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SOME PROPERTIES OF LOGARITHMIC p–LAPLACE OPERATORS AND APPLICATIONS

MINH TRIET LUONG*, THI THUY DUNG DUONG AND THI THAO NHI VU

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Abstract. A study of the existence and uniqueness of the weak solution to a class of nonlinear elliptic equations governed by the logarithmic perturbation is offered. We exploit interesting properties of the new modular function involving $L^p\log^\alpha L$-growth and the optimal embedding theorem for Orlicz-Sobolev spaces. We are concerned with these properties in analyzing the existence and uniqueness of the solution by the theory of pseudo-monotone operators proposed in [2, 3] combined with variational methods. Our approach deals not only with problems of the p-Laplacian type but also yields a slight extension of the results for more general differential operators with a similar structure.

1. Introduction

In this paper, we study the following nonlinear boundary value problem of the type: for a given domain $\Omega \subset \mathbb{R}^n$ with $n \geqslant 2$, a given datum $f : \Omega \times \mathbb{R} \to \mathbb{R}$, to find $u : \Omega \to \mathbb{R}$ satisfying

$$-\operatorname{div}\big(\mathscr{A}_{p,\alpha}(\nabla u)\big) = f(x,u) \ \text{ in } \Omega, \ \text{ and } \ u = 0 \text{ on } \partial\Omega, \tag{1}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function; the mapping $\mathscr{A}_{p,\alpha} : \Omega \to \mathbb{R}^n$, for $1 < p < n$, $\alpha \geqslant 0$ is defined by

$$\mathscr{A}_{p,\alpha}(\zeta) := \partial_\zeta\left(\frac{1}{p}|\zeta|^p \log^\alpha(e + |\zeta|)\right), \quad \zeta \in \mathbb{R}^n. \tag{2}$$

An important question that immediately arises is whether such problems are well-posed. To the best of our knowledge, the literature contains various techniques for obtaining general existence results for solutions to nonlinear problems. To mention a few, the studies with the Schauder and Banach Fixed Point Theorem, the Contraction Mapping Principle, or subjectivity theorem for monotone operators, see for instance [10, 12, 14, 20]. It is worth mentioning that the nice subjectivity property related to monotone operators has become an important tool concerning the existence of solutions from a variety of perspectives.

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* Corresponding author.

Here, we shall deal with the existence and uniqueness of weak solutions when the left-hand side is modeled around the p-Laplacian as a reference operator under some appropriate assumptions on a given datum f. To be more specific, in our study, the left-hand side of $(1)_1$ has logarithmic order, nature. When $\alpha = 0$, the interest in studying such a problem is the p-Laplacian. The p-Laplace and p-Laplace type equations are the subjects of several physical modeling and mechanical branches, such as resonance problems, electricity, electro-rheological fluid dynamics, glaciology, elasticity theory, and many others (see for example [5, 8, 15, 19] and references therein). As far as we are concerned in the literature, there are much less known existence and uniqueness results concerning this problem for the general case that involves logarithmic order (1).

To give the reader a key tool underlying the existence theory with pseudo-monotone operators, let us first summarize and exploit the proof idea for the p-Laplacian case. The weak solution $u \in W_0^{1,p}(\Omega)$ to $-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f$ provided

$$\int_\Omega \big(|\nabla u|^{p-2}\nabla u \cdot \nabla v - fv\big)dx = 0, \tag{3}$$

holds for every $v \in W_0^{1,p}(\Omega)$. The existence theorem is the usage of surjectivity results for pseudo-monotone operators, which reads as

THEOREM 1.1. (see [3]) *If $\mathscr{H} : \mathbb{X} \to \mathbb{X}^*$ is a bounded, coercive and pseudo-monotone, then $\mathscr{H}(u) = 0$ has a solution $u \in \mathbb{X}$.*

Here, for notational purposes, we regard $\mathbb{X}$ as a real reflexive Banach space, and $\mathbb{X}^*$ is its dual space in the entirety of the paper. Theorem 1.1 yields the existence of a solution to the classical homogeneous p-Laplace equation, where $\mathscr{H}$ connecting with this problem, is defined by

$$\langle \mathscr{H}(u), v\rangle := \int_\Omega \big(|\nabla u|^{p-2}\nabla u \cdot \nabla v - f(x,u)v\big)dx, \quad v \in W_0^{1,p}(\Omega). \tag{4}$$

This theorem sets forth an important surjectivity result that the famous Lax-Milgram's theorem and the Main Theorem on Monotone Operators will be constructed as consequences. Moreover, it can be applied directly to the model problems involving p-Laplace type operator under appropriate boundary conditions. Inspired by several subsequent developments regarding the solutions to several classes of equations, whose nonlinearity even does not satisfy the standard ellipticity conditions, we are devoted to a more general operator governed by logarithmic order, i.e. $\mathscr{A}_{p,\alpha}$ as in (2), also considered as the borderline case of p-Laplacian.

In the spirit of the Main Theorem on Pseudomonotone Operators stated in Theorem 1.1, we first consider the corresponding operator $\mathscr{H}_{p,\alpha}$ satisfying

$$\langle \mathscr{H}_{p,\alpha}(u), v\rangle := \int_\Omega \big[\mathscr{A}_{p,\alpha}(\nabla u)\cdot \nabla v - f(x,u)v\big]dx, \quad v \in \mathbb{X}, \tag{5}$$

where $\mathbb{X}$ denotes Orlicz-Sobolev space, which plays as a generalization of Sobolev space built upon the Orlicz spaces. Its definition will be clarified in Section 2.

With the wealth of the previous literature, the proof of existence exploits an argument similar to that in [3, 13], however, our approach is somewhat different, as we employ a new modular function, the equivalence between the new norm and the prescribed norm in Orlicz-Sobolev spaces. Furthermore, with an interesting technical comparison, it allows treating the existence and uniqueness results for (1), where the case is more general and has been less investigated. These features make our main results in this paper somewhat interesting and challenging.

Before presenting the main results, we premise some notations and assumptions used throughout this paper. As a preliminary step, we briefly discuss the notation adopted in our results. For $1 < p < n$ and $\alpha \geqslant 0$, let us consider $G_{p,\alpha} : [0,+\infty) \to [0,+\infty)$ a Young function defined by

$$G_{p,\alpha}(\sigma) = \frac{1}{p}\sigma^p \log^\alpha(e+\sigma), \quad \sigma \geqslant 0, \tag{6}$$

and $G^*_{p,\alpha}$ the Young's conjugate of $G_{p,\alpha}$. It is worth noticing that the operator $\mathscr{A}_{p,\alpha}$ in (2) can be rewritten as

$$\mathscr{A}_{p,\alpha}(\zeta) = \partial_\zeta G_{p,\alpha}(|\zeta|), \quad \zeta \in \mathbb{R}^n.$$

Here, we shall carry out the details of the Young function, its conjugate, and the properties between them in Section 2, for the convenience of the reader. Moreover, in this respect, we also connect the Δ_2-Young functions with Orlicz spaces denoted by $L^{G_{p,\alpha}}(\Omega)$, and Orlicz-Sobolev spaces $W^{1,G_{p,\alpha}}(\Omega)$, that is made up of a function $h \in W^{1,1}(\Omega)$ such that $\nabla h \in L^{G_{p,\alpha}}(\Omega)$. For the sake of readability, from now on we will denote by $\mathbb{X} := W^{1,G_{p,\alpha}}(\Omega)$ and $\mathbb{X}_0 := W_0^{1,G_{p,\alpha}}(\Omega)$, its closure of $C_0^\infty(\Omega)$ in $W^{1,G_{p,\alpha}}(\Omega)$, and when needed, further notations will be introduced step by step in the arguments. We devote Section 2 for the reader to have a place to look things up in the details.

Let us now state the main results of this paper via two following theorems, where the existence and uniqueness of weak solutions are presented.

THEOREM 1.2. (Existence) *Let* $p \in (1,n)$, $\alpha \geqslant 0$ *and* $g \in \mathbb{L}^* := L^{G^*_{p,\alpha}}(\Omega)$. *Then, one can find a constant* $\mu_0 \geqslant 0$ *such that if*

$$|f(x,t)| \leqslant g(x) + \mu_0 |t|^{p-1} \log^\alpha(e+|t|), \quad \textit{for all } x \in \Omega,\ t \in \mathbb{R}, \tag{7}$$

and so, there exists at least a weak solution $u \in \mathbb{X}_0$ *to* (1).

Here, for the sake of brevity, we use $\mathbb{L}$ as the Orlicz space $L^{G_{p,\alpha}}(\Omega)$ connected with Young function $G_{p,\alpha}$ and $\mathbb{L} := L^{G^*_{p,\alpha}}(\Omega)$ corresponds to the its Young's conjugate. These notations will be specified in Section 2.

THEOREM 1.3. (Uniqueness) *Let* $p \in (1,n)$ *and* $\alpha \geqslant 0$ *and assume that* $f(\cdot,u) \equiv g \in \mathbb{L}^*$. *Then, the Dirichlet problem* (1) *admits a unique weak solution* $u \in \mathbb{X}_0$.

The plan of our paper is as follows. In the next section, we first briefly recapitulate some standard definitions and preliminary results. In Section 3, we shall introduce the main tool of this paper: the modular and some interesting properties, proving the relation to the corresponding norm in Orlicz-Sobolev spaces in a standard way. The next section is devoted to establishing some important properties of the p-Laplacian with logarithmic order based on the subjectivity theorem for pseudo-monotone operators. Finally, in Section 5 we end up by proving main results stated above.

2. Preparatory and preliminary materials

In this section, we collect some basic definitions and necessary auxiliary results which will be employed later.

2.1. Pseudo-monotone operators

First, by $\left(\mathbb{X}, \|\cdot\|_{\mathbb{X}}\right)$, we denote a general reflexive Banach space and $\mathbb{X}^*$ its dual space. The notation $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathbb{X}^*$ and $\mathbb{X}$. For ease of notation in the argument, we shall make use of $\to$ and $\rightharpoonup$ the norm convergence and weak convergence, respectively. To be more precise, for a sequence $(\varphi_k)_{k\in\mathbb{N}} \subset \mathbb{X}$, one has

$$\varphi_k \to \varphi \Longleftrightarrow \lim_{k\to\infty} \|\varphi_k - \varphi\|_{\mathbb{X}} = 0,$$

and

$$\varphi_k \rightharpoonup \varphi \Longleftrightarrow \lim_{k\to\infty} \langle T, \varphi_k\rangle = \langle T, \varphi\rangle \text{ for any } T \in \mathbb{X}^*.$$

DEFINITION 2.1. (see [3]) Let us consider a continuous operator $\mathscr{H} : \mathbb{X} \to \mathbb{X}^*$. For any sequence $(\varphi_k)_{k\in\mathbb{N}} \subset \mathbb{X}$ and $\varphi \in \mathbb{X}$, we simply write $\varphi_k \xrightarrow{(\mathscr{H})} \varphi$ if

$$\varphi_k \rightharpoonup \varphi \text{ in } \mathbb{X}, \text{ and } \limsup_{k\to+\infty} \langle \mathscr{H}(\varphi_k), \varphi_k - \varphi\rangle \leqslant 0. \tag{8}$$

Moreover, we say that:

(i) $\mathscr{H}$ is bounded if $\mathscr{H}(S)$ is bounded in $\mathbb{X}^*$ for any bounded subset $S \subset \mathbb{X}$;

(ii) $\mathscr{H}$ is coercive if $\displaystyle\lim_{\|\varphi\|_{\mathbb{X}}\to\infty} \frac{\langle \mathscr{H}(\varphi), \varphi\rangle}{\|\varphi\|_{\mathbb{X}}} = +\infty$;

(iii) $\mathscr{H}$ is monotone if $\langle \mathscr{H}(\varphi) - \mathscr{H}(\psi), \varphi - \psi\rangle \geqslant 0$, for each $\varphi, \psi \in \mathbb{X}$;

(iv) $\mathscr{H}$ is pseudo-monotone if

$$\varphi_k \xrightarrow{(\mathscr{H})} \varphi \Longrightarrow \langle \mathscr{H}(\varphi), \varphi - w\rangle \leqslant \liminf_{k\to\infty} \langle \mathscr{H}(\varphi_k), \varphi_k - w\rangle, \quad \text{for all } w \in \mathbb{X};$$

(v) $\mathscr{H}$ satisfies the so-called (S_+)-property if

$$\varphi_k \xrightarrow{(\mathscr{H})} \varphi \Longrightarrow \varphi_k \to \varphi \text{ in } \mathbb{X}.$$

2.2. Young functions

Throughout the paper, we consider a function $G:[0,\infty)\to[0,\infty)$ satisfying $G(0)=0$ and $\lim_{\sigma\to\infty} G(\sigma)=\infty$. We then say that G is a Young function, usually denoted by $G\in\mathscr{Y}$, if G is non-decreasing, convex and

$$\lim_{\sigma\to 0^+}\sigma^{-1}G(\sigma)=0, \text{ and } \lim_{\sigma\to\infty}\sigma^{-1}G(\sigma)=\infty.$$

The Young's conjugate of Young function G is defined as follows

$$G^*(\sigma)=\sup\{\sigma r-G(r):\ r\geqslant 0\},\quad \sigma\geqslant 0.$$

As the reader may easily check that if $G\in\mathscr{Y}$ then $G^*\in\mathscr{Y}$ and $(G^*)^*=G$. In addition, for $G\in\mathscr{Y}$, we say that $G\in\Delta_2$ if there exists a constant $\Delta_2^G>1$ such that

$$G(2\sigma)\leqslant\Delta_2^G G(\sigma), \text{ for all } \sigma\geqslant 0.$$

Otherwise, we will further often write $G\in\nabla_2$ if there is a constant $\nabla_2^G>1$ such that

$$G(\nabla_2^G\sigma)\geqslant 2\nabla_2^G G(\sigma), \text{ for all } \sigma\geqslant 0.$$

At the same time, the interesting point reads

$$G\in\Delta_2 \Leftrightarrow G^*\in\nabla_2, \text{ with } \Delta_2^G=2\nabla_2^{G^*}. \tag{9}$$

Moreover, if $G\in\Delta_2$, then one can find some constants $\nu_2\geqslant\nu_1>1$ and $C>1$ independent of a and σ, such that

$$C^{-1}\min\{a^{\nu_1},a^{\nu_2}\}G(\sigma)\leqslant G(a\sigma)\leqslant C\max\{a^{\nu_1},a^{\nu_2}\}G(\sigma),$$

for all $\sigma,a\geqslant 0$. Our last preparatory step regarding Young's function is contained in the following lemma. It is known as a generalized version of Young's inequality. The proof of this lemma relies on the definitions of Δ_2^G-condition and ∇_2^G-condition and G^*. We revisit the complete proof in [7, 11].

LEMMA 2.2. (Young inequality, see [11]) *Let $G\in\Delta_2\cap\nabla_2$. For every $\sigma>0$, the following inequality holds*

$$G^*\big(\sigma^{-1}G(\sigma)\big)\leqslant G(\sigma)\leqslant G^*\big(2\sigma^{-1}G(\sigma)\big). \tag{10}$$

Moreover, for every $\varepsilon\in(0,1)$, there exists $C_\varepsilon>0$ such that

$$r\sigma^{-1}G(\sigma)\leqslant\varepsilon G(r)+C_\varepsilon G(\sigma), \quad \textit{for all } r\geqslant 0,\ \sigma>0. \tag{11}$$

2.3. Orlicz and Orlicz-Sobolev spaces

In what follows, we denote by $\mathscr{M}(\Omega)$ the set of all measurable functions $\varphi : \Omega \to \mathbb{R}$ and a Young function $G \in \mathscr{Y}$. We here introduce the Orlicz class $\mathscr{O}^G(\Omega)$, defined by $\mathscr{O}^G(\Omega) := \{\varphi \in \mathscr{M}(\Omega) : \ m_G(\varphi) < \infty\}$, where $m_G(\varphi)$ is the modular function with respect to G, given by

$$m_G(\varphi) := \fint_{\Omega} G(|\varphi(x)|)dx := \frac{1}{|\Omega|}\int_{\Omega} G(|\varphi(x)|)dx. \tag{12}$$

Here, $|\Omega|$ denotes the finite n-dimensional Lebesgue measure of Ω. The linear hull of $O^G(\Omega)$ will be called the Orlicz space $L^G(\Omega)$, that is equipped with the following Luxemburg norm

$$\|\varphi\|_{L^G(\Omega)} = \inf\left\{\sigma > 0 : \ m_G(\sigma^{-1}\varphi) \leqslant 1\right\}.$$

Let us recall the Hölder's inequality in Orlicz spaces.

LEMMA 2.3. (Hölder's inequality, see [16]) *If $G \in \Delta_2 \cap \nabla_2$, then $(L^G(\Omega), \|\cdot\|_{L^G(\Omega)})$ is a reflexive Banach space. Moreover, for every $\varphi \in L^G(\Omega)$ and $\psi \in L^{G^*}(\Omega)$, there exists a constant $C > 0$ such that*

$$\|\varphi\psi\|_{L^1(\Omega)} \leqslant 2\|\varphi\|_{L^G(\Omega)}\|\psi\|_{L^{G^*}(\Omega)}. \tag{13}$$

In two next lemmas, some useful relations between the norm in $L^G(\Omega)$ and the corresponding modular in (12) will be established.

LEMMA 2.4. *Let $G \in \Delta_2 \cap \nabla_2$ with condition $\Delta_2^G > 2$. Then, one can find a constant $C > 1$ such that*

$$C^{-1}\left(\|\varphi\|^{\theta_2}_{L^G(\Omega)} - 1\right) \leqslant m_G(\varphi) \leqslant C\left(\|\varphi\|^{\theta_1}_{L^G(\Omega)} + 1\right), \tag{14}$$

for every $\varphi \in L^G(\Omega)$, where θ_1, θ_2 defined by

$$\theta_1 = \log_2\left(\Delta_2^G\right), \text{ and } \theta_2 = 1 + \left[\log_2\left(\nabla_2^G\right)\right]^{-1}. \tag{15}$$

Proof. Since $G \in \Delta_2 \cap \nabla_2$, it enables us to check that

$$G(at) \leqslant \Delta_2^G a^{\theta_1} G(\sigma), \ \text{ and } \ G(b\sigma) \leqslant 2\nabla_2^G b^{\theta_2} G(\sigma),$$

for all $a \geqslant 1$, $b \in (0,1]$ and $\sigma \geqslant 0$, where θ_1, θ_2 are defined as in (15). We refer the interested reader to [18, Lemma 2.3] for related properties. As a consequence, inequality (14) can be obtained by applying [17, Lemma 4.11]. $\square$

LEMMA 2.5. *Let $G \in \Delta_2$ and $(\varphi_k)_{k\in\mathbb{N}}$ be a sequence in $L^G(\Omega)$. Then, the following statement holds*

$$\lim_{k\to\infty} \|\varphi_k\|_{L^G(\Omega)} = 0 \Leftrightarrow \lim_{k\to\infty} m_G(\varphi_k) = 0. \tag{16}$$

Proof. Thanks to statement *iii*) in Lemma 2.2, since $G \in \Delta_2$, it is possible to find some constants $1 < \nu_1 \leqslant \nu_2$ and $C > 1$ such that

$$C^{-1}\min\{a^{\nu_1}, a^{\nu_2}\}G(\sigma) \leqslant G(a\sigma) \leqslant C\max\{a^{\nu_1}, a^{\nu_2}\}G(\sigma),$$

for all $a, \sigma \in \mathbb{R}^+$. Plugging these inequalities into the integrals, we infer that if $0 < \|\varphi\|_{L^G(\Omega)} < 1$, there holds

$$C^{-1}\|\varphi\|_{L^G(\Omega)}^{-\nu_1} m_G(\varphi) \leqslant \int_\Omega G\big(\|\varphi\|_{L^G(\Omega)}^{-1}|\varphi(x)|\big)dx = 1 \leqslant C\|\varphi\|_{L^G(\Omega)}^{-\nu_2} m_G(\varphi),$$

which is equivalent to

$$C^{-1}\|\varphi\|_{L^G(\Omega)}^{\nu_2} \leqslant m_G(\varphi) \leqslant C\|\varphi\|_{L^G(\Omega)}^{\nu_1}. \tag{17}$$

In a similar way, for the remaining case when $\|\varphi\|_{L^G(\Omega)} \geqslant 1$, it yields

$$C^{-1}\|\varphi\|_{L^G(\Omega)}^{\nu_1} \leqslant m_G(\varphi) \leqslant C\|\varphi\|_{L^G(\Omega)}^{\nu_2}. \tag{18}$$

The validity of (16) is obtained by combining (17) and (18). And the proof is complete. □

DEFINITION 2.6. (Orlicz-Sobolev spaces) Given $G \in \mathscr{Y}$, the Orlicz-Sobolev space, written by $W^{1,G}(\Omega)$, is a generalization of Sobolev space connected with Orlicz space $L^G(\Omega)$, defined as

$$W^{1,G}(\Omega) := \left\{\varphi \in L^G(\Omega) : \ |\nabla\varphi| \in L^G(\Omega)\right\},$$

is also a Banach space equipped with the norm

$$\|\varphi\|_{W^{1,G}(\Omega)} = \|\varphi\|_{L^G(\Omega)} + \||\nabla\varphi|\|_{L^G(\Omega)}.$$

For the sake of simplicity, we will denote

$$\|\nabla\varphi\|_{L^G(\Omega)} := \||\nabla\varphi|\|_{L^G(\Omega)}.$$

The closure of $C_0^\infty(\Omega)$ in $W^{1,G}(\Omega)$ will be denoted by $W_0^{1,G}(\Omega)$.

3. New modular functions and properties

This section sets forth some interesting properties of modular functions modeled around the Young's function regarding the nonlinearity $\mathscr{A}_{p,\alpha}$ as described above. Furthermore, we also state and prove the equivalence, reveal the relationship between this modular function and norm function in the context.

Firstly, back to the definition of function $G_{p,\alpha}$ in (6), it is clear that $G_{p,\alpha} \in \mathscr{Y}$ and $G_{p,\alpha} \in \Delta_2 \cap \nabla_2$ as a consequence of the following lemma.

LEMMA 3.1. *For every* $\sigma_1, \sigma_2 \geqslant 0$, *there holds*

$$G_{p,\alpha}(\sigma_1+\sigma_2) \leqslant 2^{p+\alpha}\left[G_{p,\alpha}(\sigma_1)+G_{p,\alpha}(\sigma_2)\right].$$

Moreover, there holds

$$G_{p,\alpha}(\sigma) \sim \sigma G'_{p,\alpha}(\sigma) \sim \sigma^2 G''_{p,\alpha}(\sigma), \quad \sigma \in \mathbb{R}^+. \tag{19}$$

In the sequel, let us consider the operator $\mathscr{L}_{p,\alpha}: \mathbb{X}_0 \to \mathbb{X}_0^*$ as follows

$$\langle \mathscr{L}_{p,\alpha} u, v\rangle := \int_\Omega \mathscr{A}_{p,\alpha}(\nabla u)\cdot \nabla v dx, \text{ for } u,v \in \mathbb{X}_0, \tag{20}$$

ans we now focus on some significant properties of $\mathscr{L}_{p,\alpha}$ that are useful to our main proofs later.

The presence of $G_{p,\alpha}$ brings us the corresponding Orlicz space $L^{G_{p,\alpha}}(\Omega)$ with the following norm

$$\|\varphi\|_{\mathbb{L}} = \inf\left\{\sigma>0: \ m_{G_{p,\alpha}}\left(\sigma^{-1}\varphi\right) \leqslant 1\right\}, \quad \varphi \in \mathbb{L}. \tag{21}$$

However, it could be difficult to handle the norm in $\mathbb{L}$ by itself nature when dealing with the problem (21). The novelty of our study here lies in the construction of a new modular function having some equivalent properties to the norm. This idea was found in our previous work [21] when treating a related problem. In this paper, we shall consider the following modular function:

$$[\varphi]_{\mathbb{L}} := \left(\fint_\Omega |\varphi(x)|^p \log^\alpha\left(e+\|\varphi\|_p^{-1}|\varphi(x)|\right)dx\right)^{1/p}, \tag{22}$$

and it is provided whenever $\|\varphi\|_p > 0$, where

$$\|\varphi\|_p := \left(\fint_\Omega |\varphi(x)|^p dx\right)^{\frac{1}{p}}.$$

At this stage, there are three interesting points with this new modular function. On the one hand, it allows us to derive a comparison between the proposed new modular and the former one given in (12), stated in the next lemma, whose proof is based on the following basic facts:

$$\log^\alpha(e+s\sigma) \leqslant 2^\alpha\left[\log^\alpha\sigma+\log^\alpha(e+s)\right], \quad \log^\alpha\sigma \leqslant \left(\frac{\alpha}{eq}\right)^\alpha \sigma^q, \tag{23}$$

for any $\alpha \geqslant 0$, $\sigma \geqslant 1$ and $q>0$.

LEMMA 3.2. *For every* $\varphi \in \mathbb{L}$, *the following relation holds*

$$\Lambda^{-1} m_{G_{p,\alpha}}(\varphi) - \|\varphi\|_p^{p+1} \leqslant [\varphi]_{\mathbb{L}}^p \leqslant p^2\Lambda\left[\|\varphi\|_p^{p-1}+m_{G_{p,\alpha}}(\varphi)\right], \tag{24}$$

where Λ *is defined by*

$$\Lambda := 2^\alpha \max\left\{\left(\frac{\alpha}{e}\right)^\alpha; 1\right\}. \tag{25}$$

Proof. We now distinguish two cases. For the first case, we assume that $\|\varphi\|_p \geqslant 1$ and it is clear to obtain

$$[\varphi]_{\mathbb{L}}^p = \fint_\Omega |\varphi(x)|^p \log^\alpha \left(e + \|\varphi\|_p^{-1}|\varphi(x)|\right) dx \leqslant 2^\alpha p m_{G_{p,\alpha}}(\varphi). \tag{26}$$

Moreover, applying (23) for $\sigma = \|\varphi\|_p \geqslant 1$, it yields

$$\begin{aligned}
m_{G_{p,\alpha}}(\varphi) &= \frac{1}{p}\fint_\Omega |\varphi(x)|^p \log^\alpha \left(e+|\varphi(x)|\right) dx \\
&\leqslant \frac{2^\alpha}{p}\left[\log^\alpha \left(\|\varphi\|_p\right)\fint_\Omega |\varphi(x)|^p dx + \fint_\Omega |\varphi(x)|^p \log^\alpha \left(e+\|\varphi\|_p^{-1}|\varphi(x)|\right) dx\right] \\
&\leqslant \frac{2^\alpha}{p}\left[\left(\frac{\alpha}{e}\right)^\alpha \|\varphi\|_p^{p+1} + [\varphi]_{\mathbb{L}}^p\right] \\
&\leqslant \Lambda\left[\|\varphi\|_p^{p+1} + [\varphi]_{\mathbb{L}}^p\right],
\end{aligned} \tag{27}$$

where Λ is given as in (25). Combining two estimates in (26) and (27) we may conclude that

$$\Lambda^{-1} m_{G_{p,\alpha}}(\varphi) - \|\varphi\|_p^{p+1} \leqslant [\varphi]_{\mathbb{L}}^p \leqslant p^2 \Lambda m_{G_{p,\alpha}}(\varphi), \quad \text{if } \|\varphi\|_p \geqslant 1. \tag{28}$$

Hence, the inequality (24) holds true for the first case. For the remaining case when $0 < \|\varphi\|_p < 1$ by applying a similar argument as in the previous one. Firstly, there holds

$$m_{G_{p,\alpha}}(\varphi) = \frac{1}{p}\fint_\Omega |\varphi(x)|^p \log^\alpha \left(e+|\varphi(x)|\right) dx \leqslant \frac{1}{p}[\varphi]_{\mathbb{L}}^p \leqslant \Lambda[\varphi]_{\mathbb{L}}^p.$$

On the other hand, we take (23) into account for $\sigma = \|\varphi\|_p^{-1} > 1$ to arrive at

$$\begin{aligned}
[\varphi]_{\mathbb{L}}^p &\leqslant 2^\alpha \left[\log^\alpha \left(\|\varphi\|_p^{-1}\right)\fint_\Omega |\varphi(x)|^p dx + p\fint_\Omega G_{p,\alpha}(|\varphi(x)|) dx\right] \\
&\leqslant p 2^\alpha \left[\frac{1}{p}\left(\frac{\alpha}{e}\right)^\alpha \|\varphi\|_p^{p-1} + m_{G_{p,\alpha}}(\varphi)\right] \\
&\leqslant p^2 \Lambda \left[\|\varphi\|_p^{p-1} + m_{G_{p,\alpha}}(\varphi)\right].
\end{aligned} \tag{29}$$

Invoking two above estimates, one gets that

$$\Lambda^{-1} m_{G_{p,\alpha}}(\varphi) \leqslant [\varphi]_{\mathbb{L}}^p \leqslant p^2 \Lambda \left[\|\varphi\|_p^{p-1} + m_{G_{p,\alpha}}(\varphi)\right], \quad \text{if } 0 < \|\varphi\|_p < 1. \tag{30}$$

Finally, the assertion of (24) is concluded from (28) and (30). □

The second feature of the new modular function we would like to emphasize here is the *equivalence*, which is stated in the following lemma. This nice property was in fact pointed out in [21], we shortly write $\|\cdot\|_{\mathbb{L}} \sim [\cdot]_{\mathbb{L}}$.

LEMMA 3.3. (see [21]) *With Λ defined as in (25), one has*

$$\|\varphi\|_{\mathbb{L}} \leqslant [\varphi]_{\mathbb{L}} \leqslant \Lambda \|\varphi\|_{\mathbb{L}}, \quad \textit{for every } \varphi \in \mathbb{L}. \tag{31}$$

On the other hand, an additional interesting point is that the norm $\|\varphi\|_{\mathbb{X}} = \|\varphi\|_{\mathbb{L}} + \|\nabla\varphi\|_{\mathbb{L}}$ is equivalent to $\|\nabla\varphi\|_{\mathbb{L}}$, for all $\varphi \in \mathbb{X}_0$. More precisely, this result may be directly obtained by the fact that the embedding $\mathbb{X}_0 \hookrightarrow \mathbb{L}$ is compact. This feature can be obtained by applying the compact embedding result discussed in [4, Theorem 1 and Theorem 3]. We send the reader to [9] and [21] for detailed proofs in some specific cases. Therefore, in $\mathbb{X}_0$, we may consider the following norm

$$\|\varphi\|_{\mathbb{X}_0} = \|\nabla\varphi\|_{\mathbb{L}}, \quad \varphi \in \mathbb{X}_0, \tag{32}$$

which is also equivalent to the modular $[\nabla\varphi]_{\mathbb{L}}$. In particular, combining (31) together with (32), there holds

$$\|\varphi\|_{\mathbb{X}_0} \leqslant [\nabla\varphi]_{\mathbb{L}} \leqslant \Lambda\|\varphi\|_{\mathbb{X}_0}, \quad \text{for every } \varphi \in \mathbb{X}_0. \tag{33}$$

4. Some properties of $L^p \log^\alpha L$ operators

In this section, once having the preparatory lemmas at hand, we exploit some necessary properties of the left-hand side $L^p\log^\alpha L$ operators to establish the existence and uniqueness results along the lines of the theory studied in the Main Theorem on Pseudo-monotone operators in [3]. It is also worth mentioning that during chains of our technical comparison estimates throughout this section, constants will be denoted by the same letter C. Their value is unimportant and may change from line to line, sometimes, within the same line.

LEMMA 4.1. *The $L^p\log^\alpha L$-Laplace operator $\mathscr{L}_{p,\alpha}$ defined as in* (20) *is bounded and coercive.*

Proof. First, it is clear to see that the integral $\mathscr{L}_{p,\alpha}$ is well-defined, which means that the integral on the right-hand side of (20) is finite. Indeed, from the definition of $\mathscr{A}_{p,\alpha}$ in (2), for $u \in \mathbb{X}_0$, one has

$$\begin{aligned}
|\mathscr{A}_{p,\alpha}(\nabla u)| &\leqslant |\nabla u|^{p-1}\log^\alpha(e+|\nabla u|) + \frac{\alpha}{p}|\nabla u|^p \frac{\log^{\alpha-1}(e+|\nabla u|)}{e+|\nabla u|} \\
&\leqslant \left(1 + \frac{\alpha}{p}\frac{|\nabla u|}{(e+|\nabla u|)\log(e+|\nabla u|)}\right)|\nabla u|^{p-1}\log^\alpha(e+|\nabla u|) \\
&\leqslant \left(1+\frac{\alpha}{p}\right)|\nabla u|^{p-1}\log^\alpha(e+|\nabla u|) \\
&= (\alpha+p)\frac{G_{p,\alpha}(|\nabla u|)}{|\nabla u|},
\end{aligned}$$

where $G_{p,\alpha}$ defined in (6). Combining this estimate together with the fact that $G^*_{p,\alpha} \in \Delta_2 \cap \nabla_2$, and then applying (10), we arrive at

$$\int_\Omega G^*_{p,\alpha}\big(|\mathscr{A}_{p,\alpha}(\nabla u)|\big)dx \leqslant C\int_\Omega G^*_{p,\alpha}\left(\frac{G_{p,\alpha}(|\nabla u|)}{|\nabla u|}\right)dx \leqslant C\int_\Omega G_{p,\alpha}\big(|\nabla u|\big)dx. \tag{34}$$

Since $|\nabla u| \in L^{G_{p,\alpha}}(\Omega)$, inequality (34) ensures that $\mathscr{A}_{p,\alpha}(\nabla u) \in L^{G^*_{p,\alpha}}(\Omega)$. For this reason, it allows us to apply Hölder's inequality (13) to obtain

$$\begin{aligned}\left|\langle \mathscr{L}_{p,\alpha}u, v\rangle\right| &\leqslant \int_\Omega \left|\mathscr{A}_{p,\alpha}(\nabla u)\right| |\nabla v| dx \\ &\leqslant C\|\mathscr{A}_{p,\alpha}(\nabla u)\|_{L^{G^*_{p,\alpha}}(\Omega)} \|\nabla v\|_{L^{G_{p,\alpha}}(\Omega)} \\ &= C\|\mathscr{A}_{p,\alpha}(\nabla u)\|_{L^{G^*_{p,\alpha}}(\Omega)} \|v\|_{\mathbb{X}_0},\end{aligned}$$

for all $u, v \in \mathbb{X}_0$. Gathering all these estimates, one may conclude that $\mathscr{L}_{p,\alpha}$ is well-defined. In addition, the above inequality also yields

$$\|\mathscr{L}_{p,\alpha}u\|_{\mathbb{X}_0^*} = \sup_{v\neq 0} \frac{|\langle \mathscr{L}_{p,\alpha}u, v\rangle|}{\|v\|_{\mathbb{X}_0}} \leqslant C\|\mathscr{A}_{p,\alpha}(\nabla u)\|_{L^{G^*_{p,\alpha}}(\Omega)}, \quad \text{for every } u \in \mathbb{X}_0.$$

Next, for the boundedness of $\mathscr{L}_{p,\alpha}$, it is sufficient to ask for the following term

$$M := \|\mathscr{A}_{p,\alpha}(\nabla u)\|_{L^{G^*_{p,\alpha}}(\Omega)}$$

to be bounded. By Lemma 3.1, one can choose $\Delta_2^{G_{p,\alpha}} = 2^{p+\alpha+1} > 2$. Thanks to (9), it gets $\nabla_2^{G^*_{p,\alpha}} = 2^{p+\alpha}$. Then, applying inequality (14) in Lemma 2.4, we infer that

$$M = \|\mathscr{A}_{p,\alpha}(\nabla u)\|_{L^{G^*_{p,\alpha}}(\Omega)} \leqslant C\left(1 + \int_\Omega G^*_{p,\alpha}\big(|\mathscr{A}_{p,\alpha}(\nabla u)|\big) dx\right)^{\frac{p+\alpha}{p+\alpha+1}}, \tag{35}$$

and

$$\int_\Omega G_{p,\alpha}\big(|\nabla u|\big) dx \leqslant C\left(1 + \|\nabla u\|^{p+\alpha+1}_{L^{G_{p,\alpha}}(\Omega)}\right). \tag{36}$$

Combining (34), (35) and (36), it enables us to estimate M as follows

$$M \leqslant C\big(1 + \|\nabla u\|_{L^{G_{p,\alpha}}(\Omega)}\big)^{p+\alpha} = C\big(1 + \|u\|_{\mathbb{X}_0}\big)^{p+\alpha}.$$

Notice that $p+\alpha > 0$, it is enough to conclude that $\mathscr{L}_{p,\alpha}$ is bounded with the following estimate

$$\|\mathscr{L}_{p,\alpha}u\|_{\mathbb{X}_0^*} \leqslant C\big(1 + \|u\|_{\mathbb{X}_0}\big)^{p+\alpha}, \quad \text{for every } u \in \mathbb{X}_0.$$

At this stage, for the proof of coercive property, let us write

$$\begin{aligned}\langle \mathscr{L}_{p,\alpha}u, u\rangle &= p\int_\Omega G_{p,\alpha}\big(|\nabla u|\big) dx + \int_\Omega \frac{\alpha}{p} \frac{|\nabla u|^{p+1} \log^{\alpha-1}(e + |\nabla u|)}{e + |\nabla u|} dx \\ &\geqslant p\int_\Omega G_{p,\alpha}\big(|\nabla u|\big) dx.\end{aligned}$$

In virtue of the Lemma 2.4 again, it yields to

$$\langle \mathscr{L}_{p,\alpha}u, u\rangle \geqslant C\left(\|u\|_{\mathbb{X}_0}^{\theta} - 1\right), \text{ where } \theta := 1 + \left(\log_2 \nabla_2^{G_{p,\alpha}}\right)^{-1}. \tag{37}$$

Since $\theta > 1$, inequality (37) guarantees that

$$\lim_{\|u\|_{\mathbb{X}_0}\to\infty} \frac{\langle \mathscr{L}_{p,\alpha}u, u\rangle}{\|u\|_{\mathbb{X}_0}} = \infty,$$

and the conclusion reads $\mathscr{L}_{p,\alpha}$ is coercive. □

LEMMA 4.2. *The $L^p\log^\alpha L$-Laplace operator $\mathscr{L}_{p,\alpha}$ defined as in (20) is continuous.*

Proof. Let $(u_k)_{k\in\mathbb{N}} \subset \mathbb{X}_0$ such that $u_k \to u$ in $\mathbb{X}_0$. It is necessary to prove that $\mathscr{L}_{p,\alpha}u_k \to \mathscr{L}_{p,\alpha}u$ in $\mathbb{X}_0^*$. First, let us consider for every $k \in \mathbb{N}$ and $v \in \mathbb{X}_0$, then, there holds

$$\begin{aligned} |\langle \mathscr{L}_{p,\alpha}u_k - \mathscr{L}_{p,\alpha}u, v\rangle| &= \left|\int_\Omega \left(\mathscr{A}_{p,\alpha}(\nabla u_k) - \mathscr{A}_{p,\alpha}(\nabla u)\right)\cdot \nabla v dx\right| \\ &\leqslant \int_\Omega \left|\mathscr{A}_{p,\alpha}(\nabla u_k) - \mathscr{A}_{p,\alpha}(\nabla u)\right| |\nabla v| dx. \end{aligned} \tag{38}$$

Furthermore, we recall that

$$\left|\mathscr{A}_{p,\alpha}(\zeta_1) - \mathscr{A}_{p,\alpha}(\zeta_2)\right| \sim G''_{p,\alpha}(|\zeta_1| + |\zeta_2|)|\zeta_1 - \zeta_2|, \quad \zeta_1, \zeta_2 \in \mathbb{R}^n.$$

Thus, one obtains from (38) that

$$|\langle \mathscr{L}_{p,\alpha}u_k - \mathscr{L}_{p,\alpha}u, v\rangle| \leqslant C\int_\Omega G''_{p,\alpha}(|\nabla u_k| + |\nabla u|)|\nabla u_k - \nabla u||\nabla v| dx.$$

Here, it notices that the positive constant C depends on p and α. Thanks to Hölder's inequality (13), we arrive at

$$|\langle \mathscr{L}_{p,\alpha}u_k - \mathscr{L}_{p,\alpha}u, v\rangle| \leqslant C\|T_k\|_{L^{G^*_{p,\alpha}}(\Omega)}\|v\|_{\mathbb{X}_0}, \tag{39}$$

where T_k is defined by

$$T_k := G''_{p,\alpha}(|\nabla u_k| + |\nabla u|)|\nabla u_k - \nabla u|.$$

Using (39), we readily infer that

$$|\langle \mathscr{L}_{p,\alpha}u_k - \mathscr{L}_{p,\alpha}u, v\rangle| \leqslant C\|T_k\|_{L^{G^*_{p,\alpha}}(\Omega)}\|v\|_{\mathbb{X}_0}, \text{ for all } v \in \mathbb{X}_0,$$

which is equivalent to

$$\|\mathscr{L}_{p,\alpha}u_k - \mathscr{L}_{p,\alpha}u\|_{\mathbb{X}_0^*} \leqslant C\|T_k\|_{L^{G^*_{p,\alpha}}(\Omega)}.$$

To complete the proof, it is sufficient to show that $\lim_{k\to\infty}\|T_k\|_{L^{G^*_{p,\alpha}}(\Omega)}=0$. Thanks to Lemma 2.5, we only need to show that

$$\lim_{k\to\infty}\fint_\Omega G^*_{p,\alpha}(T_k)dx=0.$$

At this stage, thanks to Lemma 2.2, one can find $\nu>1$ such that

$$G^*_{p,\alpha}(\sigma t)\leqslant C\sigma^\nu G^*_{p,\alpha}(t),\qquad \text{for all } t\geqslant 0 \text{ and } \sigma\in(0,1].$$

Applying this inequality and using (19), the following estimate

$$T_k\leqslant CG'_{p,\alpha}(|\nabla u_k|+|\nabla u|)\frac{|\nabla u_k-\nabla u|}{|\nabla u_k|+|\nabla u|},$$

holds, it moreover gets

$$\fint_\Omega G^*_{p,\alpha}(T_k)dx\leqslant C\fint_\Omega G^*_{p,\alpha}\big(G'_{p,\alpha}(|\nabla u_k|+|\nabla u|)\big)\left[\frac{|\nabla u_k-\nabla u|}{|\nabla u_k|+|\nabla u|}\right]^\nu dx. \tag{40}$$

Here, recall that $G'_{p,\alpha}(\sigma)\sim G_{p,\alpha}(\sigma)/\sigma$ by Lemma 3.1. Therefore, applying inequality (10) in Lemma 2.2 and Hölder's inequality (13) again, estimate (40) deduces to

$$\begin{aligned}\fint_\Omega G^*_{p,\alpha}(T_k)dx&\leqslant C\fint_\Omega G_{p,\alpha}(|\nabla u_k|+|\nabla u|)\frac{|\nabla(u_k-u)|}{|\nabla u_k|+|\nabla u|}dx\\&\leqslant C\|(|\nabla u_k|+|\nabla u|)^{-1}G_{p,\alpha}(|\nabla u_k|+|\nabla u|)\|_{L^{G^*_{p,\alpha}}(\Omega)}\|\nabla(u_k-u)\|_{L^{G_{p,\alpha}}(\Omega)}\\&\leqslant C\|u_k-u\|_{\mathbb{X}_0}.\end{aligned}\tag{41}$$

Turn our attention to the last estimate in (41), it allows us to conclude the desired result with the fact that $\|(|\nabla u_k|+|\nabla u|)^{-1}G_{p,\alpha}(|\nabla u_k|+|\nabla u|)\|_{L^{G^*_{p,\alpha}}(\Omega)}$ is bounded. In a completely similar way, we can estimate M as in the previous lemma. The proof is now complete. □

LEMMA 4.3. *The $L^p\log^\alpha L$-Laplace operator $\mathscr{L}_{p,\alpha}$ defined as in (20) is monotone, pseudo-monotone and satisfies $(S)_+$-condition.*

Proof. First, we may rewrite the operator $\mathscr{L}_{p,\alpha}$ in (20) as follows

$$\langle\mathscr{L}_{p,\alpha}u,v\rangle=\int_\Omega\left(|\nabla u|^{p-2}\nabla u\log^\alpha(e+|\nabla u|)+\frac{\alpha}{p}|\nabla u|^{p-1}\nabla u\frac{\log^{\alpha-1}(e+|\nabla u|)}{e+|\nabla u|}\right)\cdot\nabla v dx.$$

The proof is divided into several steps. In the first step, we will show that $\mathscr{L}_{p,\alpha}$ is monotone. Indeed, for all $u_1,u_2\in\mathbb{X}$, by [1, Section 3.2] it is readily verified that

$$\begin{aligned}\langle\mathscr{L}_{p,\alpha}u_1-\mathscr{L}_{p,\alpha}u_2,u_1-u_2\rangle&=\int_\Omega\big(\mathscr{A}_{p,\alpha}(\nabla u_1)-\mathscr{A}_{p,\alpha}(\nabla u_2)\big)\cdot(\nabla u_1-\nabla u_2)dx\\&\geqslant C\int_\Omega\big|V_{p,\log^\alpha}(\nabla u_1)-V_{p,\log^\alpha}(\nabla u_2)\big|^2dx,\end{aligned}$$

where the auxiliary vector field $V_{p,\log^\alpha}:\mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$V_{p,\log^\alpha}(\zeta) := \left(|\zeta|^{p-2}\log^\alpha(e+|\zeta|) + \frac{\alpha}{p}\frac{|\zeta|^{p-1}\log^{\alpha-1}(e+|\zeta|)}{e+|\zeta|}\right)^{\frac{1}{2}}\zeta,$$

whenever $\zeta \in \mathbb{R}^n$. Thus, we are able to claim that $\mathscr{L}_{p,\alpha}$ is monotone.

In the second step, let us assume that $u_k \xrightharpoonup{(\mathscr{L}_{p,\alpha})} u$ in the sense of (8). In order to show $\mathscr{L}_{p,\alpha}$ satisfies the $(S)_+$ condition, we have to point out that $\lim_{k\to\infty}\|u_k-u\|_{\mathbb{X}_0}$. Since $u_k \rightharpoonup u$ in $\mathbb{X}$ and $\mathscr{L}_{p,\alpha} \in \mathbb{X}^*$, one has

$$\lim_{k\to\infty}\langle \mathscr{L}_{p,\alpha}u, u_k-u\rangle = 0,$$

which implies to

$$\limsup_{k\to\infty}\langle \mathscr{L}_{p,\alpha}u_k - \mathscr{L}_{p,\alpha}u, u_k-u\rangle \leqslant 0.$$

On the other hand, since $\mathscr{L}_{p,\alpha}$ monotone, one has

$$0 \leqslant \liminf_{k\to\infty}\langle \mathscr{L}_{p,\alpha}u_k - \mathscr{L}_{p,\alpha}u, u_k-u\rangle \leqslant \limsup_{k\to\infty}\langle \mathscr{L}_{p,\alpha}u_k - \mathscr{L}_{p,\alpha}u, u_k-u\rangle \leqslant 0.$$

Once having this estimate, it allows us to conclude

$$\liminf_{k\to\infty}\langle \mathscr{L}_{p,\alpha}u_k - \mathscr{L}_{p,\alpha}u, u_k-u\rangle = \limsup_{k\to\infty}\langle \mathscr{L}_{p,\alpha}u_k - \mathscr{L}_{p,\alpha}u, u_k-u\rangle = 0,$$

which ensures that

$$\lim_{k\to\infty}\langle \mathscr{L}_{p,\alpha}u_k - \mathscr{L}_{p,\alpha}u, u_k-u\rangle = 0.$$

This is equivalent to

$$\lim_{k\to\infty}\fint_\Omega \big(\mathscr{A}_{p,\alpha}(\nabla u_k) - \mathscr{A}_{p,\alpha}(\nabla u)\big)\cdot(\nabla u_k - \nabla u)dx = 0. \tag{42}$$

Making use of [1, Section 3.2] and Lemma 3.1, there holds

$$\fint_\Omega \big(\mathscr{A}_{p,\alpha}(\nabla u_k) - \mathscr{A}_{p,\alpha}(\nabla u)\big)\cdot(\nabla u_k - \nabla u)dx \geqslant C\fint_\Omega G''_{p,\alpha}(|\nabla u_k|+|\nabla u|)|\nabla u_k-\nabla u|^2dx,$$

in which from (42), it leads to

$$\lim_{k\to\infty}\fint_\Omega G''_{p,\alpha}(|\nabla u_k|+|\nabla u|)|\nabla u_k-\nabla u|^2dx = 0. \tag{43}$$

Next, we show the following inequality which can be verified following Young's inequality

$$\begin{aligned}\fint_\Omega G_{p,\alpha}\big(|\nabla u_k-\nabla u|\big)dx \leqslant \varepsilon\left(\fint_\Omega G_{p,\alpha}\big(|\nabla u|\big)dx + \fint_\Omega G_{p,\alpha}\big(|\nabla u_k|\big)dx\right)\\ + C_\varepsilon\fint_\Omega G''_{p,\alpha}(|\nabla u_k|+|\nabla u|)|\nabla u_k-\nabla u|^2dx,\end{aligned} \tag{44}$$

for every $\varepsilon > 0$, where $C_\varepsilon = C_\varepsilon(p,\alpha,\varepsilon) > 0$. Indeed, it is worth mentioning that the next inequality

$$\mathbb{I} := G_{p,\alpha}\big(|\nabla u_k - \nabla u|\big) \leqslant C\,|\nabla u_k - \nabla u|^2\, G''_{p,\alpha}(|\nabla u_k - \nabla u|)$$

holds by Lemma 3.1. Moreover, there holds

$$G''_{p,\alpha}(|\nabla u_k - \nabla u|) \leqslant C G''_{p,\alpha}(|\nabla u_k| + |\nabla u|)$$

whenever $p \geqslant 2$. Therefore, (44) is obviously valid in this case by applying Lemma 3.1. Otherwise, for $1 < p < 2$, we first decompose $\mathbb{I}$ as follows

$$\mathbb{I} = \big(|\nabla u_k| + |\nabla u|\big)^{\frac{p(2-p)}{2}} \Big(\big(|\nabla u_k| + |\nabla u|\big)^{p-2} |\nabla u_k - \nabla u|^2\Big)^{\frac{p}{2}} \log^{\alpha}(e + |\nabla u_k - \nabla u|).$$

We then apply Young's inequality (11) in Lemma 2.2 for $\tilde{\varepsilon} > 0$, it gives us

$$\begin{aligned}\mathbb{I} &\leqslant \tilde{\varepsilon} G_{p,\alpha}\big(|\nabla u_k| + |\nabla u|\big)\\ &\quad + C_{\tilde{\varepsilon}}\,|\nabla u_k - \nabla u|^2 \big(|\nabla u_k| + |\nabla u|\big)^{p-2} \log^{\alpha}\big(e + |\nabla u_k| + |\nabla u|\big).\end{aligned}$$

Using fundamental inequalities in Lemma 3.1, it yields to

$$\mathbb{I} \leqslant C\tilde{\varepsilon}\left[G_{p,\alpha}\big(|\nabla u|\big) + G_{p,\alpha}\big(|\nabla u_k|\big)\right] + C_{\tilde{\varepsilon}}\,|\nabla u_k - \nabla u|^2\, G''_{p,\alpha}\big(|\nabla u_k| + |\nabla u|\big).$$

By changing $C\tilde{\varepsilon} = \dfrac{\varepsilon}{2}$ in this inequality, it leads to (44). Thanks to Lemma 3.1, one obtains the following inequality

$$G_{p,\alpha}\big(|\nabla u_k|\big) \leqslant C\left[G_{p,\alpha}\big(|\nabla u|\big) + G_{p,\alpha}\big(|\nabla u_k - \nabla u|\big)\right].$$

Thus, we can deduce from (44) to

$$\begin{aligned}\fint_{\Omega} G_{p,\alpha}\big(|\nabla u_k - \nabla u|\big)dx \leqslant \varepsilon \fint_{\Omega} G_{p,\alpha}\big(|\nabla u|\big)dx&\\ + C_\varepsilon \fint_{\Omega} |\nabla u_k - \nabla u|^2 G''_{p,\alpha}\big(|\nabla u_k| + |\nabla u|\big)dx&,\end{aligned} \tag{45}$$

for all $\varepsilon > 0$. Let us take $\delta_0 = 1/(2p^2\Lambda)$, where Λ is defined by (25). For all $\delta \in (0,\delta_0)$, it is possible to fix $\varepsilon > 0$ in (45) satisfying

$$\varepsilon\left(\fint_{\Omega} G_{p,\alpha}\big(|\nabla u|\big)dx + 1\right) < \frac{1}{p}\delta^{\frac{p(p+1)}{p-1}}. \tag{46}$$

Thanks to (43), one can find $k_0 \in \mathbb{N}$ such that

$$\fint_{\Omega} |\nabla u_k - \nabla u|^2 G''_{p,\alpha}\big(|\nabla u_k| + |\nabla u|\big)dx \leqslant \varepsilon C_\varepsilon^{-1}, \ \text{ for all } k \geqslant k_0.$$

Combining (45) and (46), for every $k \geqslant k_0$, there holds

$$\fint_{\Omega} G_{p,\alpha}\big(|\nabla u_k - \nabla u|\big)dx < \frac{1}{p}\delta^{\frac{p(p+1)}{p-1}}. \tag{47}$$

From now on, we always consider k bigger than k_0. From (47), one has

$$0 \leqslant \fint_{\Omega} |\nabla u_k - \nabla u|^p dx \leqslant p \fint_{\Omega} G_{p,\alpha}\big(|\nabla u_k - \nabla u|\big)dx < \delta^{\frac{p(p+1)}{p-1}},$$

which ensures that

$$0 \leqslant \|\nabla u_k - \nabla u\|_p < \delta^{\frac{p+1}{p-1}}. \tag{48}$$

On the other hand, thanks to Lemma 3.2, one has

$$[\nabla u_k - \nabla u]_{\mathbb{L}}^p \leqslant p^2 \Lambda \left[\|\nabla u_k - \nabla u\|_p^{p-1} + \frac{1}{p} \fint_{\Omega} G_{p,\alpha}\big(|\nabla u_k - \nabla u|\big)dx \right].$$

Substituting (47) and (48) into this inequality, it gives us

$$[\nabla u_k - \nabla u]_{\mathbb{L}}^p \leqslant p^2 \Lambda \left[\delta^{p+1} + \delta^{\frac{p(p+1)}{p-1}} \right] \leqslant 2p^2 \Lambda \delta^{p+1} \leqslant \delta^p.$$

Applying inequality (31) in Lemma 3.3, we then obtain that

$$\|u_k - u\|_{\mathbb{X}_0} = \|\nabla u_k - \nabla u\|_{\mathbb{L}} \leqslant [\nabla u_k - \nabla u]_{\mathbb{L}} \leqslant \delta.$$

Therefore, it allows us to conclude that

$$\lim_{k \to \infty} \|u_k - u\|_{\mathbb{X}_0} = 0,$$

or $u_k \to u$ in $\mathbb{X}_0$. Hence, $\mathscr{L}_{p,\alpha}$ satisfies the $(S)_+$ condition.

Finally, we show that $\mathscr{L}_{p,\alpha}$ is pseudo-monotone. Assume that $u_k \xrightarrow{(\mathscr{L}_{p,\alpha})} u$. According to the above proof, one has $u_k \to u$ in $\mathbb{X}_0$. Moreover, combining with the fact that $\mathscr{L}_{p,\alpha}$ is continuous, one gets that $\mathscr{L}_{p,\alpha}(u_k) \to \mathscr{L}_{p,\alpha}(u)$ in $\mathbb{X}_0^*$. Thus, $\mathscr{L}_{p,\alpha}$ is pseudo-monotone. □

5. Existence and uniqueness of weak solution to the main problem

After dealing with some technical lemmas that play crucial ingredients in this paper, we are now in the position to accomplish our main results regarding the problem (1) that stated in Theorems 1.2 and 1.3. Let us first revisit the following lemma to handle the difficulties that may happen during the arguments.

LEMMA 5.1. *Let $G \in \Delta_2 \cap \nabla_2$. Then, there exists $\theta \in (0,1)$ such that*

$$\fint_{\Omega} G(|\varphi|)dx \leqslant C_\theta \left[\fint_{\Omega} [G(|\nabla \varphi|)]^\theta \, dx \right]^{\frac{1}{\theta}}, \tag{49}$$

for every $\varphi \in W_0^{1,G}(\Omega)$.

The proof of this lemma can be found in [6, Theorem 7]. At this stage, we are allowed to apply the assertion of this lemma and Hölder inequality to conclude that: there exists $C_0 > 0$ such that

$$\fint_{\Omega} G_{p,\alpha}(|\varphi|)dx \leqslant C_0 \fint_{\Omega} G_{p,\alpha}(|\nabla\varphi|)dx, \tag{50}$$

for every $\varphi \in \mathbb{X}_0$.

5.1. The existence

Proof of Theorem 1.2. Let us consider the new operator $\mathscr{H}_{p,\alpha} : \mathbb{X}_0 \to \mathbb{X}_0^*$ defined by

$$\langle \mathscr{H}_{p,\alpha}(u), v\rangle = \langle \mathscr{L}_{p,\alpha}(u), v\rangle - \int_{\Omega} f(x,u)v dx, \quad u,v \in \mathbb{X}_0. \tag{51}$$

We will split the proof into three steps. In the first step, we show that $\mathscr{H}_{p,\alpha}$ is well-defined and bounded. Indeed, it is clear to see that

$$\fint_{\Omega} G^*_{p,\alpha}\left(\frac{G_{p,\alpha}(|u|)}{|u|}\right)dx \leqslant \fint_{\Omega} G_{p,\alpha}(|u|)dx < \infty,$$

which means that $\frac{G_{p,\alpha}(|u|)}{|u|} \in \mathbb{L}^*$. Combing with the fact that $g \in \mathbb{L}^*$, one obtains

$$g(x) + p\mu_0 \frac{G_{p,\alpha}(|u|)}{|u|} \in \mathbb{L}^*.$$

Moreover, assumption (7) gives us

$$|f(x,u)| \leqslant g(x) + \mu_0|u|^{p-1}\log^{\alpha}(e+|u|) = g(x) + p\mu_0\frac{G_{p,\alpha}(|u|)}{|u|},$$

which implies to

$$\fint_{\Omega} G^*_{p,\alpha}(|f(x,u)|)dx \leqslant \fint_{\Omega} G^*_{p,\alpha}\left(g(x) + p\mu_0\frac{G_{p,\alpha}(|u|)}{|u|}\right)dx < \infty.$$

It follows that $f(x,u) \in \mathbb{L}^*$. Now we are able to apply Hölder's inequality (13) in Lemma 2.3, it holds

$$\begin{aligned}
|\langle \mathscr{H}_{p,\alpha}(u), v\rangle| &\leqslant \int_{\Omega} \left|\mathscr{A}_{p,\alpha}(\nabla u)\cdot v\right| dx + \int_{\Omega} |f(x,u)||v|dx \\
&\leqslant C\left[\left\|\mathscr{A}_{p,\alpha}(\nabla u)\right\|_{\mathbb{L}^*}\|v\|_{\mathbb{X}_0} + \|f\|_{\mathbb{L}^*}\|v\|_{\mathbb{L}}\right] \\
&\leqslant C\left[\left\|\mathscr{A}_{p,\alpha}(\nabla u)\right\|_{\mathbb{L}^*} + \|f\|_{\mathbb{L}^*}\right]\|v\|_{\mathbb{X}_0},
\end{aligned}$$

for all $u, v \in \mathbb{X}_0$. Then, we arrive at

$$\left\|\mathscr{H}_{p,\alpha}(u)\right\|_{\mathbb{X}_0^*} = \sup_{v\neq 0} \frac{|\langle \mathscr{H}_{p,\alpha}(u), v\rangle|}{\|v\|_{\mathbb{X}_0}} \leqslant C\left[\left\|\mathscr{A}_{p,\alpha}(\nabla u)\right\|_{\mathbb{L}^*} + \|f\|_{\mathbb{L}^*}\right]$$

which allows us to conclude that $\mathscr{H}_{p,\alpha}$ is well-defined and bounded.

In the second step, we prove that $\mathscr{H}_{p,\alpha}$ is pseudo-monotone. Let (u_k) be a sequence in $\mathbb{X}_0$ such that $u_k \xrightarrow{(\mathscr{H}_{p,\alpha})} u$ in the sense of (8). Since $\mathscr{H}_{p,\alpha}$ is continuous, which can be implied by the continuity of $\mathscr{L}_{p,\alpha}$ in Lemma 4.2. Therefore, it is sufficient to prove u_k converges to u strongly in $\mathbb{X}_0$. Let us introduce a bounded sequence T_k defined by

$$T_k = \|g\|_{\mathbb{L}^*} + \mu_0 \left\||u_k|^{p-1}\log^{\alpha}(e+|u_k|)\right\|_{\mathbb{L}^*}.$$

Combining assumption (7) and Hölder's inequality (13) in Lemma 2.3, one obtains

$$\begin{aligned}\left|\int_\Omega f(x,u_k)(u_k-u)dx\right| &\leqslant \int_\Omega g(x)|u_k-u|dx \\ &\quad + \mu_0\int_\Omega |u_k|^{p-1}\log^{\alpha}(e+|u_k|)|u_k-u|dx \\ &\leqslant T_k\|u_k-u\|_{\mathbb{L}}. \end{aligned} \tag{52}$$

The fact that $\mathbb{X}_0 \hookrightarrow \mathbb{L}$ compactly and $u_k \rightharpoonup u$ in $\mathbb{X}_0$ give us $u_k \to u$ in $\mathbb{L}$. Hence, sending k to ∞ in (52), it follows that

$$\lim_{k\to\infty}\int_\Omega f(x,u_k)(u_k-u)dx = 0.$$

Again, combining with assumption $u_k \xrightarrow{(\mathscr{H}_{p,\alpha})} u$, it leads to

$$\limsup_{k\to\infty}\langle \mathscr{L}_{p,\alpha}(u_k), u_k-u\rangle = \limsup_{k\to\infty}\langle \mathscr{H}_{p,\alpha}(u_k), u_k-u\rangle \leqslant 0.$$

Thanks to $(S)_+$ property of $\mathscr{L}_{p,\alpha}$ in Lemma 4.3, we may conclude $u_k \to u$ in $\mathbb{X}_0$. As our discussion above, it is enough to conclude that $\mathscr{H}_{p,\alpha}$ is pseudo-monotone.

Let us move to the last step which is the proof of coercive property for the operator $\mathscr{H}_{p,\alpha}$. For all $u \in \mathbb{X}_0$, one has

$$\begin{aligned}\frac{1}{|\Omega|}\langle \mathscr{H}_{p,\alpha}(u), u\rangle &= p\fint_\Omega G_{p,\alpha}(|\nabla u|)dx + \alpha\fint_\Omega \frac{|\nabla u|G_{p,\alpha}(|\nabla u|)}{(e+|\nabla u|)\log(e+|\nabla u|)}dx \\ &\quad - \fint_\Omega f(x,u)u(x)dx \\ &\geqslant p\fint_\Omega G_{p,\alpha}(|\nabla u|)dx - \fint_\Omega f(x,u)u(x)dx. \end{aligned} \tag{53}$$

Taking the assumption (7) and Poincaré's inequality into account, there holds

$$\begin{aligned}\fint_{\Omega} f(x,u)u(x)dx &\leqslant \fint_{\Omega} g(x)u(x)dx + p\mu_0 \fint_{\Omega} G_{p,\alpha}(|u|)dx \\ &\leqslant C\|g\|_{\mathbb{L}^*}\|u\|_{\mathbb{L}} + p\mu_0 C_0 \fint_{\Omega} G_{p,\alpha}(|\nabla u|)dx \\ &\leqslant C\|g\|_{\mathbb{L}^*}\|u\|_{\mathbb{X}_0} + p\mu_0 C_0 \fint_{\Omega} G_{p,\alpha}(|\nabla u|)dx. \end{aligned} \tag{54}$$

It is noticed that the constant C_0 is given in (50). Collecting two estimates in (54) and (53), one gets that

$$\frac{1}{|\Omega|}\langle \mathscr{H}_{p,\alpha}(u), u\rangle \geqslant p\big(1-\mu_0 C_0\big) \fint_{\Omega} G_{p,\alpha}(|\nabla u|)dx - C\|g\|_{\mathbb{L}^*}\|u\|_{\mathbb{X}_0}. \tag{55}$$

On the other hand, one can see that (29) leads to $[\varphi]^p_{\mathbb{L}} \leqslant p^2\Lambda\left[1+m_{G_{p,\alpha}}(\varphi)\right]$ for all $\varphi \in \mathbb{X}_0$. Combining with (33), one has

$$\fint_{\Omega} G_{p,\alpha}(|\nabla u|)dx \geqslant \frac{1}{p^2\Lambda}\|u\|^p_{\mathbb{X}_0} - 1.$$

By fixing μ_0 such that $0 \leqslant \mu_0 < \frac{1}{C_0}$ and substituting above inequality into (55), it yields

$$\frac{1}{|\Omega|}\langle \mathscr{H}_{p,\alpha}(u), u\rangle \geqslant \frac{1}{p\Lambda}\big(1-\mu_0 C_0\big)\|u\|^p_{\mathbb{X}_0} - C\|g\|_{\mathbb{L}^*}\|u\|_{\mathbb{X}_0} - p\big(1-\mu_0 C_0\big). \tag{56}$$

Since $p>1$, inequality (56) implies that $\frac{\langle \mathscr{H}_{p,\alpha}(u), u\rangle}{\|u\|_{\mathbb{X}_0}}$ goes to infinity as $\|u\|_{\mathbb{X}_0} \to \infty$. Thus, $\mathscr{H}_{p,\alpha}$ is coercive.

Finally, thanks to Theorem 1.1, equation $\mathscr{H}_{p,\alpha}(u) = 0$ has a solution $u \in \mathbb{X}_0$. In other words, equation (1) has a weak solution in $u \in \mathbb{X}_0$. The proof of existence result is complete. □

5.2. The uniqueness

Proof of Theorem 1.3. Thanks to Theorem 1.2, equation (1) has a weak solution in $\mathbb{X}_0$. Assume that $u_1, u_2 \in \mathbb{X}_0$ are two weak solutions to this equation. It is sufficient to show that $u_1 = u_2$ in $\mathbb{X}_0$. Testing the variational formulas by $v = u_1 - u_2$, one gets that

$$\fint_{\Omega} \big(\mathscr{A}_{p,\alpha}(\nabla u_1) - \mathscr{A}_{p,\alpha}(\nabla u_2)\big)\cdot(\nabla u_1 - \nabla u_2)dx = 0.$$

Similar to the proof of (44), there holds

$$\begin{aligned}\fint_{\Omega} G_{p,\alpha}\big(|\nabla u_1 - \nabla u_2|\big)dx &\leqslant \varepsilon\Big(\fint_{\Omega} G_{p,\alpha}\big(|\nabla u_1|\big)dx + \fint_{\Omega} G_{p,\alpha}\big(|\nabla u_2|\big)dx\Big) \\ &\quad + C_\varepsilon \fint_{\Omega} \big(\mathscr{A}_{p,\alpha}(\nabla u_1) - \mathscr{A}_{p,\alpha}(\nabla u_2)\big)\cdot(\nabla u_1 - \nabla u_2)dx,\end{aligned}$$

for every $\varepsilon > 0$. It yields

$$\fint_\Omega G_{p,\alpha}\big(|\nabla u_1 - \nabla u_2|\big)dx = 0,$$

which allows us to conclude that $u_1 = u_2$ in $\mathbb{X}_0$. Indeed, it can be obtained by the following inequality which is deduced from Lemma 3.2 and Lemma 3.3:

$$\begin{aligned}\|u_1 - u_2\|_{\mathbb{X}_0} &\leqslant p^2\Lambda\left[\left(\fint_\Omega |\nabla u_1 - \nabla u_2|^p dx\right)^{\frac{p}{p-1}} + \fint_\Omega G_{p,\alpha}\Big(|\nabla u_1 - \nabla u_2|\Big)dx\right]\\ &\leqslant p^2\Lambda\left[\left(\fint_\Omega G_{p,\alpha}\big(|\nabla u_1 - \nabla u_2|\big)dx\right)^{\frac{p}{p-1}} + \fint_\Omega G_{p,\alpha}\big(|\nabla u_1 - \nabla u_2|\big)dx\right].\end{aligned}$$

The proof is now complete. □

Conflict of interest and declarations

The authors declared that they have no conflict of interest. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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Minh Triet Luong
Department of Mathematics
Ho Chi Minh City University of Education
Ho Chi Minh City, Vietnam
e-mail: luongminhtriet3006@gmail.com

Thi Thuy Dung Duong
Department of Mathematics
Ho Chi Minh City University of Education
Ho Chi Minh City, Vietnam
e-mail: dungduong362@gmail.com

Thi Thao Nhi Vu
Department of Mathematics
Ho Chi Minh City University of Education
Ho Chi Minh City, Vietnam
e-mail: nhivu2001@gmail.com