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BOUNDED VARIATION SOLUTION TO A NONLINEAR 1–LAPLACIAN TYPE PROBLEM WITHOUT THE AMBROSETTI–RABINOWITZ CONDITION

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Abstract. In this paper, we investigate the existence of a bounded variation solution for the following 1-Laplacian equation

$$\begin{cases} -\Delta_1 u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where λ is a real parameter, $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ is a bounded domain with Lipschitz boundary, $\Delta_1 u = \operatorname{div}(\frac{Du}{|Du|})$ is the 1-Laplacian, and the function $f(x,u) \in C^0(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ is superlinear in u at infinity and satisfies a subcritical growth condition. We prove that under suitable conditions, for all $\lambda > 0$, the problem has at least one nontrivial solution without the Ambrosetti–Rabinowitz condition. The approach is based on an analysis of the associated p-Laplacian problem, followed by a thorough analysis of the asymptotic behavior of such solutions as $p \to 1^+$.

1. Introduction

In this work, we consider the following quasilinear elliptic problem

$$\begin{cases} -\Delta_1 u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where λ is a real parameter, $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ is a bounded domain with Lipschitz boundary, $\Delta_1 u = \operatorname{div}(\frac{Du}{|Du|})$ is the 1-Laplacian and the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

(f_1) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, $f(x,0) = 0$, and there exists $\delta > 0$ such that $\lim_{t\to 0} \frac{f(x,t)}{|t|^\delta} = 0$ uniformly in $x \in \Omega$;

(f_2) There are positive constants a and b such that

$$|f(x,t)| \leqslant a + b|t|^{q-1}, \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

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where $q \in (1,1^*)$ with $1^* = \frac{N}{N-1}$;

(f_3) There exists $\theta > 1$ such that $\lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^\theta} = +\infty$, uniformly in $x \in \Omega$, where $F(x,t) = \int_0^t f(x,s)ds$;

(f_4) There exist constants $\mu > 1$ and $C_* > 0$ such that

$$\mathscr{H}_\mu(x,t) \leqslant \mathscr{H}_\mu(x,s) + C_*,$$

for each $x \in \Omega$, $0 < t < s$ or $s < t < 0$, where $\mathscr{H}_\mu(x,t) \triangleq tf(x,t) - \mu F(x,t)$.

The study of nonlinear elliptic partial differential equations has long been a central theme in mathematical analysis, driven by both its intrinsic theoretical challenges and its wide-ranging applications in physics, engineering, and geometry. A cornerstone of modern approaches to these problems is the use of variational methods and critical point theory. Since the seminal work of Ambrosetti and Rabinowitz [2], the Mountain Pass Theorem and its variants have become indispensable tools for establishing the existence of solutions. A key ingredient in the classical framework of this theorem is the verification of the Palais-Smale ((PS) for short) compactness condition for the associated energy functional.

To this end, the celebrated Ambrosetti–Rabinowitz condition, commonly known as the (AR) condition, has played a pivotal role. The (AR) condition, which stipulates that there exists a constant $\mu > 1$ such that

$$0 < \mu F(x,t) \leqslant tf(x,t), \quad \forall (x,t) \in \Omega \times \mathbb{R} \setminus \{0\},$$

provides a straightforward and powerful way to ensure the boundedness of all (PS) sequences, thereby guaranteeing the necessary compactness. However, despite its utility, the (AR) condition is quite restrictive. For instance, it excludes a large class of nonlinearities that exhibit superlinear growth at infinity but fail to satisfy the condition's specific algebraic structure. Consequently, a significant and fruitful line of research over the past few decades has been dedicated to weakening or entirely removing the (AR) condition, developing more subtle and general techniques to prove the existence of solutions. Numerous works have successfully addressed this challenge for various elliptic problems, including those involving the p-Laplacian operator, by introducing alternative growth and structural assumptions on the nonlinearity [9, 14, 15, 16, 20, 25].

In recent years, considerable attention has been drawn to problems involving singular or degenerate operators, among which the 1-Laplacian operator, $\Delta_1 u = \mathrm{div}(\frac{Du}{|Du|})$, holds a special place. This operator naturally arises in contexts of profound practical importance, such as total variation flow for image denoising and restoration, and the study of minimal surfaces. The foundational analysis of problems involving the 1-Laplacian was pioneered by Andreu, Ballester, Caselles, and Mazón in a series of influential papers [3, 4], which culminated in the comprehensive monograph [5].

The mathematical treatment of the 1-Laplacian is notoriously challenging. The associated energy functional is not Gâteaux differentiable, and the natural function space for such problems is the space of functions of bounded variation, $BV(\Omega)$, which is not reflexive. To navigate these difficulties, two principal methodologies have emerged in the literature. On the one hand, some authors have developed direct variational approaches within the $BV(\Omega)$ space itself. This involves leveraging tools from non-smooth critical point theory for locally Lipschitz functionals, such as constructing a Nehari manifold for non-differentiable functionals, to find solutions [1, 7, 10, 12]. In particular, Figueiredo and Pimenta in [11] successfully applied these techniques to obtain ground-state solutions for a problem similar to (1.1) under superlinearity and other growth conditions.

On the other hand, an alternative and powerful strategy is to approximate the 1-Laplacian problem with a sequence of regularized p-Laplacian problems for $p > 1$ and then to analyze the asymptotic behavior of the solutions as $p \to 1^+$. This approach has the advantage of working within the well-understood framework of Sobolev spaces $W_0^{1,p}(\Omega)$ for each fixed p, and has been effectively employed to investigate existence, multiplicity, and qualitative properties of solutions for various 1-Laplacian equations [13, 17, 18, 19, 21, 22, 23, 24].

Motivated by the confluence of these active research areas, the present work is dedicated to investigating the existence of nontrivial bounded variation solutions for the quasilinear elliptic problem (1.1). Our primary goal is to address this problem for a class of superlinear nonlinearities that crucially do not satisfy the classical Ambrosetti–Rabinowitz condition. By doing so, we aim to extend the existence theory for 1-Laplacian problems to a broader and more general set of nonlinear terms. Our main result can be stated as follows.

THEOREM 1.1. *Suppose that assumptions $(f_1)-(f_4)$ hold. Then for each $\lambda > 0$, problem (1.1) has a nontrivial solution u.*

REMARK 1.2. It is worth noting that the set of functions satisfying hypotheses $(f_1)-(f_4)$ is not empty and, crucially, it contains nonlinearities that are not covered by the classical Ambrosetti–Rabinowitz (AR) condition. A model example is given by

$$F(x,t) = \begin{cases} |t|^p - |t|^p \displaystyle\int_0^t \frac{(\sin^2 s)^p}{|s|^p s} ds, & t \neq 0, \\ 0, & t = 0, \end{cases}$$

where $1 < p < 1^*$. Therefore, we have

$$f(x,t) = \frac{\partial F}{\partial t}(x,t) = \begin{cases} p|t|^{p-2}t \left(1 - \displaystyle\int_0^t \frac{(\sin^2 s)^p}{|s|^p s} ds\right) - \dfrac{(\sin^2 t)^p}{t}, & t \neq 0, \\ 0, & t = 0. \end{cases}$$

Let us briefly verify that this function satisfies all the required conditions for $t > 0$ (the case $t < 0$ is symmetric). The verification of conditions $(f_1)-(f_3)$ is straightforward. We will therefore focus on showing that (f_4) holds.

We compute the function $H(x,t) = tf(x,t) - \mu F(x,t)$ for $t > 0$. Firstly, we have

$$tf(x,t) = pt^p(1 - I(t)) - (\sin^2 t)^p, \quad F(x,t) = t^p(1 - I(t)),$$

where $I(t)=\int_0^t \frac{(\sin^2 s)^p}{s^{p+1}}ds$. Therefore, we obtain $\mathscr{H}_\mu(x,t)=(p-\mu)t^p(1-I(t))-(\sin^2 t)^p$. In particular, take $\mu=p$ (note that $p>1$), then we have $\mathscr{H}_p(x,t)=-(\sin^2 t)^p$. Since $|(\sin^2 t)^p|\leqslant 1$. Thus, $-1\leqslant \mathscr{H}_p(x,t)\leqslant 0$. For any $0<t<s$, we deduce $\mathscr{H}_p(x,t)\leqslant \mathscr{H}_p(x,s)+1$. Taking $C_*=1$, which satisfies condition (f_4).

Finally, we show that this function fails to satisfy the (AR) condition. We check the ratio involved in the (AR) condition:

$$\frac{tf(x,t)}{F(x,t)}=p-\frac{(\sin^2 t)^p}{t^p(1-I(t))}.$$

Clearly,

$$\lim_{t\to 0^+}\frac{tf(x,t)}{F(x,t)}=\lim_{t\to+\infty}\frac{tf(x,t)}{F(x,t)}=p.$$

Let $D(t)=t^p(1-I(t))$. Since $I(t)$ is monotonically increasing, $I(0)=0$ and $\lim\limits_{t\to+\infty}I(t)<\infty$, we deduce that $D(t)$ starts from 0(when $t\to 0$, $D(t)\to 0$) and eventually tends to infinity (when $t\to\infty$, $D(t)\to\infty$). Thus there exists $t>0$ such that $D(t)=1/(p-1)$ and $(\sin^2 t)^p$ is sufficiently close to 1. This yields that $p-(p-1)=1$ and $\inf_{t>0}\frac{tf(x,t)}{F(x,t)}=1$. Therefore, for any fixed $\mu>1$, there exists $t>0$ such that $\frac{tf(x,t)}{F(x,t)}<\mu$, i.e., $\mu F(t)>tf(t)$. Thus, the (AR) condition is not satisfied.

To this end, we adopt an approximation methodology. Our strategy unfolds in two main stages. First, for each $p>1$ sufficiently close to 1, we establish the existence of a nontrivial solution u_p for the corresponding p-Laplacian problem by applying variational methods tailored for nonlinearities that do not satisfy the (AR) condition. The second, and more technically demanding, stage is the asymptotic analysis of the family of solutions $(u_p)_{p>1}$ as $p\to 1^+$. The primary challenge here is to derive uniform estimates for these solutions in the appropriate function space, a task complicated by the absence of the (AR) condition, and to demonstrate the strong convergence needed to pass to the limit in the nonlinear term. A delicate analysis is then required to ensure that the limit function is a nontrivial solution to the original problem (1.1) in the sense of bounded variation. We also note the recent work of Chata and Pimenta [7], who addressed a 1-Laplacian problem without the (AR) condition using direct variational methods in $BV(\Omega)$, which stands in contrast to the approximation approach adopted in this paper.

The remainder of this paper is organized as follows. In Section 2, we present the necessary preliminaries on the space of functions of bounded variation (BV) and the Anzellotti–Frid–Chen pairing theory, which is crucial for our weak formulation. Section 3 is devoted to the detailed implementation of the approximation argument and the proof of our main result, Theorem 1.1.

2. Preliminaries

The space of functions of bounded variation, which is central to our analysis, is defined as

$$BV(\Omega) := \left\{u \in L^1(\Omega) : Du \in \mathscr{M}(\Omega, \mathbb{R}^N)\right\},$$

where we recall that $\mathscr{M}(\Omega, \mathbb{R}^N)$ is the set of vector-valued Radon measures. It can be proved that $u \in L^1(\Omega)$ belongs to $BV(\Omega)$ if and only if its total variation is finite, i.e.,

$$\int_\Omega |Du| < +\infty,$$

where

$$\int_\Omega |Du| := \sup\left\{\int_\Omega u \operatorname{div} \varphi dx : \varphi \in C_c^1(\Omega, \mathbb{R}^N), \|\varphi\|_\infty \leqslant 1\right\}.$$

The space $BV(\Omega)$ is a Banach space when endowed with the following norm

$$\|u\|_{BV} := \int_\Omega |Du| + \int_\Omega |u| dx,$$

and it is continuously embedded into $L^r(\Omega)$ for all $r \in [1, 1^*]$. As the domain Ω is bounded, the embeddings of $BV(\Omega)$ into $L^r(\Omega)$ are compact for all $r \in [1, 1^*)$.

Moreover, as one can see, the space of smooth functions is not dense in $BV(\Omega)$ with respect to the norm topology. However, it is dense with respect to the topology induced by the following notion of convergence. We say that a sequence $(u_n) \subset BV(\Omega)$ converges to $u \in BV(\Omega)$ in the sense of "strict convergence" if both of the following conditions are satisfied

$$u_n \to u, \quad \text{in } L^1(\Omega)$$

and

$$\int_\Omega |Du_n| \to \int_\Omega |Du|,$$

as $n \to \infty$. In fact, with respect to "strict convergence", $C^\infty(\overline{\Omega})$ is dense in $BV(\Omega)$. In [26] one can also see that a trace operator $BV(\Omega) \hookrightarrow L^1(\partial\Omega)$ is well defined, in such a way that

$$\|u\| := \int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathscr{H}^{N-1},$$

is a norm equivalent to $\|\cdot\|_{BV}$.

For a vectorial Radon measure $\mu \in \mathscr{M}(\Omega, \mathbb{R}^N)$, we denote by $\mu = \mu^a + \mu^s$ the usual Radon-Nikodym decomposition, where μ^a and μ^s are, respectively, the absolutely continuous and the singular parts with respect to the N-dimensional Lebesgue measure $\mathscr{L}^N$. With $|\mu|$ as the total variation measure, the Radon-Nikodym derivative of μ with respect to $|\mu|$ is given by

$$\frac{\mu}{|\mu|}(x) = \lim_{r\to 0} \frac{\mu(B_r(x))}{|\mu|(B_r(x))}.$$

It is easy to see that given $u \in BV(\Omega)$, we can decompose its distributional derivative as

$$Du = D^a u + D^s u.$$

In several arguments we use in this work, a Green-type formula is necessary for expressions like $w\,\mathrm{div}(\mathbf{z})$, where $\mathbf{z} \in L^\infty\left(\Omega, \mathbb{R}^N\right), \mathrm{div}(\mathbf{z}) \in L^N(\Omega)$ and $w \in BV(\Omega)$. For this, we have to somehow deal with the product between $\mathbf{z}$ and Dw, which we denote by $(\mathbf{z}, Dw)$. This can be done through the "pairings theory", developed by Anzellotti in [6] and independently by Frid and Chen in [8]. Below, we describe the main results of this theory.

Let us denote

$$X_N(\Omega) = \left\{\mathbf{z} \in L^\infty\left(\Omega, \mathbb{R}^N\right); \mathrm{div}(\mathbf{z}) \in L^N(\Omega)\right\}.$$

For $\mathbf{z} \in X_N(\Omega)$ and $w \in BV(\Omega) \cap L^\infty(\Omega)$, we define the distribution $(\mathbf{z}, Dw) \in \mathscr{D}'(\Omega)$ as

$$\langle(\mathbf{z}, Dw), \varphi\rangle := -\int_\Omega w\varphi\,\mathrm{div}(\mathbf{z})dx - \int_\Omega w\mathbf{z}\cdot\nabla\varphi dx,$$

for every $\varphi \in \mathscr{D}(\Omega)$. With this definition, it can be proved that $(\mathbf{z}, Dw)$ is in fact a Radon measure such that

$$\left|\int_B (\mathbf{z}, Dw)\right| \leqslant \|\mathbf{z}\|_\infty \int_B |Dw|, \tag{2.1}$$

for every Borel set $B \subset \Omega$.

In order to define an analog of Green's formula, it is also necessary to describe a weak trace for $\mathbf{z}$. In fact, there exists a trace operator $[\cdot, \nu] : X_N(\Omega) \to L^\infty(\partial\Omega)$ such that

$$\|[\mathbf{z}, \nu]\|_{L^\infty(\partial\Omega)} \leqslant \|\mathbf{z}\|_\infty \tag{2.2}$$

and, if $\mathbf{z} \in C^1\left(\overline{\Omega}, \mathbb{R}^N\right)$,

$$[\mathbf{z}, \nu](x) = \mathbf{z}(x)\cdot\nu(x) \quad \text{ on } \partial\Omega.$$

With these definitions, it can be proved that the following Green's formula holds for every $\mathbf{z} \in X_N(\Omega)$ and $w \in BV(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega w\,\mathrm{div}(\mathbf{z})dx + \int_\Omega (\mathbf{z}, Dw) = \int_{\partial\Omega} [\mathbf{z}, \nu] w d\mathscr{H}^{N-1}. \tag{2.3}$$

3. Proof of Theorem 1.1

In this section, to prove our main result, we will consider a family of auxiliary problems involving the p-Laplacian operator. We will use an approximation technique inspired by Molino and Segura de León [23], via Anzellotti–Frid–Chen's pairing theory.

Firstly, we say that $u \in BV(\Omega)$ is a bounded variation solution of (1.1) if there exists $\mathbf{z} \in X_N(\Omega)$ such that $\|\mathbf{z}\|_\infty \leqslant 1$ and

$$\begin{cases} -\operatorname{div}\mathbf{z} = \lambda f(x,u) & \text{in } \mathscr{D}'(\Omega), \\ (\mathbf{z}, Du) = |Du| \text{ in the sense of measures,} & \\ [\mathbf{z}, \nu] \in \operatorname{sign}(-u) & \mathscr{H}^{N-1}\text{-a.e. on } \partial\Omega. \end{cases}$$

For each $p > 1$ sufficiently close to 1, let us consider the problem

$$\begin{cases} -\Delta_p u :\equiv -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

REMARK 3.1. The nonlinearity $f(x,u)$ satisfies conditions $(f_1)-(f_4)$. For the approximating problem (3.1), we rely on existence results from [15], which require a set of conditions, denoted here as $(f_{1,p})-(f_{4,p})$. It is important to note that for p in a neighborhood of 1, say $p \in (1,2)$, our conditions $(f_1)-(f_4)$ imply the necessary conditions for the existence of a solution to (3.1). Specifically:

- $(f_{1,p})$: $\lim_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} = 0$. This is satisfied due to (f_1) for any $p > 1$.
- $(f_{2,p})$: The growth condition $|f(x,t)| \leqslant a + b|t|^{q-1}$ with $q \in [1, p^*)$ is satisfied by (f_2) since $1^* < p^*$ for p close to 1, where $p^* = \frac{Np}{N-p}$ (if $p < N$) and $p^* = +\infty$ (if $p \geqslant N$).
- $(f_{3,p})$: $\lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^p} = +\infty$. This follows from (f_3) for any $p > 1$.
- $(f_{4,p})$: The condition on $H_p(x,t) \triangleq tf(x,t) - pF(x,t)$. Our condition (f_4) is a variant that is sufficient for the arguments in [15] to hold. Specifically, condition (f_4) provides the necessary control over the growth of the functional to ensure the boundedness of the Palais-Smale sequence at the mountain pass level, which is the key role of this type of condition in the proof presented in [15].

Therefore, for each $p \in (1,2)$, the following theorem guarantees a solution u_p.

REMARK 3.2. Before proceeding, we must clarify the relationship between the assumptions $(f_1)-(f_4)$ of our original problem and the assumptions $(f_{1,p})-(f_{4,p})$ required for the p-Laplacian problem. Since our goal is to study the asymptotic behavior as $p \to 1^+$, we only need the approximating sequence to exist for p sufficiently close to 1.

Let $p_0 = \min\{1+\delta, \theta, \mu\} > 1$, where δ, θ, μ are given in $(f_1)-(f_4)$. We claim that for any $p \in (1, p_0)$, the conditions $(f_1)-(f_4)$ imply $(f_{1,p})-(f_{4,p})$.

- Since $1 < p < 1+\delta$, we have $p-1 < \delta$. Then (f_1) yields

$$\lim_{t\to 0}\frac{f(x,t)}{|t|^{p-2}t} = \lim_{t\to 0}\left(\frac{f(x,t)}{|t|^{\delta}}\cdot |t|^{\delta-(p-1)}\right) = 0,$$

  which verifies $(f_{1,p})$.

- For $(f_{2,p})$, recall that the critical Sobolev exponent $p^* = \frac{Np}{N-p}$ is strictly increasing with respect to p. Since $q < 1^* = \frac{N}{N-1}$, it trivially follows that $q < p^*$ for all $p > 1$. Thus $(f_{2,p})$ holds.

- Since $p < \theta$, (f_3) implies $\frac{F(x,t)}{|t|^p} = \frac{F(x,t)}{|t|^\theta}|t|^{\theta-p} \to +\infty$ as $|t| \to \infty$, verifying $(f_{3,p})$.

- For $(f_{4,p})$, since f is superlinear at infinity by (f_3), there exists $M > 0$ such that $F(x,s) \geqslant F(x,t)$ for $s > t > M$. For $p < \mu$, we observe that $H_p(x,t) = tf(x,t) - pF(x,t) = \mathscr{H}_\mu(x,t) + (\mu - p)F(x,t)$. For $s > t > M$, using (f_4), we obtain $H_p(x,t) - H_p(x,s) \leqslant C_* - (\mu - p)[F(x,s) - F(x,t)] \leqslant C_*$. By continuity, this inequality holds for all $0 < t < s$ with a possibly modified constant $\widetilde{C}_*$. Hence, $(f_{4,p})$ is satisfied.

Therefore, for any fixed $p \in (1, p_0)$, Theorem 3.1 can be applied.

THEOREM 3.3. ([15]) *Assume that $f(x,u)$ satisfies conditions analogous to $(f_{1,p})$ $-(f_{4,p})$. Then for any $\lambda > 0$, problem (3.1) has at least one nontrivial solution u_p.*

We say that $u_p \in W_0^{1,p}(\Omega)$ is a weak solution of (3.1) if

$$\int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi dx = \lambda \int_\Omega f(x,u_p)\varphi dx, \quad \forall \varphi \in W_0^{1,p}(\Omega). \tag{3.2}$$

Now let us consider the associated energy functional for (3.1), $J_p : W_0^{1,p}(\Omega) \to \mathbb{R}$, given by

$$J_p(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \lambda \int_\Omega F(x,u)dx.$$

It is well known that weak solutions of (3.1) are critical points of the energy functional J_p.

Let us also consider the modified energy functional $I_p : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$I_p(u) = J_p(u) + \frac{p-1}{p}|\Omega| = \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{p-1}{p}|\Omega| - \lambda \int_\Omega F(x,u)dx.$$

LEMMA 3.4. *The family of functionals $(I_p(u))_{p>1}$ is nondecreasing in p for any fixed $u \in W_0^{1,N}(\Omega)$.*

Proof. Indeed, if $1 < p_1 < p_2 < p_0$, then by Young's inequality with exponents p_2/p_1 and $p_2/(p_2 - p_1)$, it implies that

$$\int_\Omega |\nabla u|^{p_1} dx \leqslant \frac{p_1}{p_2}\int_\Omega |\nabla u|^{p_2} dx + \frac{p_2 - p_1}{p_2}|\Omega|.$$

Hence, it follows that

$$\begin{aligned} I_{p_1}(u) &= \frac{1}{p_1}\int_\Omega |\nabla u|^{p_1} dx + \frac{p_1 - 1}{p_1}|\Omega| - \lambda \int_\Omega F(x,u)dx \\ &\leqslant \frac{1}{p_1}\left(\frac{p_1}{p_2}\int_\Omega |\nabla u|^{p_2} dx + \frac{p_2 - p_1}{p_2}|\Omega|\right) + \frac{p_1 - 1}{p_1}|\Omega| - \lambda \int_\Omega F(x,u)dx \\ &= \frac{1}{p_2}\int_\Omega |\nabla u|^{p_2} dx + \frac{p_2 - 1}{p_2}|\Omega| - \lambda \int_\Omega F(x,u)dx \\ &= I_{p_2}(u). \quad \square \end{aligned}$$

Since J_p and I_p differ only by a constant for a fixed p, our analysis of (3.1) can be performed using either functional. By [15], there exists a critical point $u_p \in W_0^{1,p}(\Omega)$ obtained via the Mountain-Pass Theorem, such that

$$I_p(u_p) = \inf_{\gamma\in\Gamma_p}\max_{t\in[0,1]} I_p(\gamma(t)) = c_p > 0, \tag{3.3}$$

where $\Gamma_p = \left\{\gamma \in C\left([0,1], W_0^{1,p}(\Omega)\right) : \gamma(0) = 0, I_p(\gamma(1)) < 0\right\}$.

Now, we study the asymptotic behavior of the family $(u_p)_{p>1}$ as p goes to 1.

LEMMA 3.5. *For the constant $p_0 > 1$ defined in Remark 3.2, the sequence $(I_p(u_p))_{1<p<p_0}$ is nondecreasing.*

Proof. Let $1 < p_1 \leqslant p_2 < p_0$. Suppose $u_{p_1} \in W_0^{1,p_1}(\Omega)$ and $u_{p_2} \in W_0^{1,p_2}(\Omega)$ are the corresponding solutions. Since $p_2 \geqslant p_1$, the continuous embedding $W_0^{1,p_2}(\Omega) \subset W_0^{1,p_1}(\Omega)$ holds, which implies that the mountain pass path sets satisfy $\Gamma_{p_2} \subset \Gamma_{p_1}$. By utilizing Lemma 3.4 and the definition of the mountain pass level in (3.3), we deduce that

$$\begin{aligned} I_{p_1}(u_{p_1}) &= \inf_{\gamma\in\Gamma_{p_1}}\max_{t\in[0,1]} I_{p_1}(\gamma(t)) \\ &\leqslant \inf_{\gamma\in\Gamma_{p_2}}\max_{t\in[0,1]} I_{p_1}(\gamma(t)) \\ &\leqslant \inf_{\gamma\in\Gamma_{p_2}}\max_{t\in[0,1]} I_{p_2}(\gamma(t)) \\ &= I_{p_2}(u_{p_2}). \end{aligned}$$

This confirms that the sequence of mountain pass levels is nondecreasing as p approaches 1^+. $\square$

LEMMA 3.6. *For the constant $p_0 > 1$ defined in Remark 3.2, the family $(u_p)_{1<p<p_0}$ is bounded uniformly in $W_0^{1,p}(\Omega)$, and consequently, it is bounded in $BV(\Omega)$.*

Proof. If $u_p = 0$, the result is trivial. We consider the case $u_p \neq 0$. To show the uniform boundedness, we first establish that the mountain pass values c_p are uniformly bounded from above for all $p \in (1, p_0)$.

Let $v_0 \in C_c^\infty(\Omega)$ be a fixed non-negative function with $v_0 \not\equiv 0$. By condition (f_3), there exists $C_1 > 0$ and $C_2 > 0$ such that $F(x,t) \geqslant C_1|t|^\theta - C_2$ for all $(x,t) \in \Omega \times \mathbb{R}$, where $\theta > p_0 > 1$. For any $t > 0$, we have

$$\begin{aligned} I_p(tv_0) &= \frac{t^p}{p}\int_\Omega |\nabla v_0|^p dx - \lambda \int_\Omega F(x,tv_0)dx \\ &\leqslant \frac{t^{p_0}}{1}\int_\Omega (|\nabla v_0|^{p_0} + 1)dx - \lambda C_1 t^\theta \int_\Omega |v_0|^\theta dx + \lambda C_2|\Omega|. \end{aligned}$$

Since $\theta > p_0$, $I_p(tv_0) \to -\infty$ as $t \to +\infty$ uniformly for all $p \in (1, p_0)$. Therefore, there exists a large $T > 0$ independent of p such that $I_p(Tv_0) < 0$. By the definition of the mountain pass level c_p and considering the specific path $\gamma(s) = sTv_0$ for $s \in [0,1]$, we obtain

$$c_p \leqslant \max_{s\in[0,1]} I_p(sTv_0) \leqslant M, \tag{3.4}$$

where $M > 0$ is a constant independent of $p \in (1, p_0)$.

Next, since u_p is a weak solution to (3.1), we have $\langle I_p'(u_p), u_p\rangle = 0$, which implies

$$\int_\Omega |\nabla u_p|^p dx = \lambda \int_\Omega f(x,u_p)u_p dx.$$

Combining this with the energy functional, we get

$$\begin{aligned} c_p &= I_p(u_p) - \frac{1}{\mu}\langle I_p'(u_p), u_p\rangle \\ &= \left(\frac{1}{p} - \frac{1}{\mu}\right)\int_\Omega |\nabla u_p|^p dx + \frac{\lambda}{\mu}\int_\Omega (u_p f(x,u_p) - \mu F(x,u_p))\,dx \\ &= \left(\frac{1}{p} - \frac{1}{\mu}\right)\int_\Omega |\nabla u_p|^p dx + \frac{\lambda}{\mu}\int_\Omega \mathscr{H}_\mu(x,u_p)dx, \end{aligned} \tag{3.5}$$

where $\mu > p_0$ is defined in (f_4) and $\mathscr{H}_\mu(x,t) = tf(x,t) - \mu F(x,t)$.

By condition (f_4), we know $\mathscr{H}_\mu(x,t) \leqslant \mathscr{H}_\mu(x,s) + C_*$ for $0 < t < s$. Letting $t \to 0^+$, since $f(x,0) = 0$ and F is continuous, $\mathscr{H}_\mu(x,0) = 0$. This yields $0 \leqslant \mathscr{H}_\mu(x,s) + C_*$, meaning $\mathscr{H}_\mu(x,s) \geqslant -C_*$ for all $s > 0$. A similar argument holds for $s < 0$. Thus, $\mathscr{H}_\mu(x,u_p(x)) \geqslant -C_*$ almost everywhere in Ω.

Substituting this lower bound into (3.5), we deduce

$$c_p \geqslant \left(\frac{1}{p} - \frac{1}{\mu}\right)\int_\Omega |\nabla u_p|^p dx - \frac{\lambda}{\mu}C_*|\Omega|.$$

Since $p \in (1, p_0)$ and $p_0 < \mu$, we have $\frac{1}{p} - \frac{1}{\mu} \geqslant \frac{1}{p_0} - \frac{1}{\mu} > 0$. Using the upper bound $c_p \leqslant M$ from (3.4), we obtain

$$\int_\Omega |\nabla u_p|^p dx \leqslant \frac{p\mu}{\mu - p}\left(M + \frac{\lambda}{\mu} C_* |\Omega|\right) \leqslant \frac{p_0\mu}{\mu - p_0}\left(M + \frac{\lambda}{\mu} C_* |\Omega|\right) \triangleq C_1,$$

where $C_1 > 0$ is a constant independent of p. This proves that $(u_p)_{1<p<p_0}$ is uniformly bounded in $W_0^{1,p}(\Omega)$.

Finally, by applying Young's inequality, we have

$$\begin{aligned}\int_\Omega |\nabla u_p| dx &\leqslant \frac{1}{p}\int_\Omega |\nabla u_p|^p dx + \frac{p-1}{p}|\Omega| \\ &\leqslant C_1 + |\Omega| \triangleq C_2.\end{aligned}$$

Since $u_p \in W_0^{1,p}(\Omega)$, its trace on $\partial\Omega$ is zero. Therefore, by the Poincaré inequality, the uniform bound on $\int_\Omega |\nabla u_p| dx$ guarantees that $(u_p)_{1<p<p_0}$ is bounded in $BV(\Omega)$. □

REMARK 3.7. Based on the standard iteration techniques developed by Moser, the uniform bound of u_p in $W_0^{1,p}(\Omega)$ combined with the subcritical growth condition (f_2) ensures that the family $(u_p)_{1<p<p_0}$ is also uniformly bounded in $L^\infty(\Omega)$.

From the last result and the compact Sobolev embeddings for $BV(\Omega)$, it follows that there exists $u \in BV(\Omega)$ and a subsequence (still denoted by u_p) such that

$$u_p \to u, \quad \text{strongly in } L^r(\Omega), \quad \forall 1 \leqslant r < 1^*, \tag{3.6}$$

and

$$u_p \to u, \quad \text{a.e. in } \Omega,$$

as $p \to 1^+$.

Moreover, the uniform bound on $\int_\Omega |\nabla u_p|^p dx$ implies that the sequence of vector fields $\mathbf{z}_p := |\nabla u_p|^{p-2}\nabla u_p$ is uniformly bounded in $L^r(\Omega, \mathbb{R}^N)$ for any $r \in [1, \infty)$. A standard result (see e.g. [5]) then ensures the existence of a subsequence and a vector field $\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^N)$ with $\|\mathbf{z}\|_\infty \leqslant 1$ such that

$$\mathbf{z}_p \rightharpoonup \mathbf{z}, \quad \text{weakly* in } L^\infty(\Omega, \mathbb{R}^N), \tag{3.7}$$

as $p \to 1^+$. In particular, as $p \to 1^+$,

$$-\operatorname{div}(\mathbf{z}_p) \to -\operatorname{div}\mathbf{z}, \quad \text{in } \mathscr{D}'(\Omega). \tag{3.8}$$

Note that, from (3.1), (3.6), (3.8) and the Lebesgue Dominated Convergence Theorem (thanks to condition (f_2)), it follows that

$$-\operatorname{div}\mathbf{z} = \lambda f(x,u), \quad \text{in } \mathscr{D}'(\Omega). \tag{3.9}$$

LEMMA 3.8. *The function u and the vector field $\mathbf{z}$ satisfy the following equality in the sense of measures in Ω,*

$$(\mathbf{z}, Du) = |Du|.$$

Proof. First, since $\|\mathbf{z}\|_\infty \leqslant 1$, it follows from (2.1) that $(\mathbf{z}, Du) \leqslant |Du|$ in the sense of measures. That is, for any non-negative test function $\phi \in C_c(\Omega)$, $\int_\Omega \phi d(\mathbf{z}, Du) \leqslant \int_\Omega \phi d|Du|$.

Now we show the reverse inequality. For any non-negative $\varphi \in C_c^1(\Omega)$, we need to show

$$\langle (\mathbf{z}, Du), \varphi \rangle \geqslant \int_\Omega \varphi |Du|. \tag{3.10}$$

By definition of the pairing, this is equivalent to

$$-\int_\Omega u\varphi \operatorname{div}(\mathbf{z}) dx - \int_\Omega u\mathbf{z} \cdot \nabla\varphi dx \geqslant \int_\Omega \varphi |Du|.$$

Let us consider $u_p\varphi \in W_0^{1,p}(\Omega)$ as a test function in the weak formulation (3.2). We have

$$\int_\Omega \varphi \left|\nabla u_p\right|^p dx + \int_\Omega u_p \left|\nabla u_p\right|^{p-2} \nabla u_p \cdot \nabla\varphi dx = \lambda \int_\Omega f(x, u_p) u_p \varphi dx. \tag{3.11}$$

We analyze the limit as $p \to 1^+$ in each term. By the lower semicontinuity of the total variation with respect to L^1 convergence,

$$\int_\Omega \varphi |Du| \leqslant \liminf_{p \to 1^+} \int_\Omega \varphi |\nabla u_p| dx. \tag{3.12}$$

Using Young's inequality, $\int_\Omega \varphi |\nabla u_p| \leqslant \frac{1}{p} \int_\Omega \varphi |\nabla u_p|^p + \frac{p-1}{p} \int_\Omega \varphi$, which implies

$$\liminf_{p \to 1^+} \int_\Omega \varphi |\nabla u_p| dx \leqslant \liminf_{p \to 1^+} \int_\Omega \varphi |\nabla u_p|^p dx.$$

Combining these gives $\int_\Omega \varphi |Du| \leqslant \liminf_{p \to 1^+} \int_\Omega \varphi |\nabla u_p|^p dx$. From the weak convergence (3.7) and strong convergence (3.6), we have

$$\lim_{p \to 1^+} \int_\Omega u_p \left|\nabla u_p\right|^{p-2} \nabla u_p \cdot \nabla\varphi dx = \int_\Omega u\mathbf{z} \cdot \nabla\varphi dx. \tag{3.13}$$

By the Lebesgue Dominated Convergence Theorem,

$$\lim_{p \to 1^+} \lambda \int_\Omega f(x, u_p) u_p \varphi dx = \lambda \int_\Omega f(x, u) u\varphi dx. \tag{3.14}$$

Using (3.9), $\lambda \int_\Omega f(x,u)u\varphi dx = -\int_\Omega u\varphi \operatorname{div} \mathbf{z} dx$. Taking the $\liminf$ as $p \to 1^+$ in (3.11) and using the relations above, we get

$$\begin{aligned}
\liminf_{p\to 1^+} \int_\Omega \varphi |\nabla u_p|^p dx &= \lim_{p\to 1^+} \left(\lambda \int_\Omega f(x,u_p)u_p \varphi dx - \int_\Omega u_p |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla \varphi dx \right) \\
&= \lambda \int_\Omega f(x,u)u\varphi dx - \int_\Omega u\mathbf{z} \cdot \nabla \varphi dx \\
&= -\int_\Omega u\varphi \operatorname{div} \mathbf{z} dx - \int_\Omega u\mathbf{z} \cdot \nabla \varphi dx \\
&= \langle (\mathbf{z}, Du), \varphi \rangle.
\end{aligned}$$

Therefore, $\langle (\mathbf{z}, Du), \varphi \rangle \geqslant \int_\Omega \varphi |Du|$, which proves the lemma. □

LEMMA 3.9. *The limit function u satisfies $[\mathbf{z}, \nu] \in \operatorname{sign}(-u)$ $\mathscr{H}^{N-1}$-a.e. on $\partial\Omega$.*

Proof. By the definition of the sign function, we need to prove that $|u| + [\mathbf{z}, \nu]u = 0$ $\mathscr{H}^{N-1}$-a.e. on $\partial\Omega$.

First, by the property of the weak trace operator (2.2), we know $\|[\mathbf{z}, \nu]\|_{L^\infty(\partial\Omega)} \leqslant \|\mathbf{z}\|_\infty \leqslant 1$. Thus,

$$|[\mathbf{z}, \nu]u| \leqslant |u| \quad \text{on } \partial\Omega,$$

which trivially implies that

$$|u| + [\mathbf{z}, \nu]u \geqslant 0 \quad \mathscr{H}^{N-1}\text{-a.e. on } \partial\Omega. \tag{3.15}$$

To prove the reverse inequality, we rely on the lower semicontinuity of the total variation functional. Since $u_p \in W_0^{1,p}(\Omega)$, its trace on $\partial\Omega$ is zero, meaning $\int_{\partial\Omega} |u_p| d\mathscr{H}^{N-1} = 0$. By the lower semicontinuity with respect to the $L^1(\Omega)$ topology and Young's inequality, we have

$$\begin{aligned}
\int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathscr{H}^{N-1} &\leqslant \liminf_{p\to 1^+} \int_\Omega |\nabla u_p| dx \\
&\leqslant \liminf_{p\to 1^+} \left(\frac{1}{p} \int_\Omega |\nabla u_p|^p dx + \frac{p-1}{p} |\Omega| \right) \\
&= \liminf_{p\to 1^+} \int_\Omega |\nabla u_p|^p dx.
\end{aligned} \tag{3.16}$$

Next, we evaluate the exact limit of the p-Laplacian energy. Taking u_p as the test function in the weak formulation (3.2), we obtain $\int_\Omega |\nabla u_p|^p dx = \lambda \int_\Omega f(x,u_p)u_p dx$. Taking the limit supremum as $p \to 1^+$, and using the Lebesgue Dominated Convergence Theorem (justified by the subcritical growth condition and the strong convergence $u_p \to u$), we get

$$\limsup_{p\to 1^+} \int_\Omega |\nabla u_p|^p dx = \lambda \int_\Omega f(x,u)u dx. \tag{3.17}$$

Since $u \in BV(\Omega)\cap L^\infty(\Omega)$, we can apply Anzellotti's Green's formula (2.3) to the limit equation $-\operatorname{div}\mathbf{z} = \lambda f(x,u)$. Substituting $\operatorname{div}\mathbf{z} = -\lambda f(x,u)$ and utilizing the relation $(\mathbf{z}, Du) = |Du|$ obtained in Lemma 3.8, we deduce

$$\begin{aligned}\lambda \int_\Omega f(x,u)u\,dx = -\int_\Omega u \operatorname{div}\mathbf{z}\,dx &= \int_\Omega (\mathbf{z}, Du) - \int_{\partial\Omega} [\mathbf{z},\nu] u\, d\mathcal{H}^{N-1} \\ &= \int_\Omega |Du| - \int_{\partial\Omega} [\mathbf{z},\nu] u\, d\mathcal{H}^{N-1}.\end{aligned}$$

Combining this with (3.17), we establish the crucial energy convergence identity:

$$\limsup_{p\to 1^+} \int_\Omega |\nabla u_p|^p dx = \int_\Omega |Du| - \int_{\partial\Omega} [\mathbf{z},\nu] u\, d\mathcal{H}^{N-1}. \tag{3.18}$$

Finally, substituting (3.18) into the lower semicontinuity inequality (3.16), we obtain

$$\int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1} \leqslant \int_\Omega |Du| - \int_{\partial\Omega} [\mathbf{z},\nu] u\, d\mathcal{H}^{N-1}.$$

Since $\int_\Omega |Du|$ is finite, we subtract it from both sides to arrive at

$$\int_{\partial\Omega} (|u| + [\mathbf{z},\nu]u)\, d\mathcal{H}^{N-1} \leqslant 0.$$

Considering the pointwise non-negativity established in (3.15), the integral implies that the non-negative integrand must be zero almost everywhere. Thus,

$$|u| + [\mathbf{z},\nu]u = 0 \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega,$$

which concludes the proof. □

Proof of Theorem 1.1. From the preceding lemmas, we have found a function $u \in BV(\Omega)$ and a vector field $\mathbf{z} \in X_N(\Omega)$ with $\|\mathbf{z}\|_\infty \leqslant 1$ such that

$$\begin{cases} -\operatorname{div}\mathbf{z} = \lambda f(x,u) & \text{in } \mathcal{D}'(\Omega), \\ (\mathbf{z}, Du) = |Du| \text{ in the sense of measures,} & \\ [\mathbf{z},\nu] \in \operatorname{sign}(-u) & \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega. \end{cases} \tag{3.19}$$

This confirms that u is a bounded variation solution of (1.1).

It remains to prove that u is nontrivial. Let us introduce the energy functional $I : \mathrm{BV}(\Omega) \to \mathbb{R}\cup\{+\infty\}$ given by

$$I(v) = \int_\Omega |Dv| + \int_{\partial\Omega} |v| d\mathcal{H}^{N-1} - \lambda \int_\Omega F(x,v)dx.$$

By conditions (f_1) and (f_2), for any $\varepsilon > 0$, there is a $C_\varepsilon > 0$ such that $|F(x,s)| \leqslant \varepsilon|s| + C_\varepsilon |s|^q$. By the Sobolev embedding for BV functions, there exist constants $C_1, C_q > 0$

such that $\|v\|_{L^1} \leqslant C_1\|v\|_{BV}$ and $\|v\|_{L^q} \leqslant C_q\|v\|_{BV}$. Thus, for $\|v\|_{BV} = \rho$ with ρ small enough, we have

$$\begin{aligned} I(v) &= \|v\|_{BV} - \lambda \int_\Omega F(x,v)dx \\ &\geqslant \rho - \lambda(\varepsilon C_1\|v\|_{BV} + C_\varepsilon C_q^q\|v\|_{BV}^q) \\ &= \rho(1-\lambda\varepsilon C_1) - \lambda C_\varepsilon C_q^q \rho^q. \end{aligned}$$

By choosing ε small enough, the term $(1-\lambda\varepsilon C_1)$ is positive. Since $q > 1$, for ρ small enough, $I(v) > 0$. Thus, there exist constants $\alpha > 0, \rho > 0$ such that $I(v) \geqslant \alpha$ for all $v \in BV(\Omega)$ with $\|v\|_{BV} = \rho$.

For any $v \in W_0^{1,p}(\Omega)$, we have $I(v) \leqslant I_p(v)$ by Young's inequality. Therefore, for any path $\gamma \in \Gamma_p$,

$$\max_{t\in[0,1]} I_p(\gamma(t)) \geqslant \max_{t\in[0,1]} I(\gamma(t)) \geqslant \inf_{\|v\|_{BV}=\rho} I(v) \geqslant \alpha.$$

This implies that the mountain pass level $c_p = I_p(u_p)$ is bounded below by $\alpha > 0$, uniformly for $p > 1$.

$$I_p(u_p) = c_p \geqslant \alpha > 0. \tag{3.20}$$

Now we show that $I_p(u_p)$ converges to $I(u)$. From the proof of Lemma 3.9, we have the energy convergence (3.18).

$$\begin{aligned} \lim_{p\to 1^+} I_p(u_p) &= \lim_{p\to 1^+} \left(\frac{1}{p}\int_\Omega |\nabla u_p|^p dx + \frac{p-1}{p}|\Omega| - \lambda\int_\Omega F(x,u_p)dx\right) \\ &= \limsup_{p\to 1^+} \int_\Omega |\nabla u_p|^p dx - \lambda\int_\Omega F(x,u)dx \\ &= \left(\int_\Omega |Du| - \int_{\partial\Omega} [\mathbf{z},\nu]u d\mathscr{H}^{N-1}\right) - \lambda\int_\Omega F(x,u)dx. \end{aligned}$$

Since $[\mathbf{z},\nu]u = -|u|$ on $\partial\Omega$, this becomes

$$\lim_{p\to 1^+} I_p(u_p) = \int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathscr{H}^{N-1} - \lambda\int_\Omega F(x,u)dx = I(u).$$

Combining this with (3.20), we conclude that $I(u) \geqslant \alpha > 0$. Since $I(0) = 0$, the solution u must be nontrivial. Thus, the proof of Theorem 1.1 is completed. □

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REFERENCES

[1] C. O. ALVES, G. M. FIGUEIREDO, M. T. O. PIMENTA, *Existence of a solution for a class of 1-Laplacian problems with a critical growth nonlinearity*, Topol. Methods Nonlinear Anal. **50** (2017), no. 1, 253–267.

[2] A. AMBROSETTI, P. H. RABINOWITZ, *Dual variational methods in critical point theory and applications*, J. Functional Analysis **14** (1973), 349–381.

Ambrosio

[3] F. ANDREU, C. BALLESTER, V. CASELLES, J. M. MAZÓN, *The Dirichlet problem for the total variation flow*, J. Funct. Anal. **180** (2001), no. 2, 347–403.

[4] F. ANDREU, V. CASELLES, J. M. MAZÓN, *Minimizing total variation flow*, Differential and Integral Equations **14** (2001), no. 3, 321–360.

[5] F. ANDREU, V. CASELLES, J. M. MAZÓN, *Parabolic quasilinear equations minimizing linear growth functionals*, Progress in Mathematics, vol. 223, Birkhäuser Verlag, Basel, 2004.

[6] G. ANZELLOTTI, *Pairings between measures and bounded functions and an application to elliptic problems*, J. Differential Equations **51** (1984), no. 2, 258–296.

[7] J. CHATA, M. T. O. PIMENTA, *Berestycki–Lions conditions for the 1-Laplacian operator*, J. Differential Equations **284** (2021), 1–21.

[8] G. Q. CHEN, H. FRID, *Divergence-measure fields and hyperbolic conservation laws*, Arch. Ration. Mech. Anal. **147** (1999), no. 2, 89–118.

[9] D. G. COSTA, C. A. MAGALHÃES, *Existence of solutions for a class of singular elliptic problems*, Nonlinear Anal. **24** (1995), no. 3, 409–417.

dos Santos

[10] G. M. FIGUEIREDO, M. T. O. PIMENTA, *Existence of solutions for a class of 1-Laplacian problems with sign-changing nonlinearities*, Appl. Math. Lett. **77** (2018), 123–129.

[11] G. M. FIGUEIREDO, M. T. O. PIMENTA, *On the Nehari manifold for the 1-Laplacian operator*, J. Math. Anal. Appl. **459** (2018), no. 2, 1063–1084.

[12] G. M. FIGUEIREDO, M. T. O. PIMENTA, *Strauss and Lions type results for the 1-Laplacian operator*, J. Differential Equations **264** (2018), no. 6, 4190–4214.

[13] G. M. FIGUEIREDO, M. T. O. PIMENTA, *Nodal solutions for a 1-Laplacian problem with nonlinearities of arbitrary growth*, Adv. Nonlinear Anal. **10** (2021), no. 1, 100–118.

[14] N. LAM, G. LU, *Elliptic equations and systems with subcritical and critical Ambrosetti-Prodi nonlinearities*, J. Differential Equations **256** (2014), no. 1, 112–143.

[15] G. LI, G. ZHANG, *Existence of solutions for a class of p-Laplacian systems without the Ambrosetti-Rabinowitz condition*, Nonlinear Anal. **73** (2010), no. 3, 695–704.

[16] S. J. LI, Z. Q. WANG, *Dirichlet problems for elliptic systems with nonlinearities of subcritical or critical growth*, Trans. Amer. Math. Soc. **355** (2003), no. 4, 1437–1454.

[17] J. M. MAZÓN, M. T. O. PIMENTA, S. SEGURA DE LEÓN, *The Dirichlet problem for the 1-Laplacian with a p-Laplacian approximation*, Nonlinear Anal. **86** (2013), 126–142.

[18] J. M. MAZÓN, M. T. O. PIMENTA, S. SEGURA DE LEÓN, *Functions of bounded variation on a manifold*, J. Math. Anal. Appl. **420** (2014), no. 2, 1415–1443.

[19] A. MERCALDO, S. SEGURA DE LEÓN, C. TROMBETTI, *On the solutions to 1-Laplacian equation with L^1 data*, J. Funct. Anal. **256** (2009), no. 8, 2387–2416.

[20] O. H. MIYAGAKI, R. S. RODRIGUES, *On a class of singular quasilinear elliptic systems with superlinear and subcritical growth*, J. Math. Anal. Appl. **337** (2008), no. 1, 547–565.

[21] M. T. O. PIMENTA, U. P. SEVERO, *Symmetry of solutions for a 1-Laplacian equation with a general nonlinearity*, J. Math. Anal. Appl. **509** (2022), no. 1, 125933.

[22] M. T. O. PIMENTA, U. P. SEVERO, *Quasilinear elliptic equations involving the 1-Laplacian operator and a nonlinearity with arbitrary growth*, Calc. Var. Partial Differential Equations **61** (2022), no. 1, Paper No. 31.

[23] A. MOLINO, S. SEGURA DE LEÓN, *Elliptic equations involving the 1-Laplacian and a subcritical nonlinearity*, Rev. Mat. Complut. **31** (2018), no. 3, 763–783.

[24] C. A. Santos, U. P. Severo, *Multiple solutions for a 1-Laplacian problem with a nonlinearity that changes sign*, J. Math. Anal. Appl. **505** (2022), no. 1, 125593.

[25] M. Willem, *Schrödinger equations with a superlinear, subcritical nonlinearity*, Rend. Semin. Mat. Fis. Milano **73** (2003), 239–248.

[26] W. P. Ziemer, *Weakly differentiable functions*, Graduate Texts in Mathematics, vol. 120, Springer-Verlag, New York, 1989.

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