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A NOTE ON THE BESSEL DIFFERENTIAL EQUATION

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Abstract. In this paper, we will show the general solution of the Bessel differential equation given by $x^2y'' + xy' + (x^2 - v^2)y = 0$, where $v, x \in \mathbb{R}$ and $x > 0$, but only when $v = \frac{2m-1}{2}$ with $m \in \mathbb{N}$. Moreover, contrary to what we found in the literature, our general solution does not depend on a series of functions, our algorithm provides the exact general solution.

1. Introduction

The Bessel differential equation is one of the ordinary differential equations that appear most often in physics and engineering problems, even within mathematics itself, where it appears when applying the separation of variables technique to solve partial differential equations in polar, cylindrical, and spherical coordinates. Moreover, for example, the Bessel differential equation, or more specifically the Bessel functions, which in turn are solutions of a given Bessel differential equation, appear in the buckling of a column, the corona effect, the solutions of the Laplace and Helmholts equations, the vibration of a circular membrane, heat conduction, diffusion and propagation of electromagnetic waves, among many other natural events, see [1–5].

The Bessel differential equation has the following form:

$$x^2y'' + xy' + (x^2 - v^2)y = 0, \tag{1.1}$$

where $v, x \in \mathbb{R}$ and $x > 0$.

In this paper, we will present an algorithm that provides the general solution of the Bessel differential equation whenever $v = \frac{2m-1}{2}$ with $m \in \mathbb{N}$. However, contrary to what we found in the literature, our general solution does not depend on a series of functions, although it only refers to the parameter $v = \frac{2m-1}{2}$ with $m \in \mathbb{N}$, and keeping in mind that, in general, v is a non-negative integer in applications to physics and engineering, for example. Despite theses restrictions, our algorithm provides the exact general solution of the Bessel differential equation, and when m is small, it involves few calculations; however, as m increases, the need for calculations also increases, since it is necessary to calculate m derivatives of functions.

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2. Our main result

Let $y, z, f_1, f_2, f_3, f_4, f_5 : \mathbb{R}^+ \to \mathbb{R}$ be differentiable functions in variable x and of class C^∞.

THEOREM 1. *If* $x^2y'' + xy' + \left(x^2 - \frac{(2m-1)}{4}^2\right)y = 0$, *then*

$$y = \left[\left(c_1 \sin(2\sqrt{t}) + c_2 \cos(2\sqrt{t})\right)^{(m)}\Big|_{t=\frac{x^2}{4}} \right] x^{\frac{2m-1}{2}}$$

with $c_1, c_2 \in \mathbb{R}$ *and* $m \in \mathbb{N}$.

Proof. Let $y = f_1 z$. Then, $y' = f_1'z + f_1 z'$ and $y'' = f_1''z + 2f_1'z' + f_1 z''$. Now, consider

$$y'' + f_2 y' + f_3 y = 0 \tag{2.1}$$

Thus, $f_1''z + 2f_1'z' + f_1z'' + f_2f_1'z + f_2f_1z' + f_3f_1z = 0$, which implies

$$z'' + \left(\frac{2f_1'}{f_1} + f_2\right) z' + \left(\frac{f_1''}{f_1} + \frac{f_2f_1'}{f_1} + f_3\right) z = 0.$$

Moreover, take into account the following system:

$$\begin{cases} i) & \frac{2f_1'}{f_1} + f_2 = f_4 \\ ii) & \frac{f_1''}{f_1} + \frac{f_2f_1'}{f_1} + f_3 = f_5. \end{cases}$$

Hence, we have that

$$z'' + f_4 z' + f_5 z = 0 \tag{2.2}$$

Regarding system above, we have that, by $i)$, $f_1 = e^{\int \frac{f_4 - f_2}{2} dx}$. Replacing f_1 in $ii)$, after some calculation, we conclude that

$$\frac{f_4^2}{4} - \frac{f_2^2}{4} + \frac{f_4'}{2} - \frac{f_2'}{2} + f_3 - f_5 = 0 \tag{2.3}$$

Now, consider $f_2 = \frac{1}{x}$ and $f_3 = 1 - \frac{v^2}{x^2}$ with $v \in \mathbb{R}$, hence the equation (2.1) takes the form of the Bessel differential equation. Replacing f_2 and f_3 in equation (2.3), this one takes the following form

$$\frac{f_4^2}{4} + \frac{f_4'}{2} + \frac{1}{4x^2} - \frac{v^2}{x^2} + 1 - f_5 = 0 \tag{2.4}$$

Choosing $f_4 = \frac{2m}{x}$, with $m \in \mathbb{N}$, and $f_5 = 1$ in equation (2.4), we may conclude that $v^2 = \frac{(2m-1)^2}{4}$. Moreover, replacing f_4 and f_5 in equation (2.2), it follows that

$$z'' + \frac{2m}{x} z' + z = 0 \tag{2.5}$$

Taking into account that we still do not know the general solution of equation (2.5), we will make a variable change of the domain of the function z from x to t, and hence, from equation (2.5), we arrive at an equation whose general solution we know.

Let $g:\mathbb{R}^+\to\mathbb{R}$ be a differentiable function of class C^2 and bijective. Thus, if $x=g(t)$, then $\frac{dx}{dt}=\frac{dg(t)}{dt}\frac{dz}{dz}$, which implies $\frac{dz}{dx}=\frac{1}{g'(t)}\frac{dz}{dt}$ and $\frac{d^2z}{dx^2}=\frac{1}{(g'(t))^2}\left(\frac{d^2z}{dt^2}-\frac{g''(t)}{g'(t)}\frac{dz}{dt}\right)$. This implies that equation (2.5) may be rewritten as follows:

$$\frac{1}{(g'(t))^2}\frac{d^2z}{dt^2}+\left(\frac{2m}{g(t)g'(t)}-\frac{g''(t)}{(g'(t))^3}\right)\frac{dz}{dt}+z(t)=0 \tag{2.6}$$

Making in equation (2.6) $g(t)=2\sqrt{t}$, it follows that

$$tz''+\frac{(2m+1)}{2}z'+z=0 \tag{2.7}$$

Now, in equation (2.7), choosing $m=0$, we have that $tz''+\frac{1}{2}z'+z=0$, where $z(t)=c_1\sin(2\sqrt{t})+c_2\cos(2\sqrt{t})$ is its general solution, with $c_1,c_2\in\mathbb{R}$, since the Wronskian of the functions $\sin(2\sqrt{t})$ and $\cos(2\sqrt{t})$ is non zero for any $t\in\mathbb{R}\setminus\{0\}$.

Moreover, note that

$$\begin{aligned}
\left(tz''+\frac{1}{2}z'+z\right)' &= tz'''+\frac{3}{2}z''+z' \\
\left(tz'''+\frac{3}{2}z''+z'\right)' &= tz^{(4)}+\frac{5}{2}z'''+z'' \\
\left(tz^{(4)}+\frac{5}{2}z'''+z''\right)' &= tz^{(5)}+\frac{7}{2}z^{(4)}+z''' \\
&\vdots \\
\left(tz''+\frac{1}{2}z'+z\right)^{(m)} &= tz^{(m+2)}+\frac{(2m+1)}{2}z^{(m+1)}+z^{(m)}.
\end{aligned}$$

Let $w:\mathbb{R}^+\to\mathbb{R}$ be a differentiable function so that $w=z^{(m)}$, where m is fixed and arbitrary. Hence, regarding last equation, we may conclude that if $tw''+\frac{(2m+1)}{2}w'+w=0$, then $w(t)=(c_1\sin(2\sqrt{t})+c_2\cos(2\sqrt{t}))^{(m)}$ is its general solution, keeping in mind that the functions $\sin(2\sqrt{t})$ and $\cos(2\sqrt{t})$ are of class C^∞.

This implies that $z(x)=(c_1\sin(2\sqrt{t})+c_2\cos(2\sqrt{t}))^{(m)}\big|_{t=\frac{x^2}{4}}$ is the general solution of the equation (2.7). Moreover, $f_1=e^{\int\frac{2m-1}{2x}dx}=x^{\frac{2m-1}{2}}$. Therefore, we conclude that if $x^2y''+xy'+(x^2-\frac{(2m-1)^2}{4})y=0$, then

$$y(x)=\left[(c_1\sin(2\sqrt{t})+c_2\cos(2\sqrt{t}))^{(m)}\big|_{t=\frac{x^2}{4}}\right]x^{\frac{2m-1}{2}}$$

with $c_1,c_2\in\mathbb{R}$ and $m\in\mathbb{N}$. $\square$

REMARK 1. Let $J_\nu(x)$ be the Bessel function regarding equation (1.1). Thus,

$$J_{\frac{2m-1}{2}}(x) = \left[(c_1 \sin(2\sqrt{t}) + c_2 \cos(2\sqrt{t}))^{(m)}\Big|_{t=\frac{x^2}{4}} \right] x^{\frac{2m-1}{2}}$$

are the Bessel functions regarding equation $x^2y'' + xy' + (x^2 - \frac{(2m-1)}{4}^2)y = 0$ for any $c_1, c_2 \in \mathbb{R}$ and $m \in \mathbb{N}$.

EXAMPLE 1. Our example comes from the theory of propagation of eletromagnetic waves.

Let $w = 2\pi f$, f: frequency,

$c = \lambda f$, λ: radiation,

$K = \frac{nw}{c} = \frac{2\pi}{\lambda}$, n constant (in a vacuum, $n = 1$),

$\overrightarrow{r}$: position vector of a wave particle,

$\overrightarrow{E}$: electric field vector,

$\psi = \frac{\overrightarrow{\nabla} \times \overrightarrow{E}}{-(\frac{w}{c})^2}$.

In spherical coordinates, we have that $\psi(r,\theta,\phi) = R(r)S(\theta)T(\phi)$.

Thus,

$$\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) + K^2r^2 + \frac{1}{S(\theta)\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{dS(\theta)}{d\theta}\right) - \frac{m^2}{\sin^2(\theta)} = 0,$$

where $m^2 = \frac{1}{T(\phi)}\frac{d^2T(\phi)}{d\phi^2}$. Hence, equating the constant $q(q+1)$ to part $\frac{1}{R(r)}\frac{d}{dr}(r^2\frac{dR(r)}{dr}) + K^2r^2$, we have that $\frac{d}{dr}(r^2\frac{dR_q(r)}{dr}) - (q(q+1) - K^2r^2)R_q(r) = 0$.

Taking $Kr = x$ and $R_q(r) = x^{\frac{-1}{2}}Y_q(x)$, we arrive, after some calculation, at a Bessel differential equation given by

$$x^2Y_q''(x) + xY_q'(x) + \left(x^2 - \left(q + \frac{1}{2}\right)^2\right)Y_q(x) = 0 \tag{2.8}$$

Choosing, for example, $q = 1$ in (2.8), we have that if $x^2Y_1''(x) + xY_1'(x) + (x^2 - (\frac{3}{2})^2)Y_1(x) = 0$, then

$$Y_1(x) = \left[(c_1 \sin(2\sqrt{t}) + c_2 \cos(2\sqrt{t}))''\Big|_{t=\frac{x^2}{4}} \right] x^{\frac{3}{2}}.$$

Therefore, after some calculation, we may conclude that

$$Y_1(x) = c_1\left(\frac{\sin(x)}{\sqrt{x}} + \frac{\cos(x)}{\sqrt{x^3}}\right) + c_2\left(\frac{\sin(x)}{\sqrt{x^3}} - \frac{\cos(x)}{\sqrt{x}}\right)$$

for any $c_1, c_2 \in \mathbb{R}$.

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