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DYNAMICS AND STABILITY OF STOCHASTIC PANTOGRAPH DIFFERENTIAL EQUATION WITH COMPOSITE FRACTIONAL DERIVATIVE

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Abstract. In this paper, we study the existence and uniqueness of solutions, as well as the Ulam-Hyers and Ulam-Hyers-Rassias stability, for a class of stochastic pantograph differential equations involving the Hilfer fractional derivative. The analysis is carried out using the Picard operator theory. Furthermore, a numerical example is provided to illustrate and validate the theoretical results.

1. Introduction

Fractional calculus has garnered a lot of interest due its ability to model complex systems with memory and hereditary properties features that are prevalent in many disciplines, including biology, physics, engineering, and finance has drawn a great deal of interest in recent years; see [5, 9, 20] for more details. It is important to be aware that the physical significance of many of the emerging fractional derivatives is extremely limited. Among those that have been recognized for their authenticity are the Riemann-Liouville (R-L) and Caputo derivatives. One especially significant tool in fractional calculus is the Hilfer fractional derivative (HFD), which was introduced in [9] as a combination of the R-L and Caputo derivatives. This derivative presents a generalized form that can be used with flexibility to describe intermediate order behavior and nonlocal dynamics processes. Research on the HFD guarantees long-term validity and credibility because of its strong mathematical base and positive reputation in the scientific community.

Pantograph equations are particular types of delay differential equations that were first proposed by Balachandran [3]. They are important in many scientific fields, including biological modeling, control theory, and electrodynamics. These equations are

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further enhanced by stochastic components, which account for the random fluctuations present in many real-world systems. Understanding the dynamics and stability of these systems requires the incorporation of delays into stochastic differential equations, one can see [3, 7, 11–15, 17, 24, 27–29, 31, 33, 34].

Ulam stability is a fundamental concept in differential equations research that was first introduced by Stanislaw Ulam [25, 26] in connection with functional equations. It is about how well-suited solutions are to small perturbations in the initial parameters or situations. Applying the concept to fractional differential equations, particularly those with stochastic components, led to the development of Ulam-Hyers (U-H) stability and Ulam-Hyers-Rassias (U-H-R) stability theories, refer [1, 2, 4, 6, 8, 10, 16, 18, 21, 22, 25, 26, 30, 32].

Recenty, Vivek et al. [29] studied a class of stochastic pantograph differential equations (SPDE) via ϑ-Caputo fractional derivative given by the form

$$
{}^{c}\mathfrak{D}^{\alpha;\vartheta}x(\iota) = Ax(\iota) + f(\iota, x(\iota), x(\lambda\iota)) + \sigma(\iota, x(\iota), x(\lambda\iota))\frac{dW(\iota)}{d\iota}, \quad \iota \in J := [0,b],
$$

$$
x^{(k)}(0) = x_0^{(k)}, \quad k = 0, 1, 2, \ldots, n-1,
$$

where $0 < \lambda < 1, n-1 < \alpha \leqslant n$ and f, σ are given functions and A is the generator of strongly continuous semigroup $\{\mathscr{S}(\iota) : \iota \geqslant 0\}$ on a Hilbert space $\mathscr{X}$.

In this paper, we investigate the Ulam stability for a class of Hilfer fractional SPDE the form

$$
\begin{cases}
{}^{H}\mathfrak{D}^{p,q}_{0^+,\iota}x(\iota) = Ax(\iota) + f(\iota, x(\iota), x(\kappa\iota)) + \sigma(\iota, x(\iota), x(\kappa\iota))\frac{dW(\iota)}{d\iota}, \quad \iota \in J := (0,b], \\
\mathfrak{I}^{1-\gamma}_{0^+,\iota}x(0) = x_0, \quad \gamma = p + q - pq,
\end{cases}
\tag{1.1}
$$

where $\mathfrak{I}^{1-\gamma}_{0^+,\iota}$ and ${}^{H}\mathfrak{D}^{p,q}_{0^+,\iota}$ are the fractional integral of order $1-\gamma$ and the HFD of order p and type q, respectively. Here, $0 < p < 1, \frac{1}{2} < q \leqslant 1$. Let A be the generator of strongly continuous semigroup $\{\mathscr{S}(\iota) : \iota \geqslant 0\}$ on a Hilbert space $\mathscr{X}$, $\{W(\iota)\}_{\iota \geqslant 0}$ denotes the Q-Wiener process defined in the complete probability space $(\Omega, \mathfrak{I}, \mathbb{P})$ with a normal filteration $(\mathscr{F}_\iota)_{\iota \geqslant 0}$. $f : J \times \mathscr{X} \times \mathscr{X} \to \mathscr{X}$, $\sigma : J \times \mathscr{X} \times \mathscr{X} \to \mathscr{L}_2^0$ are continuous functions and $0 < \kappa < 1$. The space $\mathscr{L}_2^0$ will be defined subsequently.

The integrated mathematical system which combines proportional delays with HFD and stochastic disturbances demands complete analysis. Our goal is to define essential criteria that ensure the stability of these solutions. Our procedure consists of constructing appropriate function spaces followed by fixed-point applications, and finding conditions that guarantee the stability of the solutions.

2. Preliminaries

To make the paper self-contained, we first present essential notations and definitions used throughout the study. The main symbols and notations are summarized in Table 1.

Symbol	**Description**
$J=[0,b]$	Time interval of consideration
$(\Omega,\mathscr{F}_\iota,P)$	Complete probability space
$\mathscr{F}_\iota$	Filtration satisfying standard conditions
$\mathbb{L}^2(\Omega,\mathscr{F}_\iota,\mathscr{X})$	Hilbert space of square-integrable random variables
$C(J,\mathbb{L}^2(\Omega,\mathscr{F}_\iota,\mathscr{X}))$	Space of continuous stochastic processes
$C_{1-\gamma}(J,\mathbb{L}^2(\Omega,\mathscr{F}_\iota,\mathscr{X}))$	Weighted Banach norm $\|x\|_{1-\gamma}^2=\sup_{\iota\in J}\mathbb{E}\|\iota^{1-\gamma}x(\iota)\|^2$
$W(\iota)$	Q-Wiener process
Q	Covariance operator with $\mathrm{Tr}\,Q<\infty$
$\mathfrak{I}^p_{a^+,\iota}$	R-L fractional integral of order p
$\mathfrak{D}^p_{a^+,\iota}$	R-L fractional derivative of order p
${}^H\mathfrak{D}^{p,q}_{a^+,\iota}$	HFD of order p and type q
$\Gamma(\cdot)$	Gamma function
$E_q(z)$	Mittag-Leffler function
$\mathscr{S}_{p,q}(\iota),\ T_q(\iota),\ P_q(\iota)$	Operator families
$\mathbb{E}$	Expectation operator
$\|\cdot\|$	Norm in the Hilbert

Table 1: *Table of symbols and notations*

Let $W:J\times\Omega\to K$ as a standard Q-Wiener process defined on the probability space $(\Omega,\mathscr{F}_\iota,P)$. This process is associated with the filtration $(\mathscr{F}_\iota)_{\iota\in J}$. Suppose there exists a complete orthonormal basis $\{e_n\}_{n\geqslant1}$ in K and a sequence of nonnegative real numbers $\{\lambda_n\}_{n\in\mathbb{N}}$ satisfying

$$Qe_n=\lambda_n e_n,\quad \lambda_n\geqslant 0,\ \ n=1,2,\dots,$$

as well as a set of independent real-valued Brownian motions $\{\beta_n\}_{n\geqslant1}$ such that

$$\langle W(\iota),e\rangle=\sum_{n=1}^{\infty}\sqrt{\lambda_n}\langle e_n,e\rangle\beta_n(\iota),\quad e\in K,\ \iota\in J.$$

Define the Hilbert space

$$\mathscr{L}_2^0=\{f\mid f \text{ is a Hilbert-Schmidt operator from } Q^{\frac12}(K) \text{ to } X\},$$

with the inner product defined as

$$\langle\psi,\phi\rangle_{\mathscr{L}_2^0}=\mathrm{tr}[\psi Q\phi^*],\quad \psi,\phi\in\mathscr{L}_2^0.$$

DEFINITION 2.1. ([5]) For $p > 0$, the fractional R-L integral of order p for a function $x : [a,\infty) \to \mathbb{R}$ can be written as

$$\mathfrak{I}^{p}_{a^+,\iota}x(\iota) = \frac{1}{\Gamma(p)}\int_a^{\iota}(\iota - s)^{p-1}x(s)ds. \tag{2.1}$$

DEFINITION 2.2. ([5]) For $n-1 < p \leqslant n$, the fractional R-L derivative of order p for a function x is represented as

$$\mathfrak{D}^{p}_{a^+,\iota}x(\iota) = D^n\mathfrak{I}^{n-p}_{a^+,\iota}x(\iota) = \frac{1}{\Gamma(n-p)}\left(\frac{d}{\mathrm{d}\iota}\right)^n\int_a^{\iota}(\iota - s)^{n-p-1}x(s)\mathrm{d}s.$$

DEFINITION 2.3. ([9]) For $n-1 < p \leqslant n$, the HFD of order p and type $0 \leqslant q \leqslant 1$ of x is represented as

$${}^{H}\mathfrak{D}^{p,q}_{a^+,\iota}x(\iota) = \mathfrak{I}^{q(n-p)}_{a^+,\iota}D^n\mathfrak{I}^{(1-q)(n-p)}_{a^+,\iota}x(\iota) = \mathfrak{I}^{q(n-p)}_{a^+,\iota}\mathfrak{D}^{\rho}_{a^+,\iota}x(\iota), \quad \rho = p + q(n-p), \tag{2.2}$$

where $D = \frac{d}{d\iota}$.

LEMMA 2.4. ([9]) *For $n-1 < p \leqslant n$, $0 \leqslant \beta \leqslant 1$, and $\mathfrak{I}^{(1-q)(n-p)}_{a^+,\iota}x \in AC^k[a,b]$. Then,*

$$\mathfrak{I}^{p}_{a^+,\iota}\,{}^{H}\mathfrak{D}^{p,q}_{a^+,\iota}x(\iota) = x(\iota) - \sum_{k=1}^{n}\frac{(\iota - a)^{\rho - k}}{\Gamma(\rho + 1 - k)}\cdot\lim_{\iota \longrightarrow +a}\frac{d^k}{\mathrm{d}\iota^k}\mathfrak{I}^{(1-q)(n-p)}_{a^+,\iota}x(\iota), \tag{2.3}$$

where $\rho = p + q(n-p)$.

LEMMA 2.5. ([9, 28]) *Let $p > 0$ and $q > 0$. Then, for all $\iota \in J$, the following relations hold:*

$$\mathfrak{I}^{p}_{a^+,\iota}\left(\iota^{q-1}\right) = \frac{\Gamma(q)}{\Gamma(q+p)}\iota^{q+p-1}, \qquad \mathfrak{D}^{p}_{a^+,\iota}\left(\iota^{p-1}\right) = 0, \quad 0 < p < 1.$$

DEFINITION 2.6. ([34]) Consider a metric space $(\mathscr{X}, d)$. Suppose there exists a point $x^* \in \mathscr{X}$ such that:

1. $\mathscr{F}_{\mathscr{S}} = \{x^*\}$, where $\mathscr{F}_{\mathscr{S}} = \{x \in \mathscr{X} : \mathscr{S}(x) = x\}$;
2. The sequence $\{\mathscr{S}^n(x_0)\}_{n\in\mathbb{N}}$ converges to x^* for every $x_0 \in \mathscr{X}$.

Then, the operator $\mathscr{S} : \mathscr{X} \to \mathscr{X}$ is called a *Picard operator*.

LEMMA 2.7. ([23]) *Let $\mathscr{S} : \mathscr{X} \to \mathscr{X}$ be an increasing Picard operator with $\mathscr{F}_{\mathscr{S}} = \{x^*_{\mathscr{S}}\}$. If $(\mathscr{X}, d, \leqslant)$ is an ordered metric space, then for every $x \in \mathscr{X}$ such that $x \leqslant \mathscr{S}(x)$, it follows that $x \leqslant x^*_{\mathscr{S}}$.*

LEMMA 2.8. (Jensen's inequality, [25]). *Let $m \in \mathbb{N}$ and $\iota_1, \iota_2, \ldots, \iota_m$ be nonnegative real numbers, then*

$$\left(\sum_{i=1}^{m} \iota_i\right)^p \leqslant m^{p-1} \sum_{i=1}^{m} \iota_i^p, \quad \textit{for } \ p > 1.$$

LEMMA 2.9. ([12, 30]) *Let $\Psi(\iota)$ be a predictable process with values in $\mathscr{L}_2^0$, defined on $[\iota_1, \iota_2]$, such that*

$$\mathbb{E}\left(\int_{\iota_1}^{\iota_2} \|\Psi(s)\|_{\mathscr{L}_2^0}^2 ds\right) < \infty, \quad 0 \leqslant \iota_1 < \iota_2 \leqslant b.$$

Then,

$$\mathbb{E}\left\|\int_{\tau_1}^{\tau_2} \Psi(s)\, dW(s)\right\|^2 \leqslant \mathbb{E}\left(\int_{\tau_1}^{\tau_2} \|\Psi(s)\|_{\mathscr{L}_2^0}^2 ds\right).$$

LEMMA 2.10. ([7]) *A stochastic process $x \in C_{1-\gamma}(J, \mathbb{L}^2(\Omega, \mathscr{F}_\iota, \mathscr{X}))$ is called a mild solution of problem (1.1) if x satisfies the following stochastic integral equation:*

$$\begin{aligned} x(\iota) = \ & \mathscr{S}_{p,q}(\iota)x_0 + \int_0^\iota (\iota - s)^{q-1} P_q(\iota - s) f(s, x(s), x(\kappa s))ds \\ & + \int_0^\iota (\iota - s)^{q-1} P_q(\iota - s)\sigma(s, x(s), x(\kappa s))dW(s), \quad \iota \in J, \end{aligned} \tag{2.4}$$

where

$$\mathscr{S}_{p,q}(\iota) = \Im_{0^+,\iota}^{p(1-q)} T_q(\iota), \quad T_q(\iota) = \iota^{q-1} P_q(\iota), \quad P_q(\iota) = \int_0^\infty q\theta M_q(\theta)\mathscr{S}(\iota^q\theta)\, d\theta.$$

LEMMA 2.11. ([7]) *Assume that $\mathscr{S}(\iota)$ is continuous in the uniform operator topology for $\iota > 0$ and $\{\mathscr{S}(\iota)\}_{\iota \geqslant 0}$ is uniformly bounded (i.e., there exists $M > 1$ such that $\sup_{\iota \in [0,\infty)} \|\mathscr{S}(\iota)\| < M$), we have the following properties.*

(i) $P_q(\iota)$, $T_q(\iota)$, and $\mathscr{S}_{p,q}(\iota)$ are linear and bounded operators, that is, for $\forall \iota \geqslant 0, x \in \mathscr{X}$,

$$\|P_q(\iota)x\| \leqslant \frac{M\|x\|}{\Gamma(q)}, \quad \|T_q(\iota)x\| \leqslant \frac{M\iota^{q-1}\|x\|}{\Gamma(q)} \quad \textit{and}$$

$$\|\mathscr{S}_{p,q}(\iota)x\| \leqslant \frac{M\iota^{\gamma-1}\|x\|}{\Gamma(\gamma)}.$$

(ii) Operators $P_q(\iota)$, $T_q(\iota)$, and $S_{p,q}(\iota)$ are strongly continuous.

LEMMA 2.12. ([33]) *Let $q > 0$, and let $v(\iota)$ be a nonnegative, locally integrable function on J. Suppose $g(\iota)$ is a nonnegative, nondecreasing, continuous function on*

J satisfying $g(\iota) \leqslant C$, and let $x(\iota)$ be a nonnegative, locally integrable function such that

$$x(\iota) \leqslant v(\iota) + g(\iota) \int_0^\iota (\iota - s)^{q-1} x(s)\, ds, \quad \iota \in J.$$

Then,

$$x(\iota) \leqslant v(\iota) + \int_0^\iota \left[\sum_{n=1}^{\infty} \frac{[g(\iota)\Gamma(q)]^n}{\Gamma(nq)} (\iota - s)^{nq-1} v(s) \right] ds, \quad \iota \in J.$$

Moreover, if $v(\iota)$ is nondecreasing on J, then

$$x(\iota) \leqslant v(\iota)\, E_q(g(\iota)\Gamma(q)\iota^q),$$

where the Mittag-Leffler function $E_q(z)$ is defined as

$$E_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(kq+1)}.$$

3. Existence results

In this section, we establish the existence and uniqueness of a mild solution to problem (1.1) using the Banach fixed-point theorem (BFT). The following assumptions are introduced for this purpose:

(H_1): $f : J \times \mathscr{X} \times \mathscr{X} \to \mathscr{X}$ is continuous and satisfies

(i) There exist a continuous function $\varphi_1 : J \to \mathbb{R}^+$ and a constant $l_1 > 0$ such that

$$\|f(\iota, x_1, x_2)\|^2 \leqslant \varphi_1(\iota) + l_1 \iota^{2(1-\gamma)} \left(\|x_1\|^2 + \|x_2\|^2 \right),$$

$\iota \in J$ and $x_1, x_2 \in \mathscr{X}$.

(ii) There exist a constant $\mathscr{L}_1 > 0$ such that

$$\|f(\iota, x_1, x_2) - f(\iota, y_1, y_2)\|^2 \leqslant \mathscr{L}_1 \iota^{2(1-\gamma)} \left(\|x_1 - y_1\|^2 + \|x_2 - y_2\|^2 \right),$$

for each $\iota \in J$ and all $x_1, x_2, y_1, y_2 \in \mathscr{X}$.

(H_2): $\sigma : J \times \mathscr{X} \times \mathscr{X} \to \mathscr{L}_2^0$ is continuous and satisfies

(i) There exist a continuous function $\varphi_2 : J \to \mathbb{R}^+$ and a constant $l_2 > 0$ such that

$$\|\sigma(\iota, x_1, x_2)\|^2_{\mathscr{L}_2^0} \leqslant \varphi_2(\iota) + l_2 \iota^{2(1-\gamma)} \left(\|x_1\|^2 + \|x_2\|^2 \right),$$

$\iota \in J$ and $x_1, x_2 \in \mathscr{X}$.

(ii) There exist a constant $\mathscr{L}_2 > 0$ such that

$$\|\sigma(\iota,x_1,x_2)-\sigma(\iota,y_1,y_2)\|^2_{\mathscr{L}_2^0} \leqslant \mathscr{L}_2 \iota^{2(1-\gamma)}\left\{\|x_1-y_1\|^2+\|x_2-y_2\|^2\right\},$$

for each $\iota \in J$ and all $x_1,x_2,y_1,y_2 \in \mathscr{X}$.

Set

$$\Lambda_1 = \frac{4M^2}{\Gamma^2(q)}\mathbb{E}\|x_0\|^2 + \frac{4b^{2-2\gamma+2q}M^2\|\varphi_1\|}{\Gamma^2(q)2q-1} + \frac{4b^{1-2\gamma+2q}M^2\|\varphi_2\|}{\Gamma^2(q)2q-1},$$

$$\Lambda_2 = \frac{4b^{1-2\gamma+2q}(b\mathscr{L}_1+\mathscr{L}_2)M^2}{\Gamma^2(q)2q-1},$$

$$\Lambda_3 = \frac{36b^{2-4\gamma+4q}(b\mathscr{L}_1+\mathscr{L}_2)^2M^4}{\Gamma^4(q)4q-3}.$$

THEOREM 3.1. *Assume that hypotheses* (H_1) *and* (H_2) *hold. Then, problem* (1.1) *admits a unique mild solution in* $C_{1-\gamma}(J,\mathbb{L}^2(\Omega,\mathscr{X}))$, *provided that*

$$\Lambda_2 < 1. \tag{3.1}$$

Proof. Consider the operator $\mathscr{A} : C_{1-\gamma}(J,\mathbb{L}^2(\Omega,\mathscr{X})) \to C_{1-\gamma}(J,\mathbb{L}^2(\Omega,\mathscr{X}))$ given by

$$\begin{aligned}(\mathscr{A}x)(\iota) = {}& \mathscr{S}_{p,q}(\iota)x_0 + \int_0^\iota (\iota-s)^{q-1}P_q(\iota-s)f(s,x(s),x(\kappa s))\,ds \\ &+ \int_0^\iota (\iota-s)^{q-1}P_q(\iota-s)\sigma(s,x(s),x(\kappa s))\,dW(s), \quad \iota \in J.\end{aligned} \tag{3.2}$$

Define the bounded, closed, and convex subset

$$\mathscr{B}_\tau = \left\{u \in C_{1-\gamma}(J,\mathbb{L}^2(\Omega,\mathscr{X})) : \|u\|^2_{1-\gamma} \leqslant \tau\right\}, \quad \tau > 0.$$

We now proceed to verify, step by step, that the operator $\mathscr{A}$ has a fixed point, which serves as the unique solution to problem (1.1) according to BFT.

Step 1: $\mathscr{A}$ maps $\mathscr{B}_\tau$ into itself.

First, let us show that $\mathscr{A}$ defined by Eq. (3.2) satisfies $\mathscr{A}\mathscr{B}_\tau \subset \mathscr{B}_\tau$, with $\tau \geqslant \frac{\Lambda_1}{1-\Lambda_2}$.

For each $x \in \mathscr{B}_\tau$ and $\iota \in J$, we have

$$\begin{aligned}&\left\|\iota^{1-\gamma}(\mathscr{A}x)(\iota)\right\|^2 \\ &\leqslant 4\left\|\mathscr{S}_{p,q}(\iota)x_0\right\|^2 + 4\iota^{2(1-\gamma)}\left\|\int_0^\iota (\iota-s)^{q-1}P_q(\iota-s)f(s,x(s),x(\kappa s))ds\right\|^2 \\ &\quad + 4\iota^{2(1-\gamma)}\left\|\int_0^\iota (\iota-s)^{q-1}P_q(\iota-s)\sigma(s,x(s),x(\kappa s))dW(s)\right\|^2.\end{aligned}$$

Taking the expectation on the above inequality, we get

$$\begin{aligned}&\mathbb{E}\left\|\iota^{1-\gamma}(\mathscr{A}x)(\iota)\right\|^2\\ \leqslant\ &\frac{4M^2}{\Gamma^2(q)}\mathbb{E}\|x_0\|^2+4\iota^{2(1-\gamma)}\mathbb{E}\left\|\int_0^\iota(\iota-s)^{q-1}P_q(\iota-s)f(s,x(s),x(\kappa s))ds\right\|^2\\ &+4\iota^{2(1-\gamma)}\mathbb{E}\left\|\int_0^\iota(\iota-s)^{q-1}P_q(\iota-s)\sigma(s,x(s),x(\kappa s))dW(s)\right\|^2.\end{aligned}$$

By applying the Cauchy-Schwartz (C-S) inequality, along with Lemma 2.9 and hypothesis (H_1), and (H_2), we arrive at the following expression:

$$\begin{aligned}&\mathbb{E}\left\|\iota^{1-\gamma}(\mathscr{A}x)(\iota)\right\|^2\leqslant\frac{3M^2}{\Gamma^2(q)}\mathbb{E}\|x_0\|^2\\ &+\frac{3M^2b^{2(1-\gamma)}b}{\Gamma^2(q)}\int_0^\iota(\iota-s)^{2(q-1)}\left\{\varphi_1(\iota)+\mathbb{E}\left\|s^{1-\gamma}x(s)\right\|^2+\mathbb{E}\left\|s^{1-\gamma}x(\kappa s)\right\|^2\right\}ds\\ &+\frac{3M^2b^{2(1-\gamma)}}{\Gamma^2(q)}\int_0^\iota(\iota-s)^{2(q-1)}\left\{\varphi_2(\iota)+\mathbb{E}\left\|s^{1-\gamma}x(s)\right\|^2+\mathbb{E}\left\|s^{1-\gamma}x(\kappa s)\right\|^2\right\}ds,\end{aligned}$$

which simplifies to

$$\begin{aligned}&\mathbb{E}\left\|\iota^{1-\gamma}(\mathscr{A}x)(\iota)\right\|^2\leqslant\frac{3M^2}{\Gamma^2(q)}\mathbb{E}\|x_0\|^2\\ &+\frac{3b^{2(1-\gamma)}bM^2\|\varphi_1\|}{\Gamma^2(q)}\int_0^\iota(\iota-s)^{2(q-1)}ds+\frac{6b^{2(1-\gamma)}bM^2}{\Gamma^2(q)}\|x\|_{1-\gamma}^2\int_0^\iota(\iota-s)^{2(q-1)}ds\\ &+\frac{3b^{2(1-\gamma)}M^2\|\varphi_2\|}{\Gamma^2(q)}\int_0^\iota(\iota-s)^{2(q-1)}ds+\frac{6b^{2(1-\gamma)}M^2}{\Gamma^2(q)}\|x\|_{1-\gamma}^2\int_0^\iota(\iota-s)^{2(q-1)}ds.\end{aligned}\tag{3.3}$$

Given that

$$\int_0^\iota \iota^{2(q-1)}ds=\frac{\iota^{2q-1}}{2q-1}.$$

It follows that

$$\begin{aligned}&\mathbb{E}\left\|\iota^{1-\gamma}(\mathscr{A}x)(\iota)\right\|^2\\ \leqslant\ &\frac{3M^2}{\Gamma^2(q)}\mathbb{E}\|x_0\|^2+\frac{3b^{2-2\gamma+2q}M^2\|\varphi_1\|}{\Gamma^2(q)2q-1}+\frac{3b^{1-2\gamma+2q}M^2\|\varphi_2\|}{\Gamma^2(q)2q-1}\\ &+\left(\frac{6b^{2-2\gamma-2q}M^2}{\Gamma^2(q)2q-1}+\frac{6b^{1-2\gamma-2q}M^2}{\Gamma^2(q)2q-1}\right)\|x\|_{1-\gamma}^2.\end{aligned}$$

Therefore,

$$\|\mathscr{A}x\|_{1-\gamma}^2\leqslant\Lambda_1+\Lambda_2\|x\|_{1-\gamma}^2\leqslant\Lambda_1+\Lambda_2\tau.$$

Since $\tau\geqslant\frac{\Lambda_1}{1-\Lambda_2}$, we obtain

$$\|\mathscr{A}x\|_{1-\gamma}^2\leqslant\tau.$$

Hence, $\mathscr{A}$ maps $\mathscr{B}_\tau$ into itself.

Step 2: $\mathscr{A}$ is continuous.

Consider a sequence $x_n \to x$ in $C_{1-\gamma}(J, \mathbb{L}^2(\Omega, \mathscr{X}))$. Then, for each $\iota \in J$, applying Lemma 2.8 yields

$$\begin{aligned}
&\left\| \iota^{1-\gamma}((\mathscr{A}x_n)(\iota) - (\mathscr{A}x)(\iota)) \right\|^2 \\
\leqslant & 2\iota^{2(1-\gamma)} \left\| \int_0^\iota (\iota - s)^{q-1} P_q(\iota - s) \left(f(s, x_n(s), x_n(\kappa s)) - f(s, x(s), x(\kappa s)) \right) ds \right\|^2 \\
& + 2\iota^{2(1-\gamma)} \left\| \int_0^\iota (\iota - s)^{q-1} P_q(\iota - s) \left(\sigma(s, x_n(s), x_n(\kappa s)) - \sigma(s, x(s), x(\kappa s)) \right) dW(s) \right\|^2 .
\end{aligned} \tag{3.4}$$

Taking the expectation on the two sides of Eq. (3.4), we get

$$\begin{aligned}
&\mathbb{E}\left\| \iota^{1-\gamma}((\mathscr{A}x_n)(\iota) - (\mathscr{A}x)(\iota)) \right\|^2 \\
\leqslant & 2\iota^{2(1-\gamma)} \mathbb{E}\left\| \int_0^\iota (\iota - s)^{q-1} P_q(\iota - s) \left(f(s, x_n(s), x_n(\kappa s)) - f(s, x(s), x(\kappa s)) \right) ds \right\|^2 \\
& + 2\iota^{2(1-\gamma)} \mathbb{E}\left\| \int_0^\iota (\iota - s)^{q-1} P_q(\iota - s) \left(\sigma(s, x_n(s), x_n(\kappa s)) - \sigma(s, x(s), x(\kappa s)) \right) dW(s) \right\|^2 .
\end{aligned}$$

By using the C-S inequality, Lemma 2.9, (H_1) and (H_2), we obtain

$$\begin{aligned}
&\mathbb{E}\left\| \iota^{1-\gamma}((\mathscr{A}x_n)(\iota) - (\mathscr{A}x)(\iota)) \right\|^2 \\
\leqslant & \frac{4b^{2(1-\gamma)} b M^2 \mathscr{L}_1}{\Gamma^2(q)} \|x_n - x\|_{1-\gamma}^2 \int_0^\iota (\iota - s)^{2(q-1)} ds \\
& + \frac{4b^{2(1-\gamma)} M^2 \mathscr{L}_2}{\Gamma^2(q)} \|x_n - x\|_{1-\gamma}^2 \int_0^\iota (\iota - s)^{2(q-1)} ds \\
\leqslant & \left[\frac{4b^{2-2\gamma+2q} M^2 \mathscr{L}_1}{\Gamma^2(q) 2q - 1} + \frac{4b^{1-2\gamma+2q} M^2 \mathscr{L}_2}{\Gamma^2(q) 2q - 1} \right] \|x_n - x\|_{1-\gamma}^2 .
\end{aligned} \tag{3.5}$$

Thus,

$$\|\mathscr{A}x_n - \mathscr{A}x\|_{1-\gamma}^2 \to 0 \quad \text{as} \quad n \to \infty,$$

which confirms that $\mathscr{A}$ is continuous.

Step 3: $\mathscr{A}$ is a contraction on $\mathscr{B}_\tau$.

Let $x, y \in \mathscr{B}_\tau$. Then for each $\iota \in J$, we have

$$\begin{aligned}
&\mathbb{E}\left\| \iota^{1-\gamma}((\mathscr{A}x)(\iota) - (\mathscr{A}y)(\iota)) \right\|^2 \\
\leqslant & 2\iota^{2(1-\gamma)} \mathbb{E}\left\| \int_0^\iota (\iota - s)^{q-1} P_q(\iota - s) \left(f(s, x(s), x(\kappa s)) - f(s, y(s), y(\kappa s)) \right) ds \right\|^2 \\
& + 2\iota^{2(1-\gamma)} \mathbb{E}\left\| \int_0^\iota (\iota - s)^{q-1} P_q(\iota - s) \left(\sigma(s, x(s), x(\kappa s)) - \sigma(s, y(s), y(\kappa s)) \right) dW(s) \right\|^2 .
\end{aligned}$$

Utilizing the C-S inequality, Lemma 2.9 and hypothesis (H_1) and (H_2), we derive

$$\begin{aligned}
&\mathbb{E}\left\|\iota^{1-\gamma}((\mathscr{A}x)(\iota)-(\mathscr{A}y)(\iota))\right\|^2 \\
&\leqslant \frac{2b^{2(1-\gamma)}(b\mathscr{L}_1+\mathscr{L}_2)M^2}{\Gamma^2(q)}\int_0^\iota(\iota-s)^{2(q-1)}\Big(\mathbb{E}\left\|s^{1-\gamma}(x(s)-x(s))\right\|^2 \\
&\qquad +\mathbb{E}\left\|s^{1-\gamma}(x(\kappa s)-y(\kappa s))\right\|^2\Big)ds.
\end{aligned} \tag{3.6}$$

This leads to

$$\begin{aligned}
\mathbb{E}\left\|\iota^{1-\gamma}((\mathscr{A}x)(\iota)-(\mathscr{A}y)(\iota))\right\|^2 &\leqslant \frac{4b^{1-2\gamma+2q}(b\mathscr{L}_1+\mathscr{L}_2)M^2}{\Gamma^2(q)2q-1}\|x-y\|_{1-\gamma}^2, \\
&\leqslant \Lambda_2\|x-y\|_{1-\gamma}^2.
\end{aligned} \tag{3.7}$$

From inequality (3.1), the operator $\mathscr{A}$ admits a unique fixed point, which corresponds to the unique solution of problem (1.1). □

4. Stability results

Next, we establish the Ulam stability of problem (1.1). Let $\varepsilon > 0$ and let $\phi \in C_{1-\gamma}(J, \mathbb{L}^2(\Omega, \mathscr{X}))$ be a nondecreasing function. Consider

$$\mathbb{E}\left\|{}^H\mathfrak{D}_{0^+,\iota}^{p,q}x(\iota)-Ax(\iota)-f(\iota,x(\iota),x(\kappa\iota))-\sigma(\iota,x(\iota),x(\kappa\iota))\frac{dW(\iota)}{d\iota}\right\|^2 \leqslant \varepsilon, \quad \iota \in J, \tag{4.1}$$

$$\mathbb{E}\left\|{}^H\mathfrak{D}_{0^+,\iota}^{p,q}x(\iota)-Ax(\iota)-f(\iota,x(\iota),x(\kappa\iota))-\sigma(\iota,x(\iota),x(\kappa\iota))\frac{dW(\iota)}{d\iota}\right\|^2 \leqslant \varepsilon\phi(\iota). \tag{4.2}$$

DEFINITION 4.1. Problem (1.1) is U-H stable if there exists a constant $C_{f,\sigma} > 0$ such that, for every $\varepsilon > 0$ and for any solution $y \in C_{1-\gamma}(J, \mathbb{L}^2(\Omega, \mathscr{X}))$ of (4.1), there exists a solution $x \in C_{1-\gamma}(J, \mathbb{L}^2(\Omega, \mathscr{X}))$ of problem (1.1) satisfying

$$\mathbb{E}\left\|\iota^{1-\gamma}(y(\iota)-x(\iota))\right\|^2 \leqslant C_{f,\sigma}\varepsilon, \quad \iota \in J.$$

DEFINITION 4.2. Problem (1.1) is U-H-R stable with respect to $\phi(\iota)$ if there exists a constant $C_{f,\sigma,\phi} > 0$ such that, for every $\varepsilon > 0$ and any solution $y \in C_{1-\gamma}(J, \mathbb{L}^2(\Omega, \mathscr{X}))$ of (4.2), there exists a solution $x \in C_{1-\gamma}(J, \mathbb{L}^2(\Omega, \mathscr{X}))$ of problem (1.1) satisfying

$$\mathbb{E}\left\|\iota^{1-\gamma}(y(\iota)-x(\iota))\right\|^2 \leqslant C_{f,\sigma,\phi}\varepsilon\phi(\iota), \quad \iota \in J.$$

REMARK 4.3. A function $y \in C_{1-\gamma}(J, \mathbb{L}^2(\Omega, \mathscr{X}))$ is a solution of (4.2) if and only if there exists a function $g \in C_{1-\gamma}(J, \mathbb{L}^2(\Omega, \mathscr{X}))$ (depending on y) such that

- $\mathbb{E}\|g(\iota)\|^2 \leqslant \varepsilon\phi(\iota), \quad \iota \in J,$
- ${}^{H}\mathfrak{D}^{p,q}_{0^+,\iota}y(\iota) = Ay(\iota) + f\big(\iota, y(\iota), y(\kappa\iota)\big) + \sigma\big(\iota, y(\iota), y(\kappa\iota)\big)\frac{dW(\iota)}{d\iota} + g(\iota), \quad \iota \in J.$

THEOREM 4.4. *Assume that hypotheses (H_1)–(H_2) hold, and let*

(H_3) *There exists a non-decreasing function ϕ on J and a constant $\lambda_\phi > 0$ such that, for each $\iota \in J$,*

$$\int_0^\iota (\iota - s)^{2(q-1)}\phi(s)\,ds \leqslant \lambda_\phi\,\phi(\iota).$$

If $\Lambda_3 < 1$, then problem (1.1) is U-H-R stable.

Proof. Let $y \in C_{1-\gamma}(J, \mathbb{L}^2(\Omega, \mathscr{X}))$ be a solution satisfies the inequality (4.2) and let $x \in C_{1-\gamma}(J, \mathbb{L}^2(\Omega, \mathscr{X}))$ be the unique solution of the problem (1.1). From Lemma 2.10, we have

$$\begin{aligned} x(\iota) =& \mathscr{S}_{p,q}(\iota)x_0 + \int_0^\iota (\iota - s)^{q-1}P_q(\iota - s)f(s, x(s), x(\kappa s))ds \\ &+ \int_0^\iota (\iota - s)^{q-1}P_q(\iota - s)\sigma(s, x(s), x(\kappa s))dW(s), \quad \iota \in J. \end{aligned} \tag{4.3}$$

Since y satisfies the inequality (4.2), so in view of Remark 4.3, we have

$$\begin{aligned} y(\iota) =& \mathscr{S}_{p,q}(\iota)x_0 + \int_0^\iota (\iota - s)^{q-1}P_q(\iota - s)f(s, y(s), y(\kappa s))ds \\ &+ \int_0^\iota (\iota - s)^{q-1}P_q(\iota - s)\sigma(s, y(s), y(\kappa s))dW(s) \\ &+ \int_0^\iota (\iota - s)^{q-1}P_q(\iota - s)g(s)ds, \quad \iota \in J. \end{aligned} \tag{4.4}$$

Then, by (H_3), for any $\iota \in J$, we have

$$\begin{aligned} &\mathbb{E}\Big\|y(\iota) - \mathscr{S}_{p,q}(\iota)x_0 - \int_0^\iota (\iota - s)^{q-1}P_q(\iota - s)f(s, y(s), y(\kappa s))ds \\ &\qquad - \int_0^\iota (\iota - s)^{q-1}P_q(\iota - s)\sigma(s, y(s), y(\kappa s))dW(s)\Big\|^2 \\ \leqslant& \mathbb{E}\left\|\int_0^\iota (\iota - s)^{q-1}P_q(\iota - s)g(s)ds\right\|^2 \\ \leqslant& \frac{bM^2}{\Gamma^2(q)}\int_0^\iota (\iota - s)^{2(q-1)}\mathbb{E}\|g(s)\|^2\,ds \\ \leqslant& \frac{bM^2}{\Gamma^2(q)}\varepsilon\lambda_\phi\phi(\iota). \end{aligned}$$

Thus, by virtue of (H_1) and (H_2), for any $\iota \in J$, we get

$$\begin{aligned}\mathbb{E}\|y(\iota)-x(\iota)\|^2 \leqslant \frac{3bM^2}{\Gamma^2(q)}\varepsilon\lambda_\phi\phi(\iota) + \frac{3bM^2\mathscr{L}_1}{\Gamma^2(q)}\int_0^\iota(\iota-s)^{2(q-1)}\Big(\mathbb{E}\left\|s^{1-\gamma}(y(s)-x(s))\right\|^2 \\ +\mathbb{E}\left\|s^{1-\gamma}(y(\kappa s)-x(\kappa s))\right\|^2\Big)ds \\ +\frac{3M^2\mathscr{L}_2}{\Gamma^2(q)}\int_0^\iota(\iota-s)^{2(q-1)}\Big(\mathbb{E}\left\|s^{1-\gamma}(y(s)-x(s))\right\|^2 \\ +\mathbb{E}\left\|s^{1-\gamma}(y(\kappa s)-x(\kappa s))\right\|^2\Big)ds.\end{aligned}$$

Then,

$$\begin{aligned}&\mathbb{E}\left\|\iota^{1-\gamma}(y(\iota)-x(\iota))\right\|^2 \\ &\leqslant \frac{3M^2b^{3-2\gamma}}{\Gamma^2(q)}\varepsilon\lambda_\phi\phi(\iota) + \frac{3b^{2-2\gamma}(b\mathscr{L}_1+\mathscr{L}_2)M^2}{\Gamma^2(q)}\int_0^\iota(\iota-s)^{2(q-1)} \\ &\times\left\{\mathbb{E}\left\|s^{1-\gamma}(y(s)-x(s))\right\|^2+\mathbb{E}\left\|s^{1-\gamma}(y(\kappa s)-x(\kappa s))\right\|^2\right\}ds.\end{aligned} \tag{4.5}$$

Now, for every $z\in C\left([0,b],\mathbb{L}^2(\Omega,\mathscr{F}_\iota,\mathbb{R})\right)$, we define $\mathscr{S}: C\left([0,b],\mathbb{L}^2(\Omega,\mathscr{F}_\iota,\mathbb{R})\right)\to C\left([0,b],\mathbb{L}^2(\Omega,\mathscr{F}_\iota,\mathbb{R})\right)$ as

$$\begin{aligned}(\mathscr{S}z)(\iota) &= \frac{3M^2b^{3-2\gamma}}{\Gamma^2(q)}\varepsilon\lambda_\phi\phi(\iota) \\ &+\frac{3b^{2-2\gamma}(b\mathscr{L}_1+\mathscr{L}_2)M^2}{\Gamma^2(q)}\int_0^\iota(\iota-s)^{2(q-1)}(z(s)+z(\kappa s)\,ds, \quad \iota\in J.\end{aligned} \tag{4.6}$$

We prove that $\mathscr{S}$ is a Picard operator. Let $z_1,z_2\in C\left([0,b],\mathbb{L}^2(\Omega,\mathscr{F}_\iota,\mathbb{R})\right)$, for any $\iota\in J$, we have

$$\begin{aligned}&\mathbb{E}\|(\mathscr{S}z_1)(\iota)-(\mathscr{S}z_2)(\iota)\|^2 \\ &\leqslant \frac{9b^{4-4\gamma}(b\mathscr{L}_1+\mathscr{L}_2)^2M^4}{\Gamma^4(q)}\mathbb{E}\left\|\int_0^\iota(\iota-s)^{2(q-1)}\{z_1(s)-z_2(s)+z_1(\kappa s)-z_2(\kappa s)\}\,ds\right\|^2.\end{aligned}$$

Utilizing the C-S inequality in conjunction with Lemma 2.8, we get

$$\mathbb{E}\|(\mathscr{S}z_1)(\iota)-(\mathscr{S}z_2)(\iota)\|^2 \leqslant \frac{36b^{2-4\gamma+4q}(b\mathscr{L}_1+\mathscr{L}_2)^2M^4}{\Gamma^4(q)4q-3}\|z_1-z_2\|^2.$$

This leads to

$$\|(\mathscr{S}z_1)-(\mathscr{S}z_2)\|^2 \leqslant \Lambda_3\|z_1-z_2\|^2.$$

Since $\Lambda_3<1$, the operator $\mathscr{S}$ is a contraction mapping. Consequently, by [31, Theorem 2.1], $\mathscr{S}$ is a Picard operator with $\mathscr{F}_{\mathscr{S}}=z^*$. Therefore, for all $\iota\in[0,b]$,

$$z^*(\iota)=\frac{3M^2b^{3-2\gamma}}{\Gamma^2(q)}\varepsilon\lambda_\phi\phi(\iota)+\frac{3b^{2-2\gamma}(b\mathscr{L}_1+\mathscr{L}_2)M^2}{\Gamma^2(q)}\int_0^\iota(\iota-s)^{2(q-1)}(z^*(s)+z^*(\kappa s)\,ds. \tag{4.7}$$

Next, we prove that z^* is increasing. Let $\iota_1, \iota_2 \in [0,b]$ be such that $\iota_1 < \iota_2$. Define $N = \min_{s\in[0,b]} (z^*(s) + z^*(\sigma(s))) \in \mathbb{R}^+$. Then, we have

$$\begin{aligned}
z^*(\iota_2) - z^*(\iota_1) = & \frac{3M^2 b^{3-2\gamma}}{\Gamma^2(q)} \varepsilon\lambda_\phi (\phi(\iota_2) - \phi(\iota_1)) \\
& + \frac{3b^{2-2\gamma}(b\mathscr{L}_1 + \mathscr{L}_2)M^2}{\Gamma^2(q)} \int_0^{\iota_1} \left[(\iota_2 - s)^{2(q-1)} - (\iota_1 - s)^{2(q-1)}\right] (z^*(s) + z^*(\kappa s)\, ds \\
& + \frac{3b^{2-2\gamma}(b\mathscr{L}_1 + \mathscr{L}_2)M^2}{\Gamma^2(q)} \int_{\iota_1}^{\iota_2} (\iota_2 - s)^{2(q-1)} (z^*(s) + z^*(\kappa s)\, ds \\
\geqslant & \frac{3M^2 b^{3-2\gamma}}{\Gamma^2(q)} \varepsilon\lambda_\phi (\phi(\iota_2) - \phi(\iota_1)) \\
& + \frac{3Nb^{2-2\gamma}(b\mathscr{L}_1 + \mathscr{L}_2)M^2}{\Gamma^2(q)} \int_0^{\iota_1} \left[(\iota_2 - s)^{2(q-1)} - (\iota_1 - s)^{2(q-1)}\right] ds \\
& + \frac{3Nb^{2-2\gamma}(b\mathscr{L}_1 + \mathscr{L}_2)M^2}{\Gamma^2(q)} \int_{\iota_1}^{\iota_2} (\iota_2 - s)^{2(q-1)} ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& z^*(\iota_2) - z^*(\iota_1) \\
\geqslant & \frac{3M^2 b^{3-2\gamma}}{\Gamma^2(q)} \varepsilon\lambda_\phi (\phi(\iota_2) - \phi(\iota_1)) + \frac{3Nb^{2-2\gamma}(b\mathscr{L}_1 + \mathscr{L}_2)M^2}{\Gamma^2(q)2q-1} \left(\iota_2^{2q-1} - \iota_1^{2q-1}\right) \\
> & 0.
\end{aligned}$$

Thus, the operator z^* is increasing. Given that $\kappa\iota \leqslant \iota$, it follows that $z^*(\kappa\iota) \leqslant z^*(\iota)$ for $\iota \in [0,b]$. From Eq. (4.7), we obtain

$$z^*(\iota) \leqslant \frac{3M^2 b^{3-2\gamma}}{\Gamma^2(q)} \varepsilon\lambda_\phi \phi(\iota) + \frac{6b^{2-2\gamma}(b\mathscr{L}_1 + \mathscr{L}_2)M^2}{\Gamma^2(q)} \int_0^{\iota} (\iota - s)^{2(q-1)} z^*(s) ds. \tag{4.8}$$

By applying Lemma 2.12, for $\iota \in J$, we arrive at the following

$$\begin{aligned}
z^*(\iota) \leqslant & \frac{3M^2 b^{3-2\gamma}}{\Gamma^2(q)} \varepsilon\lambda_\phi \phi(\iota) \times E_{2q-1}\left(\frac{6b^{2-2\gamma}(b\mathscr{L}_1 + \mathscr{L}_2)M^2}{\Gamma^2(q)} \Gamma(2q-1)\iota^{2q-1}\right) \\
\leqslant & C_{f,\sigma,\phi} \varepsilon\phi(\iota), \quad \iota \in J,
\end{aligned}$$

where

$$C_{f,\sigma,\phi} = \frac{3M^2 b^{3-2\gamma}}{\Gamma^2(q)} \lambda_\phi E_{2q-1}\left(\frac{6b^{2-2\gamma}(b\mathscr{L}_1 + \mathscr{L}_2)M^2}{\Gamma^2(q)} \Gamma(2q-1) b^{2q-1}\right).$$

In particular, for $z(\iota) = \mathbb{E}\left\|\iota^{1-\gamma}(y(\iota) - x(\iota))\right\|^2$, we use Eq. (4.5) to derive $z \leqslant \mathscr{S}(z)$, where $\mathscr{S}$ is an increasing Picard operator. Subsequently, applying Lemma 2.7 results in $z \leqslant z^*$. Then, it follows that

$$\mathbb{E}\left\|\iota^{1-\gamma}(y(\iota) - x(\iota))\right\|^2 \leqslant C_{f,\sigma,\phi} \varepsilon\phi(\iota), \quad \iota \in J. \tag{4.9}$$

Hence, problem (1.1) is U-H-R stable. □

THEOREM 4.5. *Suppose that conditions (H_1)–(H_3) hold. In addition, assume:* (H_4)*: There exists a non-decreasing function* ϕ *on* J *and a constant* $\lambda_\phi > 0$ *such that*

$$\int_0^\iota (\iota - s)^{2(q-1)} \phi(s)\, ds \leqslant \lambda_\phi\, \phi(\iota), \quad \forall \iota \in J.$$

If $\Lambda_3 < 1$*, then problem* (1.1) *is U-H-R stable.*

5. An example

Consider the following SPDE with HFD

$$\begin{aligned} {}^H\mathfrak{D}^{0.25,0.85}_{0^+,\iota} x(\iota,\xi) = \frac{\partial^2}{\partial \xi^2} x(\iota,\xi) + f(\iota, x(\iota,\xi), x(\kappa\iota,\xi)) + \sigma(\iota, x(\iota,\xi), x(\kappa\iota,\xi)) \frac{dW(\iota)}{d\iota}, \\ \iota \in (0,1],\ \xi \in [0,\pi], \end{aligned} \tag{5.1}$$

with the boundary conditions

$$x(\iota,0) = x(\iota,\pi) = 0,\ \ \mathfrak{I}^{1-0.8875}_{0+,\iota} x(\iota,\xi)\big|_{\iota=0} = x_0(\xi),\ \ \iota \in (0,1],\ \ \xi \in [0,\pi].$$

Define

$$A\varsigma = \frac{\partial^2 \varsigma}{\partial \xi^2},$$

where,

$$D(A) = \Big\{ \varsigma \in \mathscr{X} : \varsigma, \frac{\partial \varsigma}{\partial \xi} \text{ are absolutely continuous }, \frac{\partial^2 \varsigma}{\partial \xi^2} \in \mathscr{X},\ \varsigma(0) = \varsigma(\pi) = 0 \Big\}.$$

It is easy to verify that A generates a strongly continuous semigroup $\{\mathscr{S}(\iota)\}_{\iota \geqslant 0}$, which is compact, analytic, and self-adjoint. We can rewrite the system into the form

$$\begin{aligned} &{}^H\mathfrak{D}^{0.25,0.85}_{0^+,\iota} x(\iota) = Ax(\iota) + f(\iota, x(\iota), x(\kappa\iota)) + \sigma(\iota, x(\iota), x(\kappa\iota)) \frac{dW(\iota)}{d\iota}, \\ &\qquad \iota \in (0,1], \\ &\mathfrak{I}^{1-0.8875}_{0+,\iota} x(\iota)\big|_{\iota=0} = x_0, \end{aligned} \tag{5.2}$$

where,

$$\begin{aligned} &x(\iota)(\xi) = x(\iota,\xi), \\ &f(\iota, x(\iota), x(\kappa\iota))(\xi) = f(\iota, x(\iota,\xi), x(\kappa\iota,\xi)), \\ &\sigma(\iota, x(\iota), x(\kappa\iota))(\xi) = \sigma(\iota, x(\iota,\xi), x(\kappa\iota,\xi)), \\ &x_0(\xi) = \sin(\xi), \end{aligned} \tag{5.3}$$

and $\mathscr{X} = \mathbb{L}^2([0,\pi],\mathbb{R})$, with the norm $\|\cdot\|$.

Define

$$\begin{aligned} f(\iota, x(\iota,\xi), x(\kappa\iota,\xi)) &= \frac{1}{\sqrt{20}} \left(x(\iota,\xi) + \sin(x(\kappa\iota,\xi)) \right), \\ \sigma(\iota, x(\iota,\xi), x(\kappa\iota,\xi)) &= \frac{1}{10\sqrt{2}} \left(x(\iota,\xi) + x(\kappa\iota,\xi) \right). \end{aligned}$$

We observe that f and σ, with $\mathscr{L}_1 = \frac{1}{10}$ and $\mathscr{L}_2 = \frac{1}{100}$, satisfy (H_1) and (H_2). We also get $\Lambda_3 = 0.704854 < 1$ and $\Lambda_2 = 0.507507 < 1$. Thus, the hypotheses of Theorem 3.1 are satisfied, and therefore, there exists a unique solution to the problem (5.1).

Furthermore, by choosing $\phi(\iota) = \iota^{0.5}$ for any $\iota \in (0,1]$, we obtain $\lambda_\phi = 1.25331$ and $C_{f,\sigma,\phi} \approx 7.186241 > 0$. Consequently, (H_3) is satisfied, and Theorem 4.5 shows that the problem (5.1) is U-H-R stable.

Next, we graph the solution $x(\iota)$ to the problem (5.1) and the perturbed solution $y(\iota)$ in Figure 1 by choosing $\varepsilon = 0.0001$ and $\phi(\iota) = \iota^{0.5}$. Figure 2 shows the upper bound $C_{f,\sigma,\phi}\varepsilon\phi(\iota)$ as well as the mean square $\mathbb{E}\left\|\iota^{1-0.8875}(y(\iota) - x(\iota))\right\|^2$. This implies that $x(\iota)$ and $y(\iota)$ have difference bounded by a constant, proving U-H-R stability of the problem (5.1).

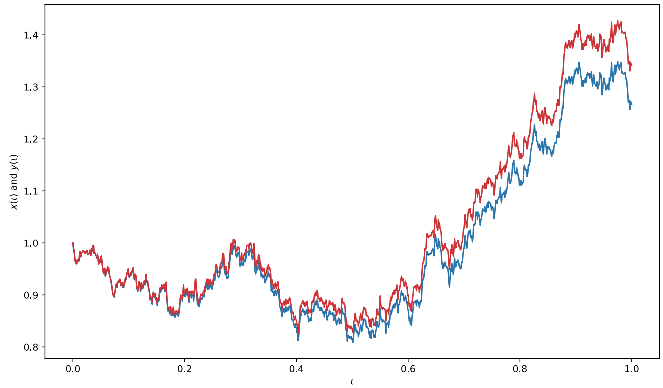

Figure 1: *Simulation of $x(\iota)$ and $y(\iota)$ of the problem (5.1) for $\varepsilon = 0.0001$ and $\phi = \iota^{0.5}$.*

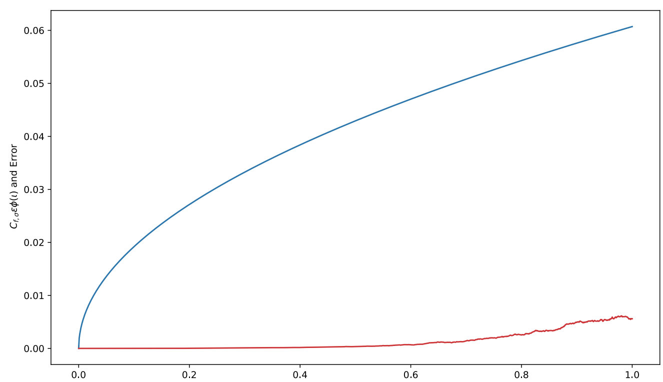

Figure 2: *U-H-R stability with respect to ε and $\phi(\iota)$.*

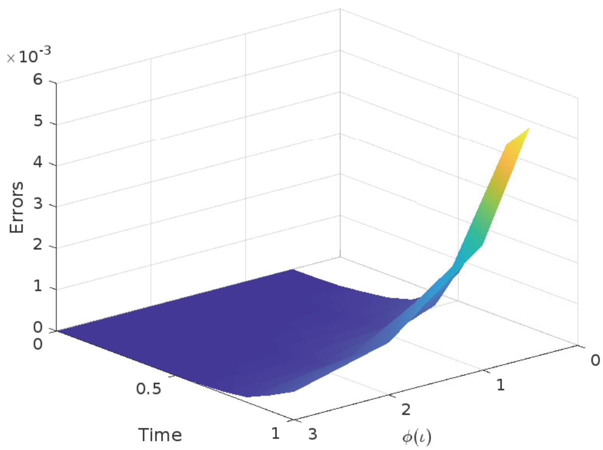

Figure 3: *Illustration of the effect of the power of* $\phi(\iota)$ *on the error.*

Time	$\phi(\iota)=\iota^{0.5}$	$\phi(\iota)=\iota$	$\phi(\iota)=\iota^2$	$\phi(\iota)=\iota^3$
0.0	0.000000e+00	0.000000e+00	0.000000e+00	0.000000e+00
0.1	2.510819e-06	1.426635e-07	6.396464e-10	3.597398e-12
0.2	2.262020e-05	2.596815e-06	4.725773e-08	1.076206e-09
0.3	9.249363e-05	1.571563e-05	6.280973e-07	3.153822e-08
0.4	1.716160e-04	3.918216e-05	2.836108e-06	2.586813e-07
0.5	3.369736e-04	9.736997e-05	1.124959e-05	1.628533e-06
0.6	6.934168e-04	2.410411e-04	4.021107e-05	8.375228e-06
0.7	1.407733e-03	5.641386e-04	1.254738e-04	3.493440e-05
0.8	2.532506e-03	1.138550e-03	3.209202e-04	1.139585e-04
0.9	4.945582e-03	2.451316e-03	8.482325e-04	3.730299e-04
1.0	5.596323e-03	3.031144e-03	1.264345e-03	6.762410e-04

Table 2: *Errors for different values of* $\phi(\iota)$.

The Table 2 demonstrates how the errors between the solutions $x(\iota)$ and $y(\iota)$ decrease significantly when the power of the function $\phi(\iota)$ increases. It implies that there is a closer approximation between the two solutions and rapid convergence of the difference between them to zero for higher-order functions of $\phi(\iota)$. Figure 3 shows the results graphically.

6. Conclusion

In this work, we investigated the Ulam stability for a class of SPDEs with HFD. We began by proving the existence and uniqueness of solutions to the problem, followed by an analysis of their stability, including U-H and U-H-R stability. The study relied on Picard operator theory to establish these results. Furthermore, a practical example was provided to illustrate the application of the theoretical findings and verify the stability criteria.

The results obtained in this study have potential applications in various fields where systems exhibit both stochastic behavior and memory effects. For instance, the modeling of biological processes, such as neuronal dynamics or population dynamics, can benefit from the inclusion of fractional operators that capture hereditary properties. Similarly, engineering systems with delays and noise, such as control systems for robotics or signal processing devices, can utilize these stability results to ensure reliable performance. Financial and economic models, where random fluctuations and memory effects play crucial roles, can also be analyzed using the proposed framework.

Appendix A. Numerical integration

Algorithm 1 *Numerical approximation*

1: Time/space discretization: $h=1/n$, $k=1/m$, $\iota_i=ih$, $\xi_j=jk$.
2: Initialize: $x_0^j=x_0(\xi_j)$, $y_0^j=x_0(\xi_j)+\varepsilon$, boundary $x_i^0=x_i^m=0$.
3: Noise: $\Delta W_i=\sqrt{h}X_i$, $X_i\sim\mathcal{N}(0,1)$.
4: Main loop: for $i=0$ to $n-1$, $j=1$ to $m-1$:
5: Spatial derivative: $(x_i^{j+1}-2x_i^j+x_i^{j-1})/k^2$
6: Hilfer derivative: $\sum_k \omega_{i,k}^{(p,q)}(x_k^j-x_{k-1}^j)$
7: Nonlocal: $\sum_k x_k^j(1-x_{0.5k}^j)h$
8: Update x_{i+1}^j, y_{i+1}^j using explicit scheme + noise.
9: Plot: $x(\iota)$, $y(\iota)$ at $\xi=0.5$.

Declaration of competing interest. There are no conflicts of interest, according to the authors.

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