Let $\mathscr{M}(H)$ be the set of all operators on H such that there exist $\alpha > 0$ such that $< Nx - Ny, x - y > \geqslant \alpha||x - y||^2$ for all $x, y \in H$.

For each $N \in \mathscr{M}(H)$ define

$$\mu(N) = inf_{x,y \in H, x \neq y} \frac{< Nx - Ny, x - y >}{||x - y||^2}.$$

The operator N is monotone (strongly monotone) if $\mu(N) \geqslant 0$ $(\mu(N) > 0)$.

We have the following note on Lipchitzian and monotone operators. Let H be a Hilbert space then

(i) $Lip(H) \subset \mathscr{M}(H)$.

(ii) $F \in Lip(H)$ implies $-||F||^* \leqslant \mu(F) \leqslant ||F||^*$.

(iii) $F, G \in Lip(H)$ implies $||FG||^* \leqslant ||F||^*||G||^*$.

DEFINITION 4. [23] Let X be a Banach space, and let X^* be its dual. Then the operator $F : X \to X^*$ is coercive if

$$\lim_{||x||\to\infty} \frac{(Fx,x)}{||x||} = \infty.$$

Here, (y,x), for $y \in X^*$ and $x \in X$, represents the evolution of y on x. In case X is Hilbert space $(y,x) = \langle y,x\rangle$ (inner product of y and x).

DEFINITION 5. [23] Let X be a Banach space and X^* be its dual then the operator $F : X \to X^*$ is to be of type(M) if the following conditions hold:

(a) If the sequence $\{x_n\}$ in X converges weakly to x in X, $\{Fx_n\}$ converges weakly to y in X^* and $\lim_n \sup(Fx_n,x_n) \leqslant (y,x)$ then $Fx = y$.

(b) F is continuous from a finite-dimensional subspace of X to X^* endowed with a weak topology.

THEOREM 3. [23] *Let $K \in \mathscr{M}(H)$ be continuous and $N \in Lip(H)$, $\mu(N) > 0$. If $(\mu(K)+\mu(N)||N||^{*^{-2}}) > 0$ then $I+KN$ is invertible with $[I+KN]^{-1} \in Lip(H)$ and*

$$||(I+KN)^{-1}||^* \leqslant \frac{1}{\mu(N)(\mu(K)+\mu(N)||N||^{*^{-2}})}.$$

THEOREM 4. [23] *Let X be Banach space and let $G : X \to X^*$ be Lipschitz on X with $||G||^* < 1$. Then the operator $N = I+G$ is invertible, N^{-1} is Lipschitz on X and $||N^{-1}||^* \leqslant \frac{1}{1-||G||^*}$.*

THEOREM 5. [23] *Let X be the Banach space and $F : X \to X^*$ be of type(M). If F is coercive, then the range of F is all of X^*.*

THEOREM 6. [23] *Let T be a continuous operator on a Banach space X such that there exists a positive number $n \geqslant 1$ such that $||T^n x - T^n y|| \leqslant k||x-y||$ for all $x,y \in X$ and for some positive number $k < 1$. Then T has a unique fixed point.*

When $n = 1$, the result becomes the Banach contraction principle.

3. Controllability of generalized linear impulsive systems

In this section, we are going to derive sufficient conditions for the controllability of the generalized impulsive system (1.1). To derive sufficient conditions, we first assume that all f_k's are identically equal to zero. By assuming all f_k's are zero the system (1.1) becomes:

$$\begin{aligned}
\dot{x}(t) &= A(t)x(t)+B_k(t)u(t) \quad t\in[t_0,T],\ t\neq t_k,\ k=1,2,\dots,p+1,\ t_{p+1}=T\\
x(t_0) &= x_0\\
x(t_k^+) &= [I_n + D^k u(t_k)]x(t_k^-).
\end{aligned} \tag{3.1}$$

The system (3.1) is a linear impulsive system with different B_k's. We first derive conditions for controllability of the linear impulsive system (3.1), subsequently we analyze the controllability of system (1.1) by assuming the system (3.1) is controllable.

One can easily varify that the solution of (3.1) in the interval $[t_0, T]$ is given by

$$\begin{aligned} x(t) = & \left(\prod_{i=1}^{k-1} I_n + D^i u(t_i) \right) \Phi(t,t_0) x_0 \\ & + \sum_{j=j}^{k-1} \left(\prod_{i=j}^{k-1} (I_n + D^i u(t_i)) \right) \int_{t_{j-1}}^{t_j} \Phi(t,s) B_j(s) u(s) ds \\ & + \int_{t_{k-1}}^{t} \Phi(t,s) B_k(s) u(s) ds, \end{aligned} \tag{3.2}$$

for all $t \in [t_{k-1}, t_k)$. Where, $\Phi(t,s)$ is the transition matrix of the system $\dot{x}(t) = A(t)x(t)$.

Various approaches are available to check the controllability of the system (3.1) and to design the particular open-loop control $u(t)$ that renders (3.1) controllable. One of the approaches is to observe the system up to the last impulse t_p and then find then check the controllability of the system (3.1) in the sub-interval $[t_p, T]$. That is $u(t) = 0$ for all $t \in [t_0, t_p]$ and $u(t_k) = 0$ for all $k = 1, 2, \cdots, p$. In this case solution (3.2) becomes

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_p}^{t} \Phi(t,s) B_{p+1}(s) u(s) ds. \tag{3.3}$$

At $t = T$ the solution (3.3) becomes:

$$x(T) = \Phi(T,t_0)x_0 + \int_{t_p}^{T} \Phi(T,s) B_{p+1}(s) u(s) ds. \tag{3.4}$$

To check controllability and derive control $u(t)$ we define operators $C_{p+1} : L^2([t_p, T]) \to \mathbb{R}^n$ by $C_{p+1}u(t) = \int_{t_p}^{T} \Phi(T,s) B_{p+1}(s) u(s) ds$ and $C^*_{p+1} : \mathbb{R}^n \to L^2([t_p, T])$ by $(C^*_{p+1}x)(t) = B^*_{p+1}\Phi^*(T,t)x$. Note that C^*_{p+1} is an adjoint of C_{p+1}. Now we have the following theorem about controllability of the system (3.1).

THEOREM 7. *System (3.1) is completely controllable over the interval $[t_0, T]$ if*

(i) The operator C_{p+1} is onto.

(ii) The operator C^*_{p+1} is one-one.

(iii) $C_{p+1}C^*_{p+1}$ is non-singular.

One can easily prove this theorem using the concept of the finite-dimensional functional analysis.

The operator $C_{p+1}C^*_{p+1}$ is called controllability grammian for the system (3.1) over the interval $[t_p, T]$ denoted by $W(t_p, T)$. If this controllability grammian is invertible, then we define the controller $u(t)$ as

$$u(t) = \begin{cases} B^*_{p+1}(t)\Phi^*(T,t)W^{-1}(t_p,T)(x_1 - \Phi(T,t_0)x_0), & \text{if } t \in [t_p, T] \\ 0, & \text{otherwise} \end{cases}$$

where x_1 is the desired final state of the system. Plugin the controller $u(t)$ in (3.4), the state at $t=T$ becomes

$$x(T) = \Phi(T,t_0)x_0 + \int_{t_p}^{T} \Phi(T,s)B_{p+1}(s)B^*_{p+1}(t)\Phi^*(T,t)W^{-1}(t_p,T)[x_1 - \Phi(T,t_0)x_0]ds,$$
$$= \Phi(T,t_0)x_0 + (C_{p+1}C^*_{p+1})W^{-1}(t_p,T)[x_1 - \Phi(T,t_0)x_0] = x_1.$$

Thus, this controller $u(t)$ drives the trajectory of the system to given initial state x_0 at $t=t_0$ to the desired final state x_1 at $t=t_1$.

If the system is time-invariant, then we have the following statement.

THEOREM 8. *The time invarying system* (3.1) *is completely controllable over the interval* $[t_0,T]$ *if* $Rank([B_{p+1}|AB_{p+1}|A^2B_{p+1}|\cdots|A^{n-1}B_{p+1}]) = n$ *(Kalmann Condition).*

Proof of Theorem 8 is on the same line as the proof of Theorem 2.

However, we have derived control for the system (3.1), which is applicable only on the final subinterval. Still, there exist many practical situations where we could not implement the controller in the last subinterval $[t_p,T]$. Because to control the system, a huge amount of potential is required during a small time interval, which may leads to failure of the system or it will affect the lifespan of the system. So it is always desirable to apply control as early as possible that is in any other subinterval $[t_{i-1},t_i]$ and it can again bring back the system as desired state by reapplying control in any other subinterval $[t_{j-1},t_j]$ $(i<j)$ if, any aberration from the expected behaviour of the system. So we discuss the controllability of the system in any of the subintervals $[t_{i-1},t_i)$.

Over any subinterval $[t_{i-1},t_i]$ let $C_i : L^2([t_{i-1},t_i],\mathbb{R}^m) \to \mathbb{R}^n$ be defined by $C_iu = \int_{t_{i-1}}^{t_i} \Phi(t_i,s)B_i(s)u(s)ds$ then we have following lemma establishes the controllability of the system (3.1) in terms of the operator C_i for $i=1,2,\cdots,p$.

LEMMA 1. *The system* (3.1) *can be steered from any initial state* $x_0 \in \mathbb{R}^n$ *to desired final state* $x_1 \in \mathbb{R}^n$ *during time interval* $[t_0,T]$ *if* $\Phi(t_i,T)\big[\prod_{t_{i-1}<t_k<T}(I_n + D^ku(t_k))\big]^{-1}x_1 \in R(C_i) + span\{\Phi(t_i,t_0)x_0\}$ *for any* $i=1,2,\cdots,p$, *where* $R(C_i)$ *denotes the range of the operator* C_i.

Proof. Since, $\Phi(t_i,T)\big[\prod_{t_{i-1}<t_k<T}(I_n+D^ku(t_k))\big]^{-1}x_1 \in R(C_i) + span\{\Phi(t_i,t_0)x_0\}$ for any $i=1,2,\cdots,p$. Therefore there exist $u_i(\cdot) \in L^2([t_{i-1},t_i],\mathbb{R}^m)$ and $n-$ vector α such that

$$x_1 = \left(\prod_{t_{i-1}<t_k<T}\big(I_n + D^ku(t_k)\big)\right)\Phi(T,t_i)\left[\int_{t_{i-1}}^{t_i}\Phi(t_i,s)B_i(s)u_i(s)ds + \alpha\Phi(t_i,t_0)x_0\right]$$
$$= \left(\prod_{t_{i-1}<t_k<T}\big(I_n + D^ku(t_k)\big)\right)\int_{t_{i-1}}^{t_i}\Phi(T,s)B_i(s)u_i(s)ds)$$
$$+\alpha\left(\prod_{t_{i-1}<t_k<T}\big(I_n + D^ku(t_k)\big)\right)\Phi(T,t_0)x_0. \tag{3.5}$$

To define control $u(\cdot) \in L^2([t_0,T],\mathbb{R}^m)$ first choose $u(t_k)$ is such that $\prod_{t_0<t_k<t_{i-1}}(I_n + D^k u(t)) = \alpha I_n$ for all $k = 1,2,\cdots,p$. This is always possible, as $(I_n + D^k u(t))$ is a diagonal matrix. For the rest of the domain, we define control as follows:

$$u(t) = \begin{cases} 0; & \text{if } t \in \cup_{k=1,k\neq i}^{p}(t_{k-1},t_k) \cup (t_p,T] \\ u_i(t); & \text{if } t \in (t_{i-1},t_i). \end{cases} \tag{3.6}$$

Plugging this control (3.6) in equation (3.2) we get

$$\begin{aligned} x(t) =& \left(\prod_{t_0<t_k<t} \left(I_n + D^k u(t_k)\right)\right)\Phi(t,t_0)x_0 \\ &+ \int_{t_{i-1}}^{t_i} \left(\prod_{t_{i-1}<t_k<t} \left(I_n + D^k u(t_k)\right)\right)\Phi(t,s)B_i(s)u_i(s)ds. \end{aligned}$$

Evaluating at $t = T$ we get

$$\begin{aligned} x(T) =& \left(\prod_{t_0<t_k<T} \left(I_n + D^k u(t_k)\right)\right)\Phi(T,t_0)x_0 \\ &+ \int_{t_{i-1}}^{t_i} \left(\prod_{t_{i-1}<t_k<T} \left(I_n + D^k u(t_k)\right)\right)\Phi(T,s)B_i(s)u_i(s)ds, \\ =& \,\alpha\left(\prod_{t_{i-1}<t_k<T} \left(I_n + D^k u(t_k)\right)\right)\Phi(T,t_0)x_0 \\ &+ \int_{t_{i-1}}^{t_i} \left(\prod_{t_{i-1}<t_k<T} \left(I_n + D^k u(t_k)\right)\right)\Phi(T,s)B_i(s)u_i(s)ds, \end{aligned}$$

as $(I_n + D^k u(t_k))$ are constant diagonal matrices. Therefore, $x(T) = x_1$ and thus $u(\cdot)$ defined by (3.6) steers the given initial state x_0 to the desired final state x_1. Hence, the proof of the lemma follows. □

LEMMA 2. *The control* $u(t) \in L^2([t_0,T],\mathbb{R}^m)$ *defined by*

$$u(t) = \begin{cases} 0, & \text{if } t \notin [t_{i-1},t_i] \\ u_i(t) & \text{if } t \in [t_{i-1},t_i] \end{cases} \tag{3.7}$$

steers the system (3.1) *from given initial state* $x_0 \in \mathbb{R}^n$ *at* $t = t_0$ *to desire final state* $x_1 \in \mathbb{R}^n$ *where,* $u_i(t) = B_i^*(t)\Phi^*(t_i,t)W^{-1}(t_{i-1},t_i)\Big[\Phi(t_i,T)\Big(\prod_{t_{i-1}<t_k<T}\left(I_n + D^k u(t_k)\right)\Big)^{-1} x_1 - \Big(\prod_{t_0<t_k<t_i}\left(I_n + D^k u(t_k)\right)\Big)\Phi(t_i,t_0)x_0\Big]$.

Proof. Plugging the given controller $u(t)$ given by (3.7) in (3.2) and evaluting the system at $t = T$, we get $x(T) = x_1$

$$
\begin{aligned}
x(T) &= \Phi(T,t_0)x_0 + \int_{t_{i-1}}^{t_i} \Phi(T,s)B_i(s)u_i(s)ds \\
&= \Phi(T,t_0)x_0 + \Phi(T,t_i)\int_{t_{i-1}}^{t_i} \Phi(t_i,s)B_i(s)B_i^*(s)\Phi^*(t_i,s)W^{-1}(t_{i-1},t_i) \\
&\quad \times \Big[\Phi(t_i,T)x_1 - \Phi(t_i,t_0)x_0\Big] \\
&= \Phi(T,t_0)x_0 + \Phi(T,t_i)(C_iC_i^*)W^{-1}(t_{i-1},t_i)\Big[\Phi(t_i,T)x_1 - \Phi(t_i,t_0)x_0\Big] \\
&= \Phi(T,t_0)x_0 + x_1 - \Phi(T,t_0)x_0 = x_1.
\end{aligned}
$$

Therefore, the controller $u(t)$ defined by (3.7) steers the system (3.1) from the given initial state x_0 to the desired final state x_1. This completes the proof of the lemma. □

The next theorem gives sufficient conditions for the controllability of the impulsive system (3.1) in terms of the operator C_i.

THEOREM 9. *The system* (3.1) *is completely controllable over the interval* $[t_0,T]$ *if one of the following condition holds:*

(i) The operator C_i *is onto.*

(ii) The operator C_i^* *is one-one.*

(iii) The controllability grammian $W(t_{i-1},t_i)$ *is invertible.*

Proof. From the lemma 1, it is clear that the system can be steered from any initial state x_0 to desired final state x_1 if $\Phi(t_i,T)\big[\prod_{t_{i-1}<t_k<T}(I_n + D^k u(t_k))\big]^{-1}x_1 \in R(C_i) + span\{\Phi(t_i,t_0)x_0\}$. Therefore the system is controllable if $\Phi(T,t_i)\big[\prod_{t_{i-1}<t_k<T}(I_n + D^k u(t_k))\big]\big(R(C_i) + span\{\Phi(t_i,t_0)x_0\}\big) = n$. Equivalently,

$$
\Big[\prod_{t_{i-1}<t_k<T}(I_n + D^k u(t_k))\Big]R(C_i) = n.
$$

Choosing control function $u(t)$ defined by (3.7) with $D^k u(t_k)$ for $k = i, i+1, \cdots, p$ such that $(I_n + D^k u(t_k))$ is invertible, range condition leads to $R(C_i) = n$. This is possible if C_i is onto.

We know that C_i is onto if and only if C_i^* is one-one and which will imply the invertibility of the controllability grammian $W(t_{i-1},t_i)$. This completes the proof of the theorem. □

Now we have the following observation regarding the controllability of the impulsive system (3.1).

REMARK 1. (1) The conditions in Theorem 9 are sufficient but not necessary, that is, if the system (3.1) is completely controllable, then there is no guarantee that any of the conditions in the theorem hold.

(2) In impulsive system (3.1), null controllability is a weaker condition than complete controllability, unlike a non-impulsive linear system. This is because 0 is always lies in $R(C_i) + span\{\Phi(t_i,t_0)x_0\}$.

THEOREM 10. *If $A(t)$ and $B_k(t)$ are time invariant matrices, then the system is completely controllable if $Rank([B_i|AB_i|\cdots|A^{n-1}B_i]) = n$ for at least one $i = 1,2,\cdots,p$.*

Proof. Let $K = Rank([B_i|AB_i|\cdots|A^{n-1}B_i])$. For time-invariant matrices A and B_i, one can easily show that $Range(K) = Range(C_i)$. This leads to surjectivity of the operator C_i and using the theorem 9 we get complete controllability of the system (3.1). □

4. Controllability of generalized semilinear impulsive systems

In this section, we discuss the controllability of the semilinear impulsive system (1.1) in terms of the solvability of the coupled equation

$$\begin{aligned} e_1 &= u_1 - G_2 e_2 \\ e_2 &= u_2 + G_1 e_1 \end{aligned} \tag{4.1}$$

for some appropriate operators $G_1 : X_1 \to X_2$ and $G_2 : X_2 \to X_1$. Here X_1 and X_2 are Hilbert spaces. Here we are looking for some sufficient conditions for the complete controllability of the system (1.1). So choose $u(t)$ such that $D^k u(t) = 0$ for all $k = 1,2,\cdots,p$ and $u(t) = 0$ for $t \notin [t_{i-1},t_i]$ that is, the control is chosen only in the time interval $[t_{i-1},t_i]$. Therefore for each $t \in [t_p,T]$ the solution of the system is given by

$$\begin{aligned} x(t) = \Phi(t,t_0)x_0 + \sum_{i=1}^{P} \int_{t_{i-1}}^{t_i} \Phi(t,s) f_i(s,x(s))ds + \int_{t_p}^{t} \Phi(t,s) f_{p+1}(s,x(s))ds \\ + \int_{t_{i-1}}^{t_i} \Phi(t,s)B_i(s)u(s)ds. \end{aligned} \tag{4.2}$$

Taking, $F(t,x(t)) = f_k(t,x(t))$ for each $t \in [t_{k-1},t_k)$, $k = 1,2,\cdots,p+1$ then the equation (4.2) becomes:

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,s)F(s,x(s))ds + \int_{t_{i-1}}^{t_i} \Phi(t,s)B_i(s)u(s)ds. \tag{4.3}$$

At $t = T$ the equation (4.3) becomes:

$$x(T) = \Phi(T,t_0)x_0 + \int_{t_0}^{T} \Phi(T,s)F(s,x(s))ds + \int_{t_{i-1}}^{t_i} \Phi(T,s)B_i(s)u(s)ds.$$

Suppose that the system (1.1) with the control $u(t)$, which is active only in the interval (t_{i-1},t_i). Therefore we have

$$x_1 = \Phi(T,t_0)x_0 + \int_{t_0}^{T} \Phi(T,s)F(s,x(s))ds + \int_{t_{i-1}}^{t_i} \Phi(T,s)B_i(s)u(s)ds.$$

That means

$$x_1 - \Phi(T,t_0)x_0 - \int_{t_0}^{T} \Phi(T,s)F(s,x(s))ds = \int_{t_{i-1}}^{t_i} \Phi(T,s)B_i(s)u(s)ds$$
$$x_1 - \Phi(T,t_0)x_0 - \int_{t_0}^{T} \Phi(T,s)F(s,x(s))ds = \Phi(T,t_i)\int_{t_{i-1}}^{t_i} \Phi(t_i,s)B_i(s)u(s)ds.$$

Taking $v = \Phi(t_i,T)\left[x_1 - \Phi(T,t_0)x_0 - \int_{t_0}^{T} \Phi(T,s)F(s,x(s))ds\right]$ we get the above equation $C_i u(t) = v$. Therefore, a suitable choice of control $u(t)$ that satisfies the equation is given by

$$u(t) = C_i^*(C_iC_i^*)^{-1}v. \tag{4.4}$$

Therefore, the control $u(t)$ in the time interval $[t_{i-1},t_i]$ is given by

$$u(t) = C_i^*(C_iC_i^*)^{-1}\left[\Phi(t_i,T)\left(x_1 - \Phi(T,t_0)x_0 - \int_{t_0}^{T} \Phi(T,s)F(s,x(s))ds\right)\right]. \tag{4.5}$$

Without loss of generality, we can take $x_0 = 0$ due to the following theorem.

THEOREM 11. *The system* (1.1) *is controllable by force only if for every* $x_1 \in \mathbb{R}^n$ *there is a control* $u(t) \in L^2([t_0,T],\mathbb{R}^m)$ *that guides* 0 *to* x_1.

We can easily prove Theorem 11 using similar arguments as in Proposition 2.2 of [24]. Thus, equations (4.3) and (4.5) can be rewritten as

$$x(t) = \int_{t_0}^{t} \Phi(t,s)F(s,x(s))ds + \int_{t_{i-1}}^{t_i} \Phi(t,s)B_i(s)u(s)ds, \tag{4.6}$$

$$u(t) = u(t) = C_i^*(C_iC_i^*)^{-1}\left[\Phi(t_i,T)\left(x_1 - \int_{t_0}^{T} \Phi(T,s)F(s,x(s))ds\right)\right]. \tag{4.7}$$

Thus, the controllability of the system (1.1) leads to the solvability of the coupled equations (4.6) and (4.7). Let $X_1 = L^2([t_{i-1},t_i],\mathbb{R}^m)$ and $X_2 = L^2([t_{i-1},t_i])$ and define operators $K,N : X_2 \to X_2$, $H_i : X_1 \to X_2$ and $R_i : X_2 \to X_1$ as follows:

$$(Kx)(t) = \int_{t_0}^{t} \Phi(t,s)x(s)ds$$
$$(Nx)(t) = F(t,x(t))$$
$$(H_iu)(t) = \int_{t_{i-1}}^{t_i} \Phi(t,s)B_i(s)u(s)ds$$
$$(R_ix)(t) = C_i^*(C_iC_i^*)^{-1}\Phi(t_i,T)\int_{t_0}^{T} \Phi(t_i,s)x(s)ds.$$

Clearly, the operators K, H_i, and R_i are continuous linear operators. The operator N is nonlinear.

With these notation, equations (4.6) and (4.7) can be written as pair of operator equations:

$$x = KNx + H_i u \tag{4.8}$$

$$u = u_1 - R_i Nx \tag{4.9}$$

where, $u_1(t) = C_i^*(C_iC_i^*)^{-1}\Phi(t_i, T)x_1$. Now we have the following theorem based on the controllability of (1.1) in terms of the solvability of coupled equations (4.8) and (4.9).

THEOREM 12. *The system* (1.1) *is completely controllable if equations* (4.8) *and* (4.9) *are uniquely globally solvable.*

Proof. Assuming the equations (4.8) and (4.9) are universally globally solvable, therefore there exists a pair (x^*, u^*) such that $x^* = KNx^* + H_i u^*$ and $u^* = u_1 - R_i N x^*$. Define control $u(t) \in L^2([t_0, T], \mathbb{R}^m)$ by

$$u(t) = \begin{cases} u^*(t), & t \in (t_{i-1}, t_i) \\ 0, & \text{otherwise.} \end{cases} \tag{4.10}$$

Putting the control in equation (4.3), we get $x(T) = x_1$, and since x_1 is arbitrarily chosen, the system (1.1) is completely controllable. This completes the proof of the theorem. □

From this theorem, we can conclude that the controllability of the impulsive system (1.1) reduces to the solvability of the coupled equations (4.8) and (4.9).

The next lemma describes the solvability of coupled equations (4.8) and (4.9) into the invertibility of the operator $(I - KN)$.

LEMMA 3. *The coupled equations* (4.8) *and* (4.9) *are uniquely globally solvable if and only if the operator* $(I - KN)$ *is invertible.*

Proof. If the coupled equations (4.8) and (4.9) are uniquely globally solvable therefore for every $u^* \in X_1$ there exists a unique $(x^*, u^*) \in (X_2, X_1)$ such that $x^* = KNx^* + H_i u^*$ and $u^* = u_1 - R_i N x^*$. This leads to invertibility of the operator $(I - KN)$.

Conversely, if the operator $(I - KN)$ is invertible then for each $u \in X_1$ the equation $(I - KN)x = H_i u$ has unique solution say x^* and choosing $u^* = u_1 - R_i N x^*$, (x^*, u^*) is solution of coupled equations (4.8) and (4.9). Hence, coupled equations (4.8) and (4.9) are uniquely globally solvable. □

5. Controllability conditions under Lipschitzian nonlinearity

In this section, we are going to derive sufficient conditions for the solvability of coupled equations (4.8) and (4.9).

From now on, we fix $b_i = \sup_{t_{i-1} \leqslant t \leqslant t_i} ||B_i(t)||$ and $b = \max\{b_i; i = 1, 2, \cdots, p+1\}$ and make the following assumptions for the discussion of the complete controllability of the system (1.1).

(A1) Let, the transition matrix for the linear system $\Phi(t,s)$ is such that $||\Phi(t,s)|| \leqslant h(t,s)$, where $h(\cdot,\cdot) : [t_0,T] \times [t_0,T] \to \mathbb{R}^+$ is the bounded function satisfying

$$\left[\int_{t_0}^{T}\int_{t_0}^{t} h^2(t,s)dsdt\right]^{\frac{1}{2}} = k < \infty.$$

(A2) Functions $f_i : [t_{i-1},t_i] \times \mathbb{R}^n$ for $i = 1, 2, \cdots, p+1$ are measurable with respect to the first argument and continuous with respect to the second argument. Moreover, there exist positive numbers α_i such that

$$||f_i(t,x) - f_i(t,y)|| \leqslant \alpha_i ||x - y||$$

and let, $\alpha = \max\{\alpha_i; i = 1, 2, \cdots, p+1\}$.

LEMMA 4. *The bounds for the operators K, H_i and R_i under the assumptions (A1)–(A2) are estimated as $||K|| \leqslant k$, $||H_i|| \leqslant bk_i \triangleq h_i$, where $k_i = \left[\int_{t_0}^{T}\int_{t_{i-1}}^{t_i} h^2(t,s)dsdt\right]^{\frac{1}{2}}$ and $||R_i|| \leqslant bl_i^2 c_i \triangleq \gamma_i$, where $c_i = ||(C_iC_i^*)^{-1}||$ and $l_i = \left[\int_{t_0}^{T} h^2(t_i,s)ds\right]^{\frac{1}{2}}$. Further, the nonlinear operator N is Lipschitz continuous with Lipschitz constant α.*

Proof. To compute a bound for K consider,

$$\begin{aligned}
||Kx||_{X_2}^2 &= \int_{t_0}^{T} ||Kx(t)||^2 dt \leqslant \int_{t_0}^{T}\left(\int_{t_0}^{t}\Phi(t,s)x(s)ds\right)^2 dt \\
&\leqslant \int_{t_0}^{T}\left(\int_{t_0}^{t}\Phi(t,s)ds\right)^2\left(\int_{t_0}^{t}x(s)ds\right)^2 dt \quad \text{(applying Holder's inequality)} \\
&\leqslant \left(\int_{t_0}^{T}\int_{t_0}^{t} h^2(t,s)dsdt\right)||x||_{X_2}^2 = k^2||x||_{X_2}^2.
\end{aligned}$$

Therefore, $||K|| \leqslant k$.

To compute a bound for H_i consider,

$$\begin{aligned}
||H_iu||_{X_2}^2 &= \int_{t_0}^{T} ||H_ix(t)||^2 dt \leqslant \int_{t_0}^{T}\left(\int_{t_{i-1}}^{t_i}\Phi(t,s)B_i(s)u(s)ds\right)^2 dt \\
&\leqslant \int_{t_0}^{T}\left(\int_{t_{i-1}}^{t_i}||\Phi(t,s)||^2 ds\right)\left(\int_{t_0}^{t}||B_i(s)u(s)||^2 ds\right)dt \\
&\qquad\qquad \text{(applying Holder's inequality)} \\
&\leqslant k_i^2 b_i^2 ||u||_{X_1}^2 \leqslant k_i^2 b^2 ||u||_{X_1}^2.
\end{aligned}$$

Therefore, $||H_i|| \leqslant bk_i = h_i$. Finally, to compute bounds for R_i consider

$$\begin{aligned}
||R_i x||^2_{X_1} &= \int_{t_{i-1}}^{t_i} ||R_i x(t)||^2 dt \\
&= \int_{t_{i-1}}^{t_i} ||(C_i^*(C_iC_i^*)^{-1}) \int_{t_0}^{T} \Phi(t_i,s)x(s)ds||^2 dt \\
&= \int_{t_{i-1}}^{t_i} ||(B_i^*(t)\Phi^*(t_i,t)(C_iC_i^*)^{-1}) \int_{t_0}^{T} \Phi(t_i,s)x(s)ds||^2 dt \\
&\leqslant \int_{t_{i-1}}^{t_i} ||B_i^*(t)\Phi^*(t_i,t)||^2 dt ||(C_iC_i^*)^{-1}||^2 \int_{t_{i-1}}^{t_i} \int_{t_0}^{T} ||\Phi(t_i,s)||^2 ds dt ||x||^2_{X_2} \\
&\leqslant c_i^2 b^2 l_i^4 ||x||^2_{X_2}.
\end{aligned}$$

Therefore, $||R_i|| \leqslant b l_i^2 c_i \triangleq \gamma_i$. Furthermore, for $x,y \in \mathbb{R}^n$ and $t \in [t_0,T]$,

$$||Nx(t) - Ny(t)|| = ||F(t,x) - F(t,y)|| = ||f_i(t,x) - f_i(t,y)|| \leqslant \alpha_i ||x-y|| \leqslant \alpha ||x-y||.$$

Hence, N is Lipschitz continuous with Lipchitz constant α. This completes the proof of the lemma. □

The following theorem gives sufficient conditions for the invertibility of the operator $(I-KN)$.

THEOREM 13. *The operator $I-KN$ is invertible if Assumptions* (A1)–(A2) *hold. Moreover $(I-KN)$ is Lipschitz continuous with Lipschitz constant $\frac{1}{1-k\alpha}$ if $k\alpha < 1$.*

Proof. To show that $I-KN$ is invertible, we first show that the equation $x_{n+1} = KNx_n + v$ has a unique solution.

For $x,y \in X_2$

$$||KNx - KNy|| \leqslant \int_{t_0}^{t} ||\Phi(t,s)||\ ||F(s,x(s)) - F(s,y(s))||ds \leqslant m\alpha(T-t_0)||x-y||$$

$$\begin{aligned}
&||(KN)^2x - (KN)^2y|| \\
&= ||(KN)\Big(\int_{t_0}^{t} \Phi(t,s)F(s,x(s))ds\Big) - (KN)\Big(\int_{t_0}^{t} \Phi(t,s)F(s,y(s))ds\Big)|| \\
&= || \int_{t_0}^{t} \Phi(t,s)F\left(s, \Big(\int_{t_0}^{\tau_1} \Phi(s,\tau_1)F(\tau_1,x(\tau_1))d\tau_1\Big)\right)ds \\
&\qquad - \int_{t_0}^{t} \Phi(t,s)F\left(s, \Big(\int_{t_0}^{\tau_1} \Phi(s,\tau_1)F(\tau_1,x(\tau_1))d\tau_1\Big)\right)ds|| \\
&\leqslant \alpha \int_{t_0}^{t} ||\Phi(t,s)|| \int_{t_0}^{\tau_1} ||\Phi(s,\tau_1)||\ ||F(\tau_1,x(\tau_1)) - F(\tau_1,y(\tau_1))||d\tau_1 ds \\
&\leqslant \alpha^2 \int_{t_0}^{t} \int_{t_0}^{\tau_1} ||\Phi(t,s)||\ ||\Phi(s,\tau_1)||d\tau_1 ds||x-y||
\end{aligned}$$

$$\leqslant \alpha^2 \left[\int_{t_0}^{T} \int_{t_0}^{\tau_1} h^2(t,s) ds dt \right] ||x-y||$$
$$\leqslant \frac{\alpha^2 m^2 (T-t_0)^2}{2} ||x-y|| \quad (m \text{ is uperbound for } h).$$

Using mathematical induction,

$$||(KN)^n x - (KN)^n y|| \leqslant \frac{\alpha^n m^n (T-t_0)^n}{n!} ||x-y|| \quad (m \text{ is uperbound for } h).$$

Since, $\frac{\alpha^n m^n (T-t_0)^n}{n!} \to 0$ as $n \to \infty$ therefore there exist n_0 such that $\frac{\alpha^{n_0} m^{n_0} (T-t_0)^{n_0}}{n_0!} < 1$. This means $(KN)^{n_0}$ is a contraction and thus by the generalized Banach fixed point theorem the equation $x_{n+1} = KNx_n + v$ has a unique solution. Therefore each $v \in X_2$ there exist unique $x_v \in X_2$ such that $x_v = KNx_v + v$. This means for every $v \in X_2$ there exists a unique $x_v \in X_2$ such that $(I-KN)x_v = v$. Therefore, the map $(I-KN) : X_2 \to X_2$ is invertible.

To show Lipschitz continuity of the operator $(I-KN)^{-1}$, consider

$$\begin{aligned}
& ||(I-KN)^{-1} v_1 - (I-KN)^{-1} v_2|| \\
&= ||x_{v_1} - x_{v_2}|| \\
&= ||KNx_{v_1} + v_1 - KNx_{v_2} - v_2|| \\
&\leqslant ||KNx_{v_1} - KNx_{v_2}|| + ||v_1 - v_2|| \\
&\leqslant ||\int_{t_0}^{t} \Phi(t,s) F(s, x_{v_1}(s)) ds - \int_{t_0}^{t} \Phi(t,s) F(s, x_{v_2}(s)) ds|| + ||v_1 - v_2|| \\
&\leqslant ||\int_{t_0}^{t} \Phi(t,s) [F(s, x_{v_1}(s)) - F(s, x_{v_2}(s))] ds|| + ||v_1 - v_2|| \\
&\leqslant ||K||\, ||Nx_{v_1} - Nx_{v_2}|| + ||v_1 - v_2|| \\
&\leqslant k\alpha ||x_{v_1} - x_{v_2}|| + ||v_1 - v_2||
\end{aligned}$$

Hence, $(1-k\alpha)||x_{v_1} - x_{v_2}|| \leqslant ||v_1 - v_2||$ which mean $(I-KN)^{-1}$ is Lipschiz continuous with Lipschitz constant $\frac{1}{(1-k\alpha)}$. Which completes the proof of the theorem. □

THEOREM 14. *Suppose that the linear system* (3.1) *is controllable and assumptions* (A1)–(A3) *are satisfied and* $\frac{\alpha \gamma_i h i}{(1-k\alpha)} < 1$ *then,*

(1) the couple equations (4.8) *and* (4.9) *are globally solvable.*

(2) the control vector $u(t)$ *defined over the interval* $[t_{i-}, t_i]$ *steers zero initial state at* t_0 *to a desired target state* x_1 *at time* T*, can be approximated by iterates* $u^{(n)}(t)$ *defined by*

$$u^{(n)}(t) = C_i^* (C_i C_i^*)^{-1} \left[\Phi(t_i, T) \left(x_1 - \int_{t_0}^{T} \Phi(T,s) F(s, x^{(n)}(s)) ds \right) \right] \tag{5.1}$$

and the state vector approximation in the time interval $[t_0, T]$ *is given by the iterates*

$$x_i^{(n+1)}(t) = \int_{t_0}^{t} \Phi(t,s) F(s, x_i^{(n)}(s)) ds + \int_{t_{i-1}}^{t_i} \Phi(t,s) B_i(s) u^{(n)}(s) ds. \tag{5.2}$$

Proof. Since, assumptions (A2) and (A3) are satisfied therefore by theorem 13 the equation $x = KNx + v$ is uniquely globally solvable for all $v \in X_2$. This mean there exist unique x such that $x = (I - KN)^{-1} H_i u$ for any fixed $H_i u \in X_2$.Therefore, the equation $u = u_1 - R_i N x$ becomes

$$u = u_1 - R_i N(1 - KN)^{-1} H_i u \tag{5.3}$$

and solvability of this equation leads to the solvability of the coupled equations (4.8) and (4.9), which leads to controllability of the semilinear system (1.1).

Define an operator $\mathscr{M} : X_1 \to X_1$ by

$$\mathscr{M} u(t) = \big(u_1 - R_i N(1 - K\alpha)^{-1} H_i u\big)(t)$$

then one can easily show that the equation (5.3) is solvable if and the operator equation $u = \mathscr{M} u$ has unique fixed point.

Since the corresponding linear system is controllable, the controllability grammian $C_i C_i^*$ is invertible, and assuming (A1)–(A3), one can get

$$|||\mathscr{M} u - \mathscr{M} v|| \leqslant \frac{\alpha \gamma_i h_i}{(1 - k\alpha)} ||u - v||$$

and according to hypotheses of the theorem $\frac{\alpha \gamma_i h_i}{(1-k\alpha)} < 1$ therefore by Banach fixed point theorem the operator $\mathscr{M}$ is contraction. Thus, the operator equation (5.3) is uniquely globally solvable.

Furthermore, starting from any initial state $x^0 \in X_1$ the iterates

$$x^{(n+1)} = (I - KN)^{-1} H_i u^{(n)} \tag{5.4}$$

$$u^{(n)} = u_1 - R_i N x^{(n)} \tag{5.5}$$

drives the system (4.2) to desired final state x_1 as $n \to \infty$.

Hence, the iterates for the control $u(t) \in L^2([t_0, T], \mathbb{R}^m)$ that steers the initial state $x_0 \in \mathbb{R}^n$ at t_0 to the desired final state $x_1 \in \mathbb{R}^n$ at time T are given by (5.1) and state of the system at any time $t \in [t_0, T]$ is given by (5.2). This completes the proof of the theorem. □

REMARK 2. When $[I_n + D^k u(t_k)] \neq 0$, controllability of the system (1.1) can also be established by modifying the operators K, H_i and R_i and using similar arguments as mentioned in the theorems 13 and 14.

EXAMPLE 1. Consider the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f_k(t, x(t)) + B_k u(t), \quad t \in [0, 0.5) \cup [0.5, 1), \\ x(0) &= x_0, \\ x(0.5^+) &= (I_2 + d^1 u(0.5) I_2) x(0.5^-) \end{aligned} \tag{5.6}$$

during the interval $[0, 1]$.

Here, the state $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $B_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $f_1(t,x(t)) = \frac{1}{2}\begin{bmatrix} \sin(x_1(T)) \\ \sin(x_2(t)) \end{bmatrix}$ and $f_2(t,x(t)) = \frac{1}{3}\begin{bmatrix} \cos(x_1(t)) \\ \cos(x_2(t)) \end{bmatrix}$. Clearly, the rank of the controllability matrices $[B_1 : AB_1] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $[B_2, AB_2] = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$ are 2, which is equal to the dimensions of the state vector. The nonlinear functions f_k are measurable with respect to t and Lipschitz continuous with respect to the state x.

Assume $u(0.5) = 0$, this means no control is applied at the impulse point. Since the rank of the controllability matrices is equal to the state vector's dimensions, a linear system corresponding to (5.6) is controllable over each subinterval. Moreover, the nonlinear terms are Lipschitz continuous with respect to x, and the nonlinear system (5.6) is controllable over the interval $[0, 0.5]$. The figure 1 shows the controlled state $x(t)$, controller $u(t)$, and phase portrait.

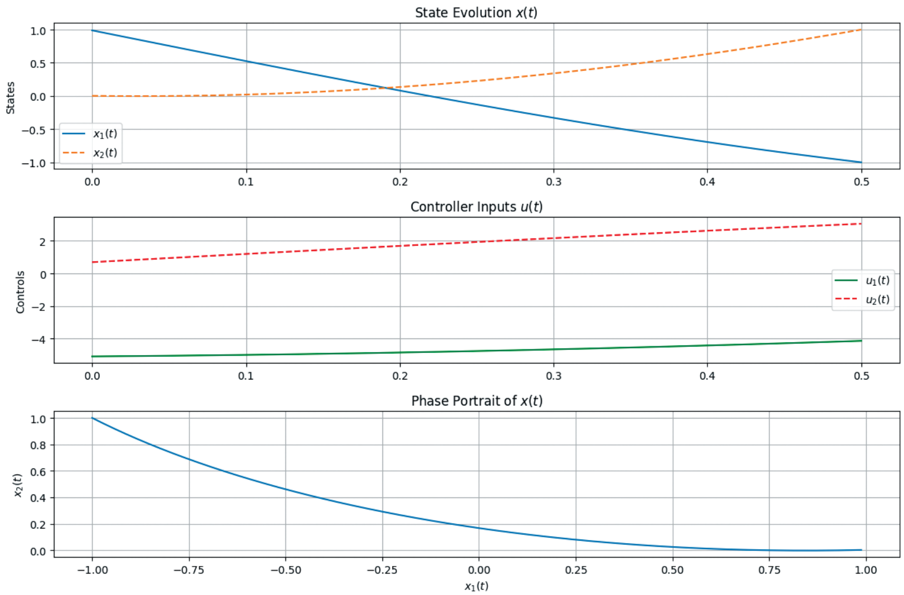

Figure 1: *Controlled State, Controller and Phase Portrait.*

At $t = 0.5$, the state of the system reaches $x(0.5) = [0, 2]'$. In order to drive the system from this state to the desired final state $x_1 = [-1, 0]'$ at $t = 1$, it is necessary to apply a control input over the time interval $[0.5, 1]$. The design and application of this control input follow the same procedure as implemented earlier over the interval $[0, 0.5]$. Specifically, the controller is designed to steer the system dynamics from the current state at $t = 0.5$ to the target state at $t = 1$, ensuring a smooth and accurate transition.

Thus, the control strategy is applied in two stages: initially over $[0, 0.5]$, where the system evolves from the initial condition to the intermediate state $x(0.5) = [0,2]'$, and later over $[0.5,1]$, where the control is used to guide the system from $x(0.5)$ to the desired final state $x_1 = [-1,0]'$. This piecewise application of control enables effective handling of the system's behavior over the entire interval $[0,1]$ and ensures that the terminal condition at $t = 1$ is successfully achieved.

6. Conclusion

In this article, we discuss the controllability of semilinear impulsive systems on a finite-dimensional space. By deriving sufficient conditions for the solvability of coupled equations, we have established the controllability of the system. Under certain assumptions and conditions, we have shown that the system can be steered from an initial state to a desired final state by appropriate controls applied in specific time intervals. The results provide insights into the controllability of complicated impulsive systems with nonlinearities, offering solutions for practical applications in various fields.

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