Differential Equations & Applications
Volume 17, Number 3 (2025), 113–131
doi:10.7153/dea-2025-17-08

EXISTENCE OF SOLUTION AND ASYMPTOTIC BEHAVIOR FOR THE NAVIER–STOKES EQUATIONS WITH GENERAL DAMPING

ANDERSON L. A. DE ARAUJO* AND MARCELO MONTENEGRO

(*Communicated by Š. Nečasová*)

Abstract. We establish the existence of solution for the Navier-Stokes equations with a general damping term, we give examples. We also derive estimates on the asymptotic behavior of the solution, for instance, energy estimate decay in time and extinction in time. We construct a sequence of solutions of auxiliary equations in finite dimension that converges to a genuine solution of the original equations.

1. Introduction

Our aim is to study the non-stationary model

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - \Delta u(x,t) + \displaystyle\sum_{i=1}^{n} u_i(x,t)\dfrac{\partial u(x,t)}{\partial x_i} \\ \qquad\qquad +\nabla p(x,t) + h(u(x,t)) = f(t) \ \text{ in } \ \Omega\times(0,T) \\ \qquad\qquad\qquad div(u(x,t)) = 0 \ \text{ in } \ \Omega\times(0,T) \\ \qquad\qquad\qquad u(x,t) = 0 \ \text{ in } \ \partial\Omega\times(0,T) \\ \qquad\qquad\qquad u(x,0) = u_0(x) \ \text{ on } \ \Omega, \end{cases} \tag{1}$$

where $u=(u_1,u_2,\ldots,u_N)$ is a vector such that $u_i=u_i(x,t)$, where $x=(x_1,x_2,\ldots,x_N)\in\mathbb{R}^N$ with $N\geqslant 2$ and $t\geqslant 0$. The function u describes the velocity field of the fluid, $p=p(x,t)$ is the pressure at point (x,t), h is a damping term which will be specified later, f is the density of the external forces, and for simplicity we consider the viscosity of the fluid $\nu=1$, it usually appears at $\nu\Delta u$, consult [3] and [4] for a research with variable ν. Since the divergence of the flow velocity is zero, we call it incompressible flow.

The study of equations describing fluids goes back to Euler, Navier and Stokes. The mathematical analysis of the Navier-Stokes equations whenever $h=0$ was first made in [15], and subsequently in [13], [17], [22], [23]. In general, the existence of regular solution for the Navier-Stokes equations is a hard task, except for very specific cases treated in [13], [16], [23], see also [8] and [9] for an historical background.

Mathematics subject classification (2020): 35Q30, 35Q35, 65N30, 76N10.
Keywords and phrases: Navier-Stokes equations, general damping, extinction in time.
Anderson L. A. de Araujo and M. Montenegro were partially supported by CNPq.
* Corresponding author.

Paper DEA-17-08

More recently, in [6] they prove that weak solutions of the $3D$ Navier-Stokes equations are not unique in the class of weak solutions with finite kinetic energy. And in [21] the author suggests a program for generating a blow-up solution for the Navier-Stokes equations.

The function $h(s)$ represents external forces, sometimes called damping, absorption or forcing term. Certain damping terms $h(u)$ are relevant in porous media models, because they produce attrition resistances. In this line, the Brinkman-Forchheimer equations with $h(u) = au + b|u|^{r-1}u,\ a,b > 0,\ r \geqslant 3$, are studied in [19]. In [11] they consider $h(u)$ with more general polynomial growth conditions. The exponential damping $h(u) = ae^{b|u|^2-1}u,\ a,b > 0$ interferes in the equations analyzed in [5]. In [12] the authors study the large time behavior of a deterministic and stochastic $3D$ convective Brinkman-Forchheimer equations in periodic domains with $h(s) = |s|^{\sigma-1}s$ and $\sigma \geqslant 3$. In [1] it was studied the modified Navier-Stokes problem by introducing in the equations the absorption term $h(u) = |u|^{\sigma-1}u$. They prove existence of weak solutions for dimension $N \geqslant 2$ and uniqueness for $N = 2$. The weak solutions extinct in a finite time if $1 < \sigma < 2$ and exponentially decay in time if $\sigma = 2$. We establish such results with a general h. In [2] the authors study the Kelvin-Voight equations with p-Laplacian and damping term $h(s) = |s|^{\sigma-1}s$ with $0 < \sigma < \infty$, they prove existence of solution, uniqueness and finite time blow-up. In [10] they show existence and uniqueness of solutions for the Navier-Stokes equations with Navier slip boundary conditions in a $3D$ bounded domain and damping term $h(s) = |s|^{\sigma-1}s$ with $\sigma \geqslant 1$.

In the present paper we study (1) with h verifying a different behavior compared to the previous listed papers.

We will study (1) with the function

$$h = (h_1, h_2, \ldots, h_N) : \mathbb{R}^N \to \mathbb{R}^N \text{ continuous} \tag{2}$$

satisfying

$$h_i(x)x_i \geqslant 0 \text{ for each } x_i \in \mathbb{R} \text{ and } i = 1,2,\ldots,N, \text{ where } x = (x_1,x_2,\ldots,x_N) \in \mathbb{R}^N. \tag{3}$$

And we assume that there exist constants $C_h > 0$ and $\sigma > 0$ such that

$$1 < \sigma \leqslant 2 \tag{4}$$

and

$$|h_i(x)| \leqslant C_h |x|^{\sigma-2}|x_i| \text{ for each } x = (x_1,x_2,\ldots,x_N) \in \mathbb{R}^N \text{ and } i = 1,2,\ldots,N. \tag{5}$$

EXAMPLES.

1. A straightforward example is $h(x) = (|x|^{\sigma-2}x_1, |x|^{\sigma-2}x_2, \ldots, |x|^{\sigma-2}x_N)$, notice that $h_i(x)x_i = |x|^{\sigma-2}x_i^2 \geqslant 0$ for each $i = 1,2,\ldots,N$, but each component h_i might change sign.

2. Another example is $h(x) = (|x|^{\sigma_1-2}x_1, |x|^{\sigma_2-2}x_2, \ldots, |x|^{\sigma_N-2}x_N)$ with $1 < \sigma_i \leqslant 2$ for $i = 1,2,\ldots,N$.

3. It is also easy to see that h may have components oscillating between two powers, for instance $c_\varsigma |x|^{\varsigma_i-2}x_i \leqslant h_i(x) \leqslant c_\vartheta |x|^{\vartheta_i-2}x_i$ with $1 < \varsigma_i, \vartheta_i \leqslant 2$ for $i = 1,2,\ldots,N$ for some constants $c_\varsigma, c_\vartheta > 0$.

These examples enable us to work with (1) in a more general or different context than in the previously quoted papers.

We state the main result.

THEOREM 1.1. *Let* $N \geqslant 2$, $f \in L^2(0,T;V'(\Omega))$ *and* $u_0 \in H(\Omega)$ *and assume* (2)–(5). *Then there is a weak solution* u *of* (1)*, such that* $u \in L^\infty(0,T;H(\Omega)) \cap L^2(0,T;V(\Omega))$.

We summarize the methods we will employ in the continuation of the paper.

- In Section 2 we define the function spaces, norms, inner products and the concept of solution. We emphasize that Theorem 1.1 is enounced for every dimension $N \geqslant 2$.
- In Section 3 we describe a way to find a sequence R_k with components $h_{i,k}$, $i = 1,\dots,N$ verifying (2)–(5). To assert that R_k converges to h, we develop detailed estimates for $h_{i,k}$ with well controlled constants, which are independent of dimension and other appearing indexes, in conformity with Lemmas 3.1, 3.2 and 3.3.
- In Section 4 we use the Galerkin method to find solutions of auxiliary equations in finite dimension, which are close to (1), see (19), (20), (21). We also construct a sequence of functions R_k converging to the damping term h. The finite dimensional problems have solutions forming a sequence that converges to a true solution of the original equations (1), leading to the proof of Theorem 1.1. The Lipschitz components of R_k are fundamental to solve (1), see Lemma 3.1 and Remark 4.1 for specific details.
- In Section 5 we present an energy estimate decay in time and an asymptotic formula, leading to the extinction in time. These are the contents of Theorems 5.1 and 5.2.

2. Space function setting

We briefly recall some function spaces used in the research of Navier-Stokes equations. Let $\Omega \subset \mathbb{R}^N$ be a nonempty bounded open set with smooth boundary. We define

$$\mathscr{C}_{div}(\Omega) = \{w \in (C_0^\infty(\Omega))^N : div(w) = 0\},$$

$$H(\Omega) = \overline{\mathscr{C}_{div}(\Omega)}^{(L^2(\Omega))^N}$$

and

$$V(\Omega) = \overline{\mathscr{C}_{div}(\Omega)}^{(H_0^1(\Omega))^N},$$

where $C_0^\infty(\Omega)$ is the set of infinitely differentiable functions with compact support in Ω. And $(C_0^\infty(\Omega))^N$, $(L^2(\Omega))^N$ and $(H_0^1(\Omega))^N$ stand for the N-Cartesian product of the inherent spaces, respectively. Properties of $\mathscr{C}_{div}(\Omega)$, $H(\Omega)$ and $V(\Omega)$ are in [23]. We denote by $V'(\Omega)$ the topological dual of $V(\Omega)$ and, $\langle .,. \rangle$ is the duality pairing between $V'(\Omega)$ and $V(\Omega)$. Everywhere in the paper we use the dot $\mathbf{.}$ to mean the Euclidean inner product in $\mathbb{R}^N$.

The inner product and norm of the spaces $V(\Omega)$ and $H(\Omega)$ are, respectively

$$((u,v)) = \sum_{i,j=1}^{N} \int_\Omega \frac{\partial u_i}{\partial x_j}(x) \frac{\partial v_i}{\partial x_j}(x) dx, \quad \|u\|^2_{V(\Omega)} = \sum_{i,j=1}^{N} \int_\Omega \left(\frac{\partial u_i}{\partial x_j}(x) \right)^2 dx$$

and

$$(u,v) = \sum_{i=1}^{N} \int_\Omega u_i(x) v_i(x) dx, \quad \|u\|^2_{H(\Omega)} = \sum_{i=1}^{N} \int_\Omega |u_i(x)|^2 dx.$$

For each $v \in V(\Omega)$ and $1 < q < \infty$, we denote the norms $\|\nabla u\|_{L^2(\Omega)}$ and $\|u\|_{L^q(\Omega)}$ by

$$\|\nabla u\|^2_{L^2(\Omega)} = \sum_{i=1}^{N} \int_\Omega |\nabla u_i|^2 dx$$

and

$$\|u\|^q_{L^q(\Omega)} = \sum_{i=1}^{N} \int_\Omega |u_i|^q dx.$$

Let $T > 0$ be a real number. We denote by $L^2(0,T;H(\Omega))$, $L^2(0,T;V(\Omega))$, $L^2(0,T;V'(\Omega))$, $L^\infty(0,T;H(\Omega))$ the Banach spaces of measurable functions $v : [0,T] \to H(\Omega)$, respectively, with the following norms

$$\|v\|_{L^2(0,T;H(\Omega))} = \left(\int_0^T |v(t)|^2_{H(\Omega)} dt \right)^{1/2},$$

$$\|v\|_{L^2(0,T;V(\Omega))} = \left(\int_0^T \|v(t)\|^2_{V(\Omega)} dt \right)^{1/2},$$

$$\|v\|_{L^2(0,T;V'(\Omega))} = \left(\int_0^T \|v(t)\|^2_{V'(\Omega)} dt \right)^{1/2},$$

$$\|v\|_{L^\infty(0,T;H(\Omega))} = \sup_{t \in [0,T]} \text{ess}\, \|v(t)\|_{H(\Omega)}.$$

We denote by $C^0([0,T];V'(\Omega))$ the Banach space of continuous functions $v : [0,T] \to V'(\Omega)$ with norm

$$\|v\|_{C^0([0,T];V'(\Omega))} = \max_{t \in [0,T]} \|v(t)\|_{V'(\Omega)}.$$

By Fubini's theorem, we identify the space $L^q(0,T;L^q(\Omega))$ with $L^q(\Omega \times (0,T))$ for $1 < q < \infty$ with norm

$$\|v\|_{L^q(0,T;L^q(\Omega))} = \left(\int_0^T \|v(t)\|^q_{L^q(\Omega)} dt \right)^{1/q} = \left(\int_0^T \int_\Omega |v(x,t)|^q dx dt \right)^{1/q}.$$

We also denote by $C_0^\infty(\Omega\times(0,T))$ the set of infinitely differentiable functions with compact support in $\Omega\times(0,T)$ and $C_0^\infty(0,T)$ is the set of infinitely differentiable functions with compact support in $(0,T)$. We also denote $\mathscr{D}'(0,T)=(C_0^\infty(0,T))'$, that is, the distribution space in the interval $(0,T)$. Analogously, $\mathscr{D}'(\Omega\times(0,T))=(C_0^\infty(\Omega\times(0,T)))'$.

Throughout this note it will be useful to define the trilinear form by

$$b(u,v,w)=\sum_{i,j=1}^{N}\int_\Omega u_j(x)\frac{\partial v_i}{\partial x_j}(x)w_i(x)dx \text{ for } u,v,w\in V(\Omega). \tag{6}$$

Let $f\in L^2(0,T;V'(\Omega))$ and $u_0\in H(\Omega)$. According to [23, p. 280–281], $u\in L^2(0,T;V(\Omega))$ is a weak solution of (1) if

$$\begin{aligned}&(u'(t),v)+((u(t),v))+b(u(t),u(t),v)+(h(u(t)),v)=(f(t),v),\\ &\forall v\in V(\Omega) \text{ and for a.e. } 0<t<T\end{aligned} \tag{7}$$

and

$$u(x,0)=u_0 \text{ for a.e. } x\in\Omega. \tag{8}$$

The mapping $-\Delta:V(\Omega)\to V'(\Omega)$ is such that

$$||-\Delta u||_{V'(\Omega)}=\sup_{\|v\|_{V(\Omega)}=1}|\langle-\Delta u,v\rangle|\leqslant\|u\|_{V(\Omega)} \tag{9}$$

and the following lemma is in [18].

LEMMA 2.1. *The mapping $-\Delta:V(\Omega)\to V'(\Omega)$ is continuous.*

LEMMA 2.2. *For all $u\in V(\Omega)$ the bilinear form B defined by $v\mapsto\langle Bu,v\rangle=b(u,u,v)$ is continuous on $V(\Omega)$, in other words, $Bu\in V'(\Omega)$ and*

$$\|Bu\|_{V'(\Omega)}\leqslant C\|u\|^2_{V(\Omega)}. \tag{10}$$

Proof. Let $u\in V(\Omega)$, then the mapping B defined by $v\mapsto\langle Bu,v\rangle=b(u,u,v)$ is linear and continuous in $V(\Omega)$. Hence

$$|\langle Bu,v\rangle|=|b(u,u,v)|\leqslant C\|u\|^2_{V(\Omega)}\|v\|_{V(\Omega)}.$$

Then

$$\sup_{\|v\|_{V(\Omega)}=1}|\langle Bu,v\rangle|\leqslant C||u||^2_{V(\Omega)}.$$

Therefore,

$$||Bu||_{V'(\Omega)}\leqslant C\|u\|^2_{V(\Omega)},\quad \forall u\in V(\Omega). \quad\square$$

The next embedding is taken form [16, Théorème 5.1].

LEMMA 2.3. *Assume that $V(\Omega), H(\Omega)$ and $V'(\Omega)$ are Banach spaces and $V(\Omega) \hookrightarrow H(\Omega) \hookrightarrow V'(\Omega)$ are continuous embeddings and the embedding $V(\Omega) \hookrightarrow H(\Omega)$ is compact. Then, it is also compact*

$$L^2(0,T;V(\Omega)) \cap \{\varphi : \varphi' \in L^2(0,T;V'(\Omega))\} \hookrightarrow L^2(0,T;H(\Omega)).$$

We remark that the dimension in the current paper is $N \geqslant 2$. Notice that $u \in V(\Omega)$ implies $u_i \in H_0^1(\Omega)$ for $i = 1,\dots,N$. By the Sobolev embedding, we conclude that $u_i \in L^q(\Omega)$ for $1/q = 1/2 - 1/N$ and $i = 1,\dots,N$. The trilinear form (6) is then well defined if, and only if, $N \leqslant 4$, because $1/q + 1/2 + 1/N = 1$, whence

$$\left| \int_\Omega u_j(x) \frac{\partial v_i}{\partial x_j}(x) w_i(x) dx \right| \leqslant \|u_j\|_{L^q(\Omega)} \left\| \frac{\partial v_i}{\partial x_j} \right\|_{L^2(\Omega)} \|w_i\|_{L^N(\Omega)}.$$

And observe that $V(\Omega) \cap (L^N(\Omega))^N = V(\Omega)$ if $2 \leqslant N \leqslant 4$. If $N \geqslant 5$, then $b(u,v,w)$ is clearly well defined in $V(\Omega) \times V(\Omega) \times (V(\Omega) \cap (L^N(\Omega))^N$.

3. Sequence converging to the damping term

Recall that $h = (h_1, h_2, \dots, h_N) : \mathbb{R}^N \to \mathbb{R}^N$ is continuous and verifies (2)–(5) with $1 < \sigma \leqslant 2$. In the course proof of Theorem 1.1 we build a sequence $h_{i,k}$ that converges to each h_i for $i = 1,2,\dots,N$, and then $R_k = (h_{1,k}, h_{2,k}, \dots, h_{N,k}) \to h$ uniformly on bounded sets of $\mathbb{R}^N$ as $k \to \infty$. Define $h_{i,k} : \mathbb{R}^N \to \mathbb{R}$ by

$$h_{i,k}(x) = \begin{cases} -k[H_i(x_1,\dots,x_{j-1},-k-\frac{1}{k},x_{j+1},\dots,x_N) - H_i(x_1,\dots,x_{j-1},-k,x_{j+1},\dots,x_N)], \\ \text{if} \quad x_i \leqslant -k, \\ -k[H_i(x_1,\dots,x_{j-1},x_i-\frac{1}{k},x_{j+1},\dots,x_N) - H_i(x_1,\dots,x_{j-1},x_i,x_{j+1},\dots,x_N)], \\ \text{if} \quad -k \leqslant x_i \leqslant -\frac{1}{k}, \\ k^2 s[H_i(x_1,\dots,x_{j-1},-\frac{2}{k},x_{j+1},\dots,x_N) - H_i(x_1,\dots,x_{j-1},-\frac{1}{k},x_{j+1},\dots,x_N)], \\ \text{if} \quad -\frac{1}{k} \leqslant x_i \leqslant 0, \\ k^2 s[H_i(x_1,\dots,x_{j-1},\frac{2}{k},x_{j+1},\dots,x_N) - H_i(x_1,\dots,x_{j-1},\frac{1}{k},x_{j+1},\dots,x_N)], \\ \text{if} \quad 0 \leqslant x_i \leqslant \frac{1}{k}, \\ k[H_i(x_1,\dots,x_{j-1},x_i+\frac{1}{k},x_{j+1},\dots,x_N) - H_i(x_1,\dots,x_{j-1},x_i,x_{j+1},\dots,x_N)], \\ \text{if} \quad \frac{1}{k} \leqslant x_i \leqslant k, \\ k[H_i(x_1,\dots,x_{j-1},k+\frac{1}{k},x_{j+1},\dots,x_N) - H_i(x_1,\dots,x_{j-1},k,x_{j+1},\dots,x_N)], \\ \text{if} \quad x_i \geqslant k, \end{cases} \tag{11}$$

where $k \in \mathbb{N}$ and $H : \mathbb{R}^N \to \mathbb{R}^N$ has components $H = (H_1, H_2, \dots, H_N)$ such that

$$H_i(x_1,\dots,x_{j-1},s,x_{j+1},\dots,x_N) = \int_0^s h_i(x_1,\dots,x_{j-1},\zeta,x_{j+1},\dots,x_N) d\zeta,$$

hence $\frac{\partial H_i}{\partial x_i}(x) = h_i(x)$ and $H_i(x_1,\dots,x_{j-1},0,x_{j+1},\dots,x_N) = 0$. Also, define

$$x_{(i)} = (x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_N),$$

which is $x = (x_1, \dots, x_i, \dots, x_N)$ with x_i coordinate omitted. Taking into account (2) and (3), by virtue of Strauss [20, Lemma 2.2] and using the explicit expression of the sequence (11), we obtain the following result.

LEMMA 3.1. *Let h and σ be as in (2)–(5). The sequence $h_{i,k} : \mathbb{R}^N \to \mathbb{R}$ of continuous functions (11), verifies*

(A) *$h_{i,k}(x)x_i \geqslant 0$ for every $x \in \mathbb{R}^N$;*

(B) *for every $k \in \mathbb{N}$ there is a continuous function $x_{(i)} \mapsto c_{i,k}(x_{(i)})$ such that*

$$|h_{i,k}(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_N) - h_{i,k}(x_1,\dots,x_{i-1},y_i,x_{i+1},\dots,x_N)| \leqslant c_{i,k}(x_{(i)})|x_i - y_i|$$

for every $x_i, y_i \in \mathbb{R}$, $x_{(i)} \in \mathbb{R}^{N-1}$ and $i = 1,2,\dots,N$;

(C) *$h_{i,k}$ converges uniformly to h_i as $k \to \infty$ in bounded sets of $\mathbb{R}^N$ for $i = 1,2,\dots,N$.*

The constants $c_{i,k}(.)$ are well behaved as it is conveyed next.

LEMMA 3.2. *Let h and σ fulfilling (2)–(5). The Lipschitz constants $c_{i,k}(.)$ from Lemma 3.1, verify*

$$c_{i,k}(x_{(i)}) \leqslant Ck \sup_{x_i} \left\{ |h_i(x_1,\dots,x_i,\dots,x_N)| : x_i \in \left[-k - \frac{1}{k}, k + \frac{1}{k} \right] \right\}, \forall x_{(i)} \in \mathbb{R}^{N-1}, \tag{12}$$

where the constant C does not depend neither on x nor on k. Moreover, the sequence

$$R_k = (h_{1,k}, h_{2,k}, \dots, h_{N,k}) \tag{13}$$

satisfies the estimate

$$|R_k(x)| \leqslant \widetilde{C}(k)(|x| + |x|^{\sigma-1}), \ \forall x \in \mathbb{R}^N, \tag{14}$$

for some constant $\widetilde{C}(k) > 0$ that depends only on $k \in \mathbb{N}$.

Proof. For a fixed $i = 1,2,\dots,N$, notice that by Lemma 3.2 and (5) we have

$$\begin{aligned} c_{i,k}(x_{(i)}) &\leqslant Ck \sup_t \left\{ |h_i(x)| : x_i \in \left[-k - \tfrac{1}{k}, k + \tfrac{1}{k} \right] \right\}, \ \forall x_{(i)} \in \mathbb{R}^{N-1} \\ &\leqslant Ck \sup_t \left\{ C_h |x|^{\sigma-2}|x_i| : x_i \in \left[-k - \tfrac{1}{k}, k + \tfrac{1}{k} \right] \right\}, \ \forall x_{(i)} \in \mathbb{R}^{N-1}. \end{aligned}$$

Since $|x| = \sqrt{x_i^2 + |x_{(i)}|^2}$, we obtain for each $k \in \mathbb{N}$ that

$$c_{i,k}(x_{(i)}) \leqslant C_h C k \left(k + \frac{1}{k} \right) \left(\left(k + \frac{1}{k} \right)^2 + |x_{(i)}|^2 \right)^{\frac{\sigma-2}{2}}, \ \forall x_{(i)} \in \mathbb{R}^{N-1}, \tag{15}$$

where C_h appeared in (5). Since $R_k = (h_{1,k}, h_{2,k}, \dots, h_{N,k})$ we obtain for the Euclidean inner product with dot . notation

$$R_k(x) \cdot x \geqslant 0, \ \forall x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N.$$

Hence

$$
\begin{aligned}
|R_k(x)-R_k(y)| &= \sum_{i=1}^{N} |h_{i,k}(x_1,x_2,\ldots,x_N)-h_{i,k}(y_1,y_2,\ldots,y_N)| \\
&\leqslant \sum_{i=1}^{N} \big(|h_{i,k}(x_1,x_2,\ldots,x_N)-h_{i,k}(y_1,x_2,\ldots,x_N)| \\
&\quad +|h_{i,k}(y_1,x_2,x_3,\ldots,x_N)-h_{i,k}(y_1,y_2,x_3,\ldots,x_N)| \\
&\quad +|h_{i,k}(y_1,y_2,x_3,\ldots,x_N)-h_{i,k}(y_1,y_2,y_3,x_3,\ldots,x_N)| \\
&\quad +\ldots \\
&\quad + |h_{i,k}(y_1,\ldots,y_{N-1},x_N)-h_{i,k}(y_1,y_2,\ldots,y_N)|\big)\,,\ \forall x,y\in\mathbb{R}^N.
\end{aligned}
$$

By Lemma 3.1 (B) we obtain

$$
\begin{aligned}
|R_k(x)-R_k(y)| &\leqslant \sum_{i=1}^{N} \big(C_{i,k}(x_2,\ldots,x_N)|x_1-y_1| \\
&\quad +C_{i,k}(y_1,x_3,\ldots,x_N|x_2-y_2| \\
&\quad +C_{i,k}(y_1,y_2,x_4,\ldots,x_N)|x_3-y_3| \\
&\quad +\ldots \\
&\quad + C_{i,k}(y_1,\ldots,y_{N-1})|x_N-y_N|\big) \\
&= \sum_{i=1}^{N}\left(\sum_{j=1}^{N} C_{i,k}(y_1,\ldots,y_{j-1},x_{j+1},\ldots,x_N)|x_j-y_j|\right) \\
&\leqslant \left(\sum_{i=1}^{N} \max_{j\in\{1,2,\ldots,N\}} C_{i,k}(y_1,\ldots,y_{j-1},x_{j+1},\ldots,x_N)\right)|x-y|.
\end{aligned}
$$

From (15) and taking $y=0$, we get

$$
\begin{aligned}
|R_k(x)| &\leqslant C_hCk\left(k+\frac{1}{k}\right)\left(N\max_{j\in\{1,2,\ldots,N\}}\left(\left(k+\frac{1}{k}\right)^2+|(0,\ldots,0,x_{j+1},\ldots,x_N)|^2\right)^{\frac{\sigma-2}{2}}\right)|x| \\
&\leqslant NC_hCk\left(k+\frac{1}{k}\right)\left(\left(k+\frac{1}{k}\right)^2+|x|^2\right)^{\frac{\sigma-2}{2}}|x|.
\end{aligned}
$$

Thus there is a constant $\widetilde{C}(k)>0$ such that

$$
|R_k(x)|^2 \leqslant \widetilde{C}(k)(|x|^2+|x|^{2\sigma-2}),\ \forall x\in\mathbb{R}^N.
$$

Since σ satifies (4), there exists another constant $\widetilde{C}(k)>0$ such that (14) holds. □

We need refined estimates on the sequences $h_{i,k}$ for $i=1,2,\ldots,N$ with constants that do depend of k.

LEMMA 3.3. *Let h and σ verifying (2)–(5). And let $h_{i,k}$ be the sequence as in Lemma 3.1. Then*

(I) for every $k \in \mathbb{N}$, one has $|h_{i,k}(x)| \leqslant 2^{\sigma-1}C_h|x|^{\sigma-1}$ for $|x_i| \geqslant \frac{1}{k}$, with $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$ and $i = 1, 2, \ldots, N$;

(II) for every $k \in \mathbb{N}$, $|h_{i,k}(x)| \leqslant C_h|(x_1, \ldots, x_{i-1}, 2, x_{i+1}, \ldots x_N)|^{\sigma-1}$ for $|x_i| \leqslant \frac{1}{k}$, with $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$ and $i = 1, 2, \ldots, N$.

Proof. The constant C_h is the one of (5) and recall that $x_{(i)} = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$. We discriminate 4 steps below.

Step 1. Suppose that $-k \leqslant x_i \leqslant -\frac{1}{k}$.

By the mean value theorem, there exists $\eta_i \in (x_i - \frac{1}{k}, x_i)$ such that

$$\begin{aligned} h_{i,k}(x) &= -k[H_i(x_1, \ldots, x_{i-1}, x_i - \tfrac{1}{k}, x_{i+1}, \ldots, x_N) - H_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_N)] \\ &= -k\tfrac{\partial}{\partial x_i} H_i(x_1, \ldots, x_{i-1}, \eta_i, x_{i+1}, \ldots, x_N)(x_i - \tfrac{1}{k} - x_i) \\ &= h_i(x_1, \ldots, x_{i-1}, \eta_i, x_{i+1}, \ldots, x_N) \end{aligned}$$

and

$$h_{i,k}(x) = h_i(x_1, \ldots, x_{i-1}, \eta_i, x_{i+1}, \ldots x_N).$$

Since $x_i - \frac{1}{k} < \eta_i < x_i < 0$, we have

$$\begin{aligned} |h_{i,k}(x)| &= |h_i(x_1, \ldots, x_{i-1}, \eta_i, x_{i+1}, \ldots x_N)| \\ &\leqslant C_h|\eta_i||(x_1, \ldots, x_{i-1}, \eta_i, x_{i+1}, \ldots x_N)|^{\sigma-2} \\ &\leqslant C_h|(x_1, \ldots, x_{i-1}, \eta_i, x_{i+1}, \ldots x_N)|^{\sigma-1} \\ &\leqslant C_h|(x_1, \ldots, x_{i-1}, x_i - \tfrac{1}{k}, x_{i+1}, \ldots x_N)|^{\sigma-1} \\ &\leqslant C_h|(x_1, \ldots, x_{i-1}, |x_i| + \tfrac{1}{k}), x_{i+1}, \ldots x_N)|^{\sigma-1} \\ &\leqslant C_h|(x_1, \ldots, x_{i-1}, 2|x_i|, x_{i+1}, \ldots x_N)|^{\sigma-1} \\ &\leqslant 2^{\sigma-1}C_h|x|^{\sigma-1}. \end{aligned}$$

Step 2. Assume $\frac{1}{k} \leqslant x_i \leqslant k$.

By the mean value theorem, there exists $\eta_i \in (x_i, x_i + \frac{1}{k})$ such that

$$\begin{aligned} h_{i,k}(x) &= k[H_i(x_1, \ldots, x_{i-1}, x_i + \tfrac{1}{k}, x_{i+1}, \ldots, x_N) - H_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_N)] \\ &= k\tfrac{\partial}{\partial x_i} H_i(x_1, \ldots, x_{i-1}, \eta_i, x_{i+1}, \ldots, x_N)(x_i + \tfrac{1}{k} - x_i) \\ &= h_i(x_1, \ldots, x_{i-1}, \eta_i, x_{i+1}, \ldots, x_N) \end{aligned}$$

and

$$h_{i,k}(x) = h_i(x_1, \ldots, x_{i-1}, \eta_i, x_{i+1}, \ldots x_N).$$

Since $0 < x_i < \eta_i < x_i + \frac{1}{k}$, we have

$$\begin{aligned}
|h_{i,k}(x)| &\leqslant C_h|\eta_i||(x_1,\ldots,x_{i-1},\eta_i,x_{i+1},\ldots x_N)|^{\sigma-2}\\
&\leqslant C_h|(x_1,\ldots,x_{i-1},\eta_i,x_{i+1},\ldots x_N)|^{\sigma-1}\\
&\leqslant C_h|(x_1,\ldots,x_{i-1},x_i+\tfrac{1}{k},x_{i+1},\ldots x_N)|^{\sigma-1}\\
&\leqslant C_h|(x_1,\ldots,x_{i-1},|x_i|+\tfrac{1}{k},x_{i+1},\ldots x_N)|^{\sigma-1}\\
&\leqslant C_h|(x_1,\ldots,x_{i-1},2x_i,x_{i+1},\ldots x_N)|^{\sigma-1}\\
&\leqslant 2^{\sigma-1}C_h|x|^{\sigma-1}.
\end{aligned}$$

Step 3. Suppose that $|x_i| \geqslant k$, then

$$h_{i,k}(x) = \begin{cases} -k[H_i(x_1,\ldots,x_{i-1},-k-\frac{1}{k},x_{i+1},\ldots,x_N) - H_i(x_1,\ldots,x_{i-1},-k,x_{i+1},\ldots,x_N)], \\ \quad \text{if } x_i \leqslant -k \\ k[H_i(x_1,\ldots,x_{i-1},k+\frac{1}{k},x_{i+1},\ldots,x_N) - H_i(x_1,\ldots,x_{i-1},k,x_{i+1},\ldots,x_N)], \\ \quad \text{if } x_i \geqslant k. \end{cases}$$

If $x_i \leqslant -k$, by the mean value theorem, there exists $\eta_i \in (-k-\frac{1}{k}, -k)$ such that

$$\begin{aligned}
h_{i,k}(x) &= -k[H_i(x_1,\ldots,x_{i-1},-k-\tfrac{1}{k},x_{i+1},\ldots,x_N) - H_i(x_1,\ldots,x_{i-1},-k,x_{i+1},\ldots,x_N)]\\
&= -k\tfrac{\partial}{\partial x_i}H_i(x_1,\ldots,x_{i-1},\eta_i,x_{i+1},\ldots,x_N)(-k-\tfrac{1}{k}-(-k))\\
&= h_i(x_1,\ldots,x_{i-1},\eta_i,x_{i+1},\ldots,x_N)
\end{aligned}$$

and

$$h_{i,k}(x) = h_i(x_1,\ldots,x_{i-1},\eta_i,x_{i+1},\ldots x_N).$$

Since $-k-\frac{1}{k} < \eta_i < -k < 0$ and $k < |\eta_i| < k+\frac{1}{k}$, we conclude that

$$\begin{aligned}
|h_{i,k}(x)| &= |h_i(x_1,\ldots,x_{i-1},\eta_i,x_{i+1},\ldots x_N)|\\
&\leqslant C_h|\eta_i||(x_1,\ldots,x_{i-1},\eta_i,x_{i+1},\ldots x_N)|^{\sigma-2}\\
&\leqslant C_h|(x_1,\ldots,x_{i-1},\eta_i,x_{i+1},\ldots x_N)|^{\sigma-1}\\
&\leqslant C_h|(x_1,\ldots,x_{i-1},k+\tfrac{1}{k},x_{i+1},\ldots x_N)|^{\sigma-1}\\
&\leqslant C_h|(x_1,\ldots,x_{i-1},|x_i|+\tfrac{1}{k},x_{i+1},\ldots x_N)|^{\sigma-1}\\
&\leqslant C_h|(x_1,\ldots,x_{i-1},2|x_i|,x_{i+1},\ldots x_N)|^{\sigma-1}\\
&\leqslant 2^{\sigma-1}C_h|x|^{\sigma-1}.
\end{aligned} \tag{16}$$

If $x_i \geqslant k$, by the mean value theorem, there exists $\eta_i \in (k, k+\frac{1}{k})$ such that

$$\begin{aligned}
h_{i,k}(x) &= k[H_i(x_1,\ldots,x_{i-1},k+\tfrac{1}{k},x_{i+1},\ldots,x_N) - H_i(x_1,\ldots,x_{i-1},k,x_{i+1},\ldots,x_N)]\\
&= k\tfrac{\partial}{\partial x_i}H_i(x_1,\ldots,x_{i-1},\eta_i,x_{i+1},\ldots,x_N)(k+\tfrac{1}{k}-k)\\
&= h_i(x_1,\ldots,x_{i-1},\eta_i,x_{i+1},\ldots,x_N).
\end{aligned}$$

By computations similar to conclude (16) one has

$$|h_{i,k}(x)| = |h_i(x_1,\ldots,x_{i-1},\eta_i,x_{i+1},\ldots x_N)| \leqslant 2^{\sigma-1}C_h|x|^{\sigma-1}.$$

Step 4. Assume $-\frac{1}{k} \leqslant x_i \leqslant \frac{1}{k}$, then

$$h_{i,k}(x) = \begin{cases} k^2x_i[H_i(x_1,\ldots,x_{i-1},\frac{-2}{k},x_{i+1},\ldots,x_N) - H_i(x_1,\ldots,x_{i-1},\frac{-1}{k},x_{i+1},\ldots,x_N)], \\ \quad \text{if } -\frac{1}{k} \leqslant x_i \leqslant 0 \\ k^2x_i[H_i(x_1,\ldots,x_{i-1},\frac{2}{k},x_{i+1},\ldots,x_N) - H_i(x_1,\ldots,x_{i-1},\frac{1}{k},x_{i+1},\ldots,x_N)], \\ \quad \text{if } 0 \leqslant x_i \leqslant \frac{1}{k}. \end{cases}$$

If $-\frac{1}{k} \leqslant x_i \leqslant 0$, by the mean value theorem, there exists $\eta_i \in (-\frac{2}{k},-\frac{1}{k})$ such that

$$\begin{aligned} h_{i,k}(x) &= k^2x_i[H_i(x_1,\ldots,x_{i-1},\tfrac{-2}{k},x_{i+1},\ldots,x_N) - H_i(x_1,\ldots,x_{i-1},\tfrac{-1}{k},x_{i+1},\ldots,x_N)] \\ &= k^2x_i\tfrac{\partial}{\partial x_i}H_i(x_1,\ldots,x_{i-1},\eta_i,x_{i+1},\ldots,x_N)(-\tfrac{2}{k}-(-\tfrac{1}{k})) \\ &= -kx_ih_i(x_1,\ldots,x_{i-1},\eta_i,x_{i+1},\ldots,x_N). \end{aligned}$$

Therefore

$$\begin{aligned} |h_{i,k}(x)| &= |-kx_ih_i(x_1,\ldots,x_{i-1},\eta_i,x_{i+1},\ldots x_N)| \\ &= k|x_i||h_i(x_1,\ldots,x_{i-1},\eta_i,x_{i+1},\ldots x_N)| \\ &\leqslant C_hk|x_i||\eta_i||(x_1,\ldots,x_{i-1},\eta_i,x_{i+1},\ldots x_N)|^{\sigma-2} \\ &\leqslant C_h|(x_1,\ldots,x_{i-1},\eta_i,x_{i+1},\ldots x_N)|^{\sigma-1} \\ &\leqslant C_h|(x_1,\ldots,x_{i-1},\frac{2}{k},x_{i+1},\ldots x_N)|^{\sigma-1} \\ &\leqslant C_h|(x_1,\ldots,x_{i-1},2,x_{i+1},\ldots x_N)|^{\sigma-1}. \end{aligned} \tag{17}$$

If $0 \leqslant x_i \leqslant \frac{1}{k}$, by the mean value theorem, there exists $\eta_i \in (\frac{1}{k},\frac{2}{k})$ such that

$$\begin{aligned} h_{i,k}(x) &= k^2x_i[H_i(x_1,\ldots,x_{i-1},\tfrac{2}{k},x_{i+1},\ldots,x_N) - H_i(x_1,\ldots,x_{i-1},\tfrac{1}{k},x_{i+1},\ldots,x_N)] \\ &= k^2x_i\tfrac{\partial}{\partial x_i}H_i(x_1,\ldots,x_{i-1},\eta_i,x_{i+1},\ldots,x_N)(\tfrac{2}{k}-\tfrac{1}{k}) \\ &= kx_ih_i(x_1,\ldots,x_{i-1},\eta_i,x_{i+1},\ldots,x_N). \end{aligned}$$

By similar computations to conclude (17) one obtains

$$|h_{i,k}(x)| \leqslant C_h|(x_1,\ldots,x_{i-1},2,x_{i+1},\ldots x_N)|^{\sigma-1}.$$

The proof of *(I)* and *(II)* is done. $\square$

4. Proof of Theorem 1.1

In this section we will employ the Galerkin method to solve (1).

Proof. We prove now Theorem 1.1. According to Lemmas 3.1, 3.2 and 3.3, there exists a sequence of functions $(R_k)_{k\in\mathbb{N}}$, with $R_k = (h_{1,k}, h_{2,k}, \dots, h_{N,k})$, where each $h_{i,k}$ was defined in (11) and Lemma 3.1 for $i = 1,2,\dots,N$, see also Lemma 3.2 and (13). Hence,

$$R_k : \mathbb{R}^N \to \mathbb{R}^N \text{ is locally Lipschitz continuous and } R_k(x) \cdot x \geqslant 0, \forall x \in \mathbb{R}^N$$

and by Lemma 3.1 (C) we conclude

$$R_k \to h \text{ uniformly on bounded sets of } \mathbb{R}^N \text{ as } k \to \infty.$$

Let $\{v_k : k \in \mathbb{N}\}$ be an orthonormal basis of $H(\Omega)$ and let $W_m(\Omega)$ be the space generated by $\{v_1, v_2, \dots, v_m\}$ according to [16, p.75]. Since $u_0 \in H(\Omega)$, there are functions $u_{0m} \in W_m(\Omega)$ which are projections of u_0 over $H(\Omega)$. Therefore

$$u_{0m} \to u_0 \text{ in } H(\Omega) \tag{18}$$

and

$$\|u_{0m}\|_{H(\Omega)} \leqslant \|u_0\|_{H(\Omega)} \text{ for every } m.$$

Firstly, we solve the finite dimensional problems (19), (20), (21) that are close to (1). For each $m \in \mathbb{N}$ we are lead to find $u_{km} \in W_m(\Omega)$ such that

$$u_{km}(t) = \sum_{i=1}^{m} g_{kim}(t) v_i, \tag{19}$$

$$(u'_{km}(t), v_i) + ((u_{km}(t), v_i)) + b(u_{km}(t), u_{km}(t), v_i) + (R_k(u_{km}), v_i) = (f(t), v_i), \tag{20}$$

$$u_{km}(0) = u_{0m} \text{ in } W_m(\Omega), \tag{21}$$

where $g_{kim} : [0,T] \to \mathbb{R}$ is a function, recall notations in Section 2. By a Theorem of Carathéodory [7], equation (20) has a maximal solution on an interval $[0, t_{km})$ with $0 < t_{km} \leqslant T$. The proof of Theorem 1.1 will be continued below.

REMARK 4.1. Notice that the equation (20) can be rewritten as

$$g'_{kim}(t) = G_i(t, g_{k1m}(t), g_{k2m}(t), \dots, g_{kmm}(t)), \quad i = 1,2,\dots,m,$$

where

$$\begin{aligned} &G_i(t, g_{k1m}(t), g_{k2m}(t), \dots, g_{kmm}(t)) \\ &= -((v_i, v_i)) g_{kim}(t) - b(u_{km}(t), u_{km}(t), v_i) - (R_k(u_{km}), v_i) + (f(t), v_i). \end{aligned}$$

Since by Lemma 2.2

$$|b(u_{km}(t), u_{km}(t), v_i)| \leqslant C \|u_{km}(t)\|^2_{V(\Omega)} \|v_i\|_{V(\Omega)},$$

and by (14) of Lemma 3.2, and by Lemma 3.3, we obtain (see more precisely (31)),

$$|R_k(u_{km}(t))|^2 \leqslant \widehat{C}(|u_{km}(t)|^2+1). \tag{22}$$

Thus G_i is a Carathéodory function, because R_k is a Lischitz function. Moreover there exists a integrable function $M_{kmi}(t)$ such that

$$|G_i(t,x)| \leqslant M_{kmi}(t), \tag{23}$$

for each (t,x) in a compact subset of $(0,T)\times\mathbb{R}^m$. Hence the theorem of Carathéodory [7] can be applied. On the other hand, without the Lipschitz sequence we constructed in Lemma 3.1, we are not able to verify the continuity of $G_i(t,x)$ in the variable x, because the function h is only assumed to be continuous.

Continuation of the Proof of Theorem 1.1. Multiplying (20) by $g_{kim}(t)$ we get for $1\leqslant i\leqslant m$ the expression

$$\begin{aligned} &(u'_{km}(t),v_j)g_{kim}(t)+((u_{km}(t),v_j))g_{kim}(t)\\ &+b((u_{km}(t),u_{km}(t),v_j)g_{kim}(t)+(R_k(u_{km}),v_i)g_{kim}(t)\\ =\ &(f(t),v_j)g_{kim}(t), \end{aligned} \tag{24}$$

Summing (24) in the index i for $i=1,2,\dots,N$, we have

$$\begin{aligned} &(u'_{km}(t),u_{km}(t))+((u_{km}(t),u_{km}(t)))\\ &+b((u_{km}(t),u_{km}(t),u_{km}(t))+(R_k(u_{km}),u_{km})\\ =\ &(f(t),u_{km}(t)). \end{aligned} \tag{25}$$

Since $R_k(u_{km})\,.\,u_{km}\geqslant 0$, we obtain

$$2\frac{d}{dt}\|u_{km}(t)\|^2_{H(\Omega)}+\|u_{km}(t)\|^2_{V(\Omega)} \leqslant \|f(t)\|_{V'(\Omega)}\|u_{km}(t)\|_{V(\Omega)}. \tag{26}$$

By Young inequality

$$\|f(t)\|_{V'(\Omega)}\|u_{km}(t)\|_{V(\Omega)} \leqslant \frac{1}{2}\|f(t)\|^2_{V'(\Omega)}+\frac{1}{2}\|u_{km}(t)\|^2_{V(\Omega)}.$$

It follows from (26), that

$$2\frac{d}{dt}\|u_{km}(t)\|^2_{H(\Omega)}+\|u_{km}(t))\|^2_{V(\Omega)} \leqslant \frac{1}{2}\|f(t)\|^2_{V'(\Omega)}+\frac{1}{2}\|u_{km}(t)\|^2_{V(\Omega)}.$$

Hence

$$2\frac{d}{dt}\|u_{km}(t)\|^2_{H(\Omega)}+\frac{1}{2}\|u_{km}(t))\|^2_{V(\Omega)} \leqslant \frac{1}{2}||f(t)||^2_{V'(\Omega)}.$$

Integrating for $t\in(0,T)$ we get

$$\|u_{km}(t)\|^2_{H(\Omega)}+\frac{1}{4}\int_0^t\|u_{km}(s))\|^2_{V(\Omega)}ds \leqslant \frac{1}{4}\int_0^t\|f(s)\|^2_{V'(\Omega)}ds+\|u_{0m}\|^2_{H(\Omega)}. \tag{27}$$

By the convergence $u_{0m} \to u_0$ in $H(\Omega)$, we conclude that (u_{0m}) is a bounded sequence in $L^2(\Omega)$ by a constant $C_0 > 0$ that is independent of t, k and m, consequently, $u_{km}(t)$ is defined up to T. Thus u_{km} is uniformly bounded in $L^\infty(0,T;H(\Omega))$. Taking $t = T$ in (27) we get

$$\|u_{km}(T)\|^2_{H(\Omega)} + \frac{1}{4}\int_0^T \|u_{km}(s))\|^2_{V(\Omega)} ds \leqslant \frac{1}{4}\int_0^T \|f(s)\|_{V'(\Omega)} ds + C_0. \tag{28}$$

Therefore,

$$u_{km} \text{ is bounded in } L^2(0,T;V(\Omega)) \cap L^\infty(0,T;H(\Omega)). \tag{29}$$

We write (20) in the form

$$u'_{km}(t) - \Delta u_{km}(t) + Bu_{km}(t) + R_k(u_{km}(t)) = f(t).$$

Let P_m be the orthogonal projection operator of $H(\Omega)$ over $W_m(\Omega)$. The adjoint operator $P^* : V'(\Omega) \to V'(\Omega)$ is uniformly bounded and linear. Therefore, by Lemmas 2.1 and 2.2,

$$u'_{km}(t) + P^*_m(-\Delta u_{km}(t)) + P^*_m Bu_{km}(t) + P^*_m(R_k(u_{km}(t))) = P^*_m f(t),$$

implying that

$$u'_{km} \text{ is bounded in } L^2(0,T;V'(\Omega)). \tag{30}$$

Notice that $P^*_m(-\Delta u_{km}(t))$ and $P^*_m f$ are uniformly bounded in $L^2(0,T;V'(\Omega))$ and by Lemma 2.2 and (29) we have that $P^*_m Bu_{km}(t)$ is uniformly bounded in $L^2(0,T;V'(\Omega))$. By (14) of Lemma 3.2, and specially by Lemma 3.3, we obtain

$$|R_k(u_{km}(t))|^2 \leqslant \widehat{C}(|u_{km}(t)|^2 + 1), \tag{31}$$

where $\widehat{C}$ is a constant independent of k and m. Hence by (29), $R_k(u_{km})$ is uniformly bounded in $L^\infty(0,T;H(\Omega))$.

The estimates before are true for all terms $(k,m) \in \mathbb{N} \times \mathbb{N}$, and in particular, for $(m,m) \in \mathbb{N} \times \mathbb{N}$. Thus by (30) we have for $t,s \in [0,T]$ that

$$\|u_{mm}(t) - u_{mm}(s)\|_{V'(\Omega)} \leqslant \left(\int_s^t \|u'_{mm}(\xi)\|^2_{V'(\Omega)} d\xi\right)^{1/2} |t-s|^{1/2} \leqslant C|t-s|^{1/4}, \tag{32}$$

where $C > 0$ is independent of m. By (29), (32) and Arzeà-Ascoli theorem, there exists a subsequence of (u_{mm}), which we simply denote by (u_m), and a function $u : \Omega \times (0,T) \to \mathbb{R}$ such that

$$u_m \to u \text{ in } C^0([0,T];V'(\Omega)). \tag{33}$$

Since $u_m \in L^\infty(0,T;H(\Omega)) \cap L^2(0,T;V(\Omega))$ see [16, p.58, Théoremè], by Lemma 2.3 and (30) we obtain a subsequence of u_m such that

$$u_m \to u \text{ in } L^2(0,T;H(\Omega)) \text{ as } m \to \infty. \tag{34}$$

Using the estimate (28) and (29) we have as $m \to \infty$

$$u_m \rightharpoonup u \text{ weak-}* \text{ in } L^\infty(0,T;H(\Omega)), \tag{35}$$
$$u_m \rightharpoonup u \text{ weakly in } L^2(0,T;V(\Omega)), \tag{36}$$
$$u_m' \rightharpoonup u' \text{ weakly in } L^2(0,T;V'(\Omega)). \tag{37}$$

From (34), there exists a subsequence of (u_m) such that

$$u_m \to u \text{ a.e. in } \Omega \times (0,T).$$

Hence, by the continuity of h, we conclude that

$$h(u_m) \to h(u) \text{ a.e. in } \Omega \times (0,T)$$

and

$$R_m(u_m) \to h(u_m) \text{ a.e. in } \Omega \times (0,T),$$

because $|u_m(x,t)|$ is bounded in $\mathbb{R}$. Therefore

$$R_m(u_m) \to h(u) \text{ a.e. in } \Omega \times (0,T). \tag{38}$$

Integrating (25) in $\Omega \times (0,T)$, we conclude

$$\begin{aligned}
&\int_0^T (R_m(u_m(t)),u_m(t))dt \\
&= \int_0^T (f(t),u_m(t))dt + \int_0^T |u_m(t)|^2 dt - |u_m(T)|^2 + |u_m(0)|^2 \\
&\quad - \int_0^T ((u_{km}(t),u_{km}(t)))dt \\
&\leqslant \frac{1}{2}\int_0^T |f(t)|^2 dt + \frac{1}{2}\int_0^T |u_m(t)|^2 dt + \int_0^T |u_m(t)|^2 dt + |u_m(0)|^2.
\end{aligned} \tag{39}$$

Using (33), we have by (39) that

$$\int_0^T (R_m(u_m(t)),u_m(t))dt \leqslant C, \tag{40}$$

where $C > 0$ is a constant independent of m. Thus, from (38) and (40), it follows by a theorem of Strauss [20] that

$$R_m(u_m) \to h(u) \text{ in } (L^1(\Omega \times (0,T)))^N. \tag{41}$$

In particular, by (4), (5), (14), (29) and Lemma 3.3 we have

$$R_m(u_m), h(u) \in L^2(0,T;H(\Omega)).$$

Multiplying both sides of (20) by $\varphi \in C_0^\infty(0,T)$ and integrating from 0 to T we get

$$\begin{aligned}
&\int_0^T (u_m'(t),w)\varphi dt + \int_0^T ((u_m(t),w))\varphi dt + \int_0^T b(u_m(t),u_m(t),w)\varphi dt \\
&+ \int_0^T (R_m(u_m(t)),w)\varphi dt = \int_0^T (f(t),w)\varphi dt,\ \forall w \in W_m(\Omega),\ \forall \varphi \in C_0^\infty(0,T).
\end{aligned} \tag{42}$$

Since $W_m(\Omega)$ is dense in $V(\Omega)$, it follows that (42) is true for all $w \in V(\Omega)$. In particular, considering $w \in C_0^\infty(\Omega)$ and integrating by parts we get

$$\begin{aligned}&\int_0^T (u_m'(t),w)\varphi dt + \int_0^T ((u_m(t),w))\varphi dt + \int_0^T b(u_m(t),u_m(t),w)\varphi dt\\&+\int_0^T (R_m(u_m(t)),w)\varphi dt = \int_0^T (f(t),w)\varphi dt,\ \forall w\in C_0^\infty(\Omega),\ \forall \varphi \in C_0^\infty(0,T).\end{aligned} \tag{43}$$

Using the convergences (34)–(37) and (41) in (43), we deduce

$$\begin{aligned}&\int_0^T (u'(t),w)\varphi dt + \int_0^T ((u(t),w))\varphi dt + \int_0^T b(u(t),u(t),w)\varphi dt\\&+\int_0^T (h(u(t)),w)\varphi dt = \int_0^T (f(t),w)\varphi dt,\ \forall w\in C_0^\infty(\Omega),\ \forall \varphi \in C_0^\infty(0,T).\end{aligned}$$

The convergence

$$\int_0^T b(u_m(t),u_m(t),w)\varphi dt \to \int_0^T b(u(t),u(t),w)\varphi dt,\ \forall w\in C_0^\infty(\Omega),\ \forall \varphi\in C_0^\infty(0,T)$$

follows from (34) and (37). Therefore

$$u' - \Delta u + Bu + h(u) = f \text{ in } \mathscr{D}'(\Omega\times(0,T)).$$

Since $-\Delta u \in L^2(0,T;V'(\Omega))$, $Bu \in L^1(0,T;V'(\Omega))$, $h(u)\in L^2(0,T;H(\Omega))$ and $f \in L^2(0,T;V'(\Omega))$ we have that $u' \in L^1(0,T;V'(\Omega))$. Therefore,

$$u' - \Delta u + Bu + h(u) = f \text{ in } L^1(0,T;V'(\Omega)) \tag{44}$$

and

$$(u'(t),v) + ((u(t),v)) + b(u(t),u(t),v) + (h(u(t)),v) = (f(t),v) \text{ in } \mathscr{D}'(0,T), \forall v\in V(\Omega). \tag{45}$$

The initial condition $u(0) = u_0$ in Ω is true. Indeed, it follows by (18) that

$$u_m(0) = u_{0m} \to u_0 \text{ in } H(\Omega). \tag{46}$$

By (33) we have

$$u_m(0) \to u(0) \text{ in } V'(\Omega). \tag{47}$$

Since $H(\Omega)\subset V'(\Omega)$, by (46) and (47) and the uniqueness of the limit we obtain that $u(0) = u_0$. Hence u is a solution by definition (7)–(8).

We mention a few words about pressure recovery. If u solves (44), since $-\Delta u \in L^2(0,T;V'(\Omega))$, $Bu \in L^1(0,T;V'(\Omega))$, $h(u)\in L^2(0,T;H(\Omega))$, $f\in L^2(0,T;V'(\Omega))$ and $u'\in L^1(0,T;V'(\Omega))$. Then, setting

$$S = u' - \nu\Delta u + Bu + h(u) - f,$$

we conclude that S belongs to $\mathscr{D}'(\Omega\times(0,T))$ and by (45) we have

$$\langle S(t),\varphi\rangle = 0 \text{ in } \mathscr{D}'(0,T),\ \forall \varphi\in V(\Omega)$$

It follows from [23, Proposition I.1.1] that for some $p \in \mathscr{D}'(\Omega \times (0,T))$, S is of the form

$$S = -\nabla p.$$

Hence,

$$u' - \nu\Delta u + Bu + h(u) + \nabla p = f.$$

The proof of Theorem 1.1 is complete. □

5. Energy estimate and time extinction

Recall, that we are working with $h = (h_1, h_2, \ldots, h_N) : \mathbb{R}^N \to \mathbb{R}^N$ continuous and $1 < \sigma \leqslant 2$ satisfying (2)–(5). We are going to obtain estimates related to a solution u obtained in Theorem 1.1. Additionally, assume that there exist constants $c_h > 0$ and $\gamma_1 > 0$ such that

$$1 < \gamma_1 \leqslant \sigma \leqslant 2 \tag{48}$$

and

$$c_h |x|^{\gamma_1 - 2} |x_i|^2 \leqslant h_i(x) x_i \text{ for each } x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N \text{ and } i = 1, 2, \ldots, N. \tag{49}$$

The next two results are similar to the ones in [1], hence we just outline the main ideas of the proofs to put evidence on the constants we work with in the present paper.

We will establish an energy estimate for a weak solution of (1).

THEOREM 5.1. *Let $E(t) = \|u(t)\|^2_{H(\Omega)}/2$ be the kinetic energy associated to (1). Assume (2)–(5), (48), (49), $f \in L^2(0,T;V'(\Omega))$ and $u_0 \in H(\Omega)$. Let u be a solution of (1), then*

$$\frac{d}{dt}E(t) + \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega)} + c_h \|u(t)\|^{\gamma_1}_{L^{\gamma_1}(\Omega)} \leqslant \frac{1}{2}\|f(t)\|^2_{V'(\Omega)} \tag{50}$$

and

$$\begin{aligned} 2E(t) &+ \int_0^t \|\nabla u(s)\|^2_{L^2(\Omega)} ds + 2c_h \int_0^t \|u(s)\|^{\gamma_1}_{L^{\gamma_1}(\Omega)} ds \\ &\leqslant \int_0^t \|f(s)\|^2_{V'(\Omega)} ds + \|u_0\|^2_{H(\Omega)} \text{ for a.e. } t \in [0,T]. \end{aligned} \tag{51}$$

Proof. We sketch the main ideas. After getting (50), we take $\liminf$ in expression (43) and let $m \to \infty$ to obtain

$$(u'(t), v) + ((u(t), v)) + b(u(t), u(t), v) + (h(u(t)), v) = (f(t), v) \text{ a.e. } t \in (0,T), \forall v \in V(\Omega).$$

Taking $v = u(t)$, we conclude that

$$(u'(t), u(t)) + \|\nabla u(t)\|_{L^2(\Omega)} + (h(u(t)), u(t)) = (f(t), u(t)). \tag{52}$$

By (49) we conclude that

$$\frac{1}{2}\frac{d}{dt}E(t)+\|\nabla u(t)\|^2_{L^2(\Omega)}+c_h\|u(t)\|^{\gamma_1}_{L^{\gamma_1}(\Omega)}\leqslant (f(t),u(t)) \text{ for a.e. } t\in[0,T]. \tag{53}$$

Using the Cauchy-Schwarz inequality, we obtain

$$(f(t),u(t))\leqslant \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega)}+\frac{1}{2}\|f(t)\|^2_{V'(\Omega)}. \tag{54}$$

Inserting (54) into (53), we get (50). The proof of (51) follows by integrating (50) between 0 and $t\leqslant T$. □

Next we state the time extinction for a weak solution of (1).

THEOREM 5.2. *Assume (2)–(5), (48), (49), $u_0\in H(\Omega)$ and let u be a solution of (1). If $f=0$ a.e. in $\Omega\times(0,T)$, where $T>0$ or $T=\infty$, then there exists $t^*>0$ such that $u=0$ a.e. in Ω and for a.e. $t\geqslant t^*$.*

Proof. Let $E(t)=\|u(t)\|^2_{H(\Omega)}/2$ be the kinetic energy. Then E might be estimated in the following way

$$E(t)\leqslant\left(E(0)^{\frac{\mu-1}{\mu}}-C't\right)^{\frac{\mu}{\mu-1}}, \tag{55}$$

where $C'=[(\mu-1)/\mu]\min(1,c_h)\left(2/C_G^2\right)^{\frac{1}{\mu}}$, $\mu=1+2(2-\gamma_1)/[(2-\gamma_1)N+2\gamma_1]$ and $C_G=C(N,\gamma_1)$ is the constant resulting from Gagliardo-Nirenberg-Sobolev inequality. Since $u_0\in H(\Omega)$, clearly $E(0)<\infty$, and by virtue of the fact that $\mu>1$ one gets

$$0<\frac{\mu}{\mu-1}=\frac{4+(2-\gamma_1)N}{2(2-\gamma_1)}.$$

Hence the right-hand side of (55) vanishes for

$$t^*=\frac{E(0)^{\frac{\mu-1}{\mu}}}{C'}=\|u_0\|_{H(\Omega)}^{\frac{4(2-\gamma_1)}{4+(2-\gamma_1)N}}\,\frac{4+(2-\gamma_1)N}{4(2-\gamma_1)}\,\frac{C_G^{\frac{2[(2-\gamma_1)N+2\gamma_1]}{4+(2-\gamma_1)N}}}{\min(1,c_h)},$$

implying that $u=0$ for $t\geqslant t^*$. □

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(Received December 9, 2024)

Anderson L. A. de Araujo
Universidade Federal de Viçosa
Departamento de Matemática
Avenida Peter Henry Rolfs, s/n, CEP 36570-900, Viçosa, MG, Brasil
e-mail: `anderson.araujo@ufv.br`

Marcelo Montenegro
Universidade Estadual de Campinas, IMECC
Departamento de Matemática
Rua Sérgio Buarque de Holanda, 651, CEP
13083-859, Campinas, SP, Brasil
e-mail: `msm@ime.unicamp.br`