Differential
Equations
& Applications
Volume 18, Number 2 (2026), 45–58
doi:10.7153/dea-2026-18-04

A NOTE ON THE BOUNDEDNESS OF VARIABLE HARDY OPERATORS ON WEIGHTED HERZ—MORREY SPACES WITH VARIABLE EXPONENTS

MEHVISH SULTAN, BABAR SULTAN* AND RENÉ ERLIN CASTILLO

(*Communicated by A. El Hamidi*)

Abstract. In this paper, we will introduce the idea of weighted Herz–Morrey spaces with variable exponents. Then the fractional Hardy-type operators of variable order is shown to be bounded in these spaces.

1. Introduction

We consider the Riesz potential operator

$$I^{\zeta(x)}f(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\zeta(x)}}dy,\quad 0<\zeta(x)<n. \tag{1.1}$$

Note that the $\zeta(x)$ is the order of the Riesz potential operator which is variable.

Variable exponent function spaces have been widely studied and have many important applications. These variable exponent function spaces are important for studying problems in partial differential equations and applied mathematics. In particular, Herz spaces with variable exponent generalize classical Herz spaces see [3]. The Herz-Morrey spaces $M\dot{K}^{\alpha,\lambda}_{q(\cdot),p}(\mathbb{R}^n)$ generalize the idea of Herz spaces with variable exponent $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. These function spaces were initially defined in [4]. Izuki and Noi introduced the concept of weighted variable Herz spaces $\dot{K}^{\alpha,p}_{q(\cdot)}(w)$ in their papers, [6].

The concept of grand Morrey, introduced in [10], has attracted significant attention from researchers. Boundedness of integral operators on grand variable Herz spaces can be seen in [17, 22, 29, 31]. Sultan et al. [13] introduced the idea of grand variable Herz-Morrey spaces and proved the boundedness of Riesz potential operators on these spaces. In [23], the definition of grand weighted Herz spaces $\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(w)$ spaces was introduced and proved the boundedness of integral operators in these spaces. In [14] it was shown that fractional integral operators are bounded in grand weighted variable Herz-Morrey spaces. Additional developments concerning grand variable Herz spaces, grand variable

Mathematics subject classification (2020): 46E30, 47B38.
Keywords and phrases: Lebesgue spaces, weighted estimates, Hardy operators, weighted variable Herz-Morrey spaces.

* Corresponding author.

© ELEMENT, Zagreb
Paper DEA-18-04

Herz–Morrey spaces, and variable exponent function spaces can be found in [2, 11, 12, 15–21, 25–28, 30, 33], where different classes of operators and related inequalities were studied.

Motivated by the aforementioned results, in this paper we will define the idea of weighted Herz–Morrey spaces with variable exponents and demonstrate the boundedness of the Hardy operators of variable order on weighted variable Herz-Morrey spaces $M\dot{K}^{\alpha(\cdot),u}_{\lambda,q(\cdot)}(w)$.

2. Preliminaries

In this paper, $|G| > 0$ and χ_G denote the Lebesgue measure and characteristic function of a measurable subset of $\mathbb{R}^n$, respectively. C is always a positive constant and its value can be changed by its appearance. Let G be a measurable subset of $\mathbb{R}^n$, and let $p : G \to [1,\infty)$ be a measurable function. We suppose that

$$1 \leqslant p_-(G) \leqslant p(y) \leqslant p_+(G) < \infty, \tag{2.1}$$

where $p_- := \operatorname{ess\,inf}_{y\in G} p(y)$, $p_+ := \operatorname{ess\,sup}_{y\in G} p(y)$.

(a) Lebesgue space with variable exponent $L^{p(\cdot)}(G)$ is the class of functions such that

$$L^{p(\cdot)}(G) = \left\{ f : \int_G \left(\frac{|f(y)|}{\gamma} \right)^{p(y)} dy < \infty, \text{ where } \gamma \text{ is a constant} \right\}.$$

Norm in $L^{p(\cdot)}(G)$ is defined as follows

$$\|f\|_{L^{p(\cdot)}(G)} := \inf \left\{ \gamma > 0 : \int_G \left(\frac{|f(y)|}{\gamma} \right)^{p(y)} dy \leqslant 1 \right\}.$$

(b) The space $L^{p(\cdot)}_{\text{loc}}(G)$ is the following class of functions:

$$L^{p(\cdot)}_{\text{loc}}(G) := \left\{ f : f \in L^{p(\cdot)}(K) \text{ for all compact subsets } K \subset G \right\}.$$

The Hardy-Littlewood maximal operator M for $f \in L^1_{\text{loc}}(G)$ is defined as

$$Mf(x) := \sup_{r>0} \frac{1}{r^n} \int\limits_{B(x,r)} |f(y)|dy \ (x \in G),$$

where $B(x,r)$ is the ball of radius r and center at x.

The set $\mathscr{P}(G)$ consists of all measurable functions $p(\cdot)$ satisfying the conditions that $p_- > 1$ and $p_+ < \infty$. The notion of $\mathscr{B}(G)$ denotes the collection of all $p(\cdot) \in \mathscr{P}(G)$ such that M is bounded on $L^{p(\cdot)}(G)$. Let's review the well-known log-Hölder

continuity condition (also known as the Dini-Lipschitz condition) for $p : G \mapsto (0,\infty)$: for any $x, y \in G$, where $|x-y| \leqslant \frac{1}{2}$, then we have a positive constant C such that

$$|p(x)-p(y)| \leqslant \frac{C}{-\ln|x-y|}. \tag{2.2}$$

Further, we say that $p(\cdot)$ satisfies the decay condition if there exists $p_\infty := p(\infty) = \lim\limits_{|x|\to\infty} p(x)$, and there is a positive constant $C_\infty > 0$ such that

$$|p(x)-p_\infty| \leqslant \frac{C_\infty}{\ln(e+|x|)}. \tag{2.3}$$

We will also need the log-Hölder continuity condition at 0 for $p(\cdot)$: There exists a constant $C_0 > 0$ such that, for all $|x| \leqslant \frac{1}{2}$,

$$|p(x)-p(0)| \leqslant \frac{C_0}{\ln|x|}. \tag{2.4}$$

The smallest admissible constant C in (2.2) (respectively, C_∞ in (2.3)) is referred to as the log-Hölder continuity constant (respectively, the log-Dini–Lipschitz decay constant) associated with the exponent $p(\cdot)$. By $\mathscr{P}^{\log} = \mathscr{P}^{\log}(G)$, we denote the functions $p(\cdot) \in \mathscr{P}(G)$ satisfying (2.1) and (2.2). Let $\mathscr{P}_\infty(G)$ (respectively, $\mathscr{P}_{0,\infty}(G)$) denote the subset of $\mathscr{P}(G)$ consisting of functions that satisfy condition (2.3) (respectively, both conditions (2.3) and (2.4)).

Now, we introduce the definition of the Herz space with variable exponent.

Let $\chi_t = \chi_{R_t}$, $R_t = B_t \setminus B_{t-1}$, and $B_t = B(0,2^t) = \{r \in \mathbb{R}^n : |r| < 2^t\}$ for all $t \in \mathbb{Z}$.

Definition 1. Let g be a measurable function, let w be a weight, and let $r(\cdot) \in \mathscr{P}(\mathbb{R}^n)$. The weighted variable exponent Lebesgue space $L^{r(\cdot)}(w)$ consists of all measurable functions g such that

$$g\, w^{\frac{1}{r(\cdot)}} \in L^{r(\cdot)}(\mathbb{R}^n).$$

The norm on $L^{r(\cdot)}(w)$ is defined by

$$\|g\|_{L^{r(\cdot)}(w)} := \left\| g\, w^{\frac{1}{r(\cdot)}} \right\|_{L^{r(\cdot)}}.$$

Definition 2. Let $\alpha(\cdot), r(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, $0 \leqslant \lambda < \infty$, and $0 < u < \infty$. The homogeneous weighted Herz–Morrey spaces with variable exponents, denoted by $M\dot{K}^{\alpha(\cdot),u}_{\lambda,r(\cdot)}(w)$, consist of all locally integrable functions $g \in L^{r(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}, w)$ such that

$$M\dot{K}^{\alpha(\cdot),u}_{\lambda,r(\cdot)}(w) := \left\{ g \in L^{r(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}, w) : \|g\|_{M\dot{K}^{\alpha(\cdot),u}_{\lambda,r(\cdot)}(w)} < \infty \right\}, \tag{2.5}$$

where the (quasi-)norm is defined by

$$\|g\|_{M\dot{K}^{\alpha(\cdot),u}_{\lambda,r(\cdot)}(w)} := \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} \left\| 2^{k\alpha(\cdot)} g \chi_k \right\|^u_{L^{r(\cdot)}(w)} \right)^{\frac{1}{u}}.$$

DEFINITION 3. Let $1 < s < \infty$ and let $s' = \frac{s}{s-1}$ be the conjugate exponent of s. Let w be a weight function. We say that $w \in A_s$ if there exists a constant $C > 0$ such that, for every cube $Q \subset \mathbb{R}^n$,

$$\left(\frac{1}{|Q|}\int_Q w(t)\,\mathrm{d}t\right)\left(\frac{1}{|Q|}\int_Q w(t)^{1-s'}\,\mathrm{d}t\right)^{s-1} \leqslant C < \infty.$$

For $s = 1$, we say that $w \in \alpha_1$ if there exists a constant $C > 0$ such that

$$Mw(t) \leqslant Cw(t) \quad \text{for a.e. } t \in \mathbb{R}^n,$$

where M denotes the Hardy–Littlewood maximal operator.

Finally, we define

$$A_\infty := \bigcup_{1 \leqslant s < \infty} A_s.$$

DEFINITION 4. A weight function w is said to belong to the variable exponent Muckenhoupt class $A_{p(\cdot)}$, where $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, if

$$\sup_{\substack{B \subset \mathbb{R}^n \\ B \text{ is a ball}}} \frac{1}{|B|} \left\| w^{\frac{1}{p(\cdot)}} \chi_B \right\|_{L^{p(\cdot)}} \left\| w^{-\frac{1}{p(\cdot)}} \chi_B \right\|_{L^{p'(\cdot)}} < \infty, \tag{2.6}$$

where $p'(\cdot)$ denotes the conjugate exponent of $p(\cdot)$ defined by

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \quad \text{for a.e. } x \in \mathbb{R}^n.$$

DEFINITION 5. A weight function w is said to belong to the class $A'_{p(\cdot)}$, where $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, if

$$\sup_{\substack{B \subset \mathbb{R}^n \\ B \text{ is a ball}}} |B|^{-p_B} \|w\chi_B\|_{L^1} \left\| w^{-1}\chi_B \right\|_{L^{p'(\cdot)/p(\cdot)}} < \infty, \tag{2.7}$$

where p_B is the harmonic average of $p(\cdot)$ over B, namely,

$$\frac{1}{p_B} = \frac{1}{|B|}\int_B \frac{1}{p(x)}\,dx.$$

The collection of all such weights is denoted by $A'_{s(\cdot)}$.

DEFINITION 6. Let $s_1(\cdot), s_2(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ satisfy

$$\frac{1}{s_1(\cdot)} - \frac{1}{s_2(\cdot)} = \frac{\gamma}{n}.$$

A weight function w is said to belong to the class $A(s_1(\cdot), s_2(\cdot))$ if, for every ball $B \subset \mathbb{R}^n$,

$$\|w\chi_B\|_{L^{s_2(\cdot)}} \left\| w^{-1}\chi_B \right\|_{L^{s'_1(\cdot)}} \leqslant |B|^{1-\frac{\gamma}{n}}, \tag{2.8}$$

where $s_1'(\cdot)$ denotes the conjugate exponent of $s_1(\cdot)$, defined by

$$\frac{1}{s_1(x)} + \frac{1}{s_1'(x)} = 1 \quad \text{for a.e. } x \in \mathbb{R}^n.$$

LEMMA 1. [1] *Let $f \in L^{r(\cdot)}(G)$ and $g \in L^{s(\cdot)}(G)$, and assume that the exponents satisfy*

$$\frac{1}{r'(\cdot)} = \frac{1}{r(\cdot)} + \frac{1}{s(\cdot)}.$$

Then there exists a constant $C > 0$ such that

$$\|fg\|_{L^{r'(\cdot)}(G)} \leqslant C \|f\|_{L^{r(\cdot)}(G)} \|g\|_{L^{s(\cdot)}(G)}.$$

LEMMA 2. [24] *Assume that $r(\cdot)$ satisfy the decay conditions are satisfied at the origin and at infinity. Then we have*

$$\left\| \chi_{B(0,2^{t+1}) \setminus B(0,2^t)} \right\|_{L^{r(\cdot)}(w)} \leqslant k_0 2^{\frac{tn}{p(0)}}, \quad 0 < t \leqslant 1, \tag{2.9}$$

and

$$\left\| \chi_{B(0,2^{t+1}) \setminus B(0,2^t)} \right\|_{L^{r(\cdot)}(w)} \leqslant k_\infty 2^{\frac{tn}{p_\infty}}, \quad t \geqslant 1, \tag{2.10}$$

where $k_0 \geqslant 1$ and $k_\infty \geqslant 1$ are constants independent of t.

DEFINITION 7. Let f is a locally integrable function on $\mathbb{R}^n$, $0 \leqslant \zeta(z) < n$. The n-dimensional fractional Hardy operators of variable order $\zeta(z)$ can be defined as

$$\mathscr{H} f(z) := \frac{1}{|z|^{n-\zeta(z)}} \int_{|x|<|z|} f(x)dx,$$

$$\mathscr{H}^* f(z) := \int_{|x| \geqslant |z|} \frac{f(x)}{|x|^{n-\zeta(z)}} dx, \quad z \in \mathbb{R}^n \setminus 0.$$

3. Main results

THEOREM 1. *Let $1 < u < \infty$, $1/q_1(x) - 1/q_2(x) = \zeta(x)/n$, $0 < \zeta(x) < n$, and $\alpha, q_2 \in \mathscr{P}_{0,\infty}(\mathbb{R}^n)$, such that*

$$\frac{-n}{q_{1\infty}} < \alpha_\infty < \frac{n}{q_{1\infty}'}, \quad \frac{-n}{q_1(0)} < \alpha(0) < \frac{n}{q_1'(0)}.$$

Then

$$\|(1+|x|)^{-\lambda(x)} \mathscr{H}(f)\|_{M\dot{K}^{\alpha(\cdot),u}_{\lambda,q_2(\cdot)}(w)} \leqslant C \|f\|_{M\dot{K}^{\alpha(\cdot),u}_{\lambda,q_1(\cdot)}(w)},$$

where $q(\cdot)$ is the variable Sobolev exponent defined by

$$\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\zeta(x)}{n},$$

and

$$\lambda(x) = C_\infty \zeta(x)\left(1 - \frac{\zeta(x)}{n}\right) \leqslant \frac{n}{4} C_\infty, \tag{3.1}$$

C_∞ is being the decay constant for $p(\cdot)$.

Proof. Let $f \in MK^{\alpha(\cdot),u}_{\lambda,q_2(\cdot)}(w)$, and $f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x)$, we have

$$\begin{aligned}
|\mathscr{H}(f)(x).\chi_\ell(x)| &\leqslant \frac{1}{|x|^{n-\zeta(x)}} \int_{R_\ell} |f(x)|dx.\chi_\ell(x) \\
&\leqslant C2^{-\ell n} \sum_{j=-\infty}^{\ell} \|f_j\|_{L^{q_1(\cdot)}(w)} \|\chi_j\|_{L^{q_1'(\cdot)}(w)}.|z|^{\zeta(x)}\chi_\ell(x).
\end{aligned}$$

It is known, see e.g. [32] that

$$\begin{aligned}
I^{\zeta(\cdot)}(\chi_\ell)(x) &\geqslant I^{\zeta(\cdot)}(\chi_\ell)(x).(\chi_\ell)(x) \\
&= \int_{R_\ell} \frac{1}{|x-y|^{\zeta(x)-n}} dy.\chi_\ell(x) \\
&\geqslant C|x|^{\zeta(x)}\chi_\ell(x).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&\|\chi_k(1+|x|)^{-\lambda(x)}\mathscr{H}(f)\|_{L^{q_2(\cdot)}(w)} \\
&\leqslant C2^{-\ell n} \sum_{j=-\infty}^{\ell} \|f_j\|_{L^{q_1(\cdot)}(w)} \|\chi_j\|_{L^{q_1'(\cdot)}(w)} \|(1+|x|)^{-\lambda(x)} I^{\zeta(\cdot)}(\chi_\ell)\|_{L^{q_2(\cdot)}(w)} \\
&\leqslant C2^{-\ell n} \sum_{j=-\infty}^{\ell} \|f_j\|_{L^{q_1(\cdot)}(w)} \|\chi_j\|_{L^{q_1'(\cdot)}(w)} \|(1+|x|)^{-\lambda(x)} I^{\zeta(\cdot)}(\chi_\ell w^{\frac{1}{q_2(\cdot)}})\|_{L^{q_2(\cdot)}} \\
&\leqslant C2^{-\ell n} \sum_{j=-\infty}^{\ell} \|f_j\|_{L^{q_1(\cdot)}(w)} \|\chi_j\|_{L^{q_1'(\cdot)}(w)} \|\chi_\ell w^{\frac{1}{q_2(\cdot)}}\|_{L^{q_1(\cdot)}} \\
&\leqslant C \sum_{j=-\infty}^{\ell} 2^{-\ell n} \|f_j\|_{L^{q_1(\cdot)}(w)} \|\chi_j\|_{L^{q_1'(\cdot)}(w)} \|\chi_\ell\|_{L^{q_1(\cdot)}(w)}.
\end{aligned}$$

$$\begin{aligned}
&\|(1+|x|)^{-\lambda(x)}\mathscr{H}(f)\|_{MK^{\alpha(\cdot),u}_{\lambda,q_2(\cdot)}(w)} \\
&= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{\ell=-\infty}^{k_0} \|2^{\ell\alpha(\cdot)}\chi_\ell(1+|x|)^{-\lambda(x)}\mathscr{H}(f)\|^u_{L^{q_2(\cdot)}(w)} \right)^{\frac{1}{u}}
\end{aligned}$$

$$\leqslant \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{\ell=-\infty}^{-1} 2^{\ell\alpha(0)u}\left(\sum_{j=-\infty}^{\ell} 2^{-\ell n}\|f_j\|_{L^{q_1(\cdot)}(w)}\|\chi_j\|_{L^{q_1'(\cdot)}(w)}\|\chi_\ell\|_{L^{q_1(\cdot)}(w)}\right)^u\right)^{\frac{1}{u}}$$

$$+ \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{\ell=-0}^{k_0} 2^{\ell\alpha_\infty u}\left(\sum_{j=-\infty}^{\ell} 2^{-\ell n}\|f_j\|_{L^{q_1(\cdot)}(w)}\|\chi_j\|_{L^{q_1'(\cdot)}(w)}\|\chi_\ell\|_{L^{q_1(\cdot)}(w)}\right)^u\right)^{\frac{1}{u}}$$

$$=: E_1 + E_2.$$

Now we will find the estimate for E_1.

By the lemma 2, we get

$$2^{-\ell n}\|\chi_j\|_{L^{q_1'(\cdot)}(w)}\|\chi_\ell\|_{L^{q_1(\cdot)}(w)} \leqslant C2^{-\ell n}2^{\frac{\ell n}{q_1(0)}}2^{\frac{jn}{q_1'(0)}} \leqslant C2^{\frac{(j-\ell)n}{q_1'(0)}}. \tag{3.2}$$

Applying above results to E_1 to get

$$E_1 \leqslant \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{\ell=-\infty}^{-1} 2^{\ell\alpha(\cdot)u}\left(\sum_{j=-\infty}^{\ell} 2^{-\ell n}\|f_j\|_{L^{q_1(\cdot)}(w)}\|\chi_j\|_{L^{q_1'(\cdot)}(w)}\|\chi_\ell\|_{L^{q_1(\cdot)}(w)}\right)^u\right)^{\frac{1}{u}}$$

$$\leqslant C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{\ell=-\infty}^{-1} 2^{\ell\alpha(0)u}\left(\sum_{j=-\infty}^{\ell} 2^{\frac{(j-\ell)n}{q_1'(0)}}\|f_j\|_{L^{q_1(\cdot)}(w)}\right)^u\right)^{\frac{1}{u}}.$$

Let $b := \frac{n}{q_1'(0)} - \alpha(0)$, applying the Hölder's inequality and Fubini's theorem to get,

$$E_1 \leqslant C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{\ell=-\infty}^{-1}\left(\sum_{j=-\infty}^{\ell} 2^{\alpha(0)j}\|f_j\|_{L^{q_1(\cdot)}(w)}2^{b(j-\ell)}\right)^u\right)^{\frac{1}{u}}$$

$$\leqslant C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{\ell=-\infty}^{-1}\left(\sum_{j=-\infty}^{\ell} 2^{\alpha(0)j}\|f_j\|_{L^{q_1(\cdot)}(w)}2^{b(j-\ell)}\right)^u\right)^{\frac{1}{u}}$$

$$\leqslant C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\sum_{\ell=-\infty}^{-1}\left(\sum_{j=-\infty}^{\ell} 2^{\alpha(0)uj}\|f_j\|^u_{L^{q_1(\cdot)}(w)}2^{bu(j-\ell)/2}\right)\right.$$

$$\left.\times\left(\sum_{j=-\infty}^{\ell} 2^{bu'(j-\ell)/2}\right)^{\frac{u}{u'}}\right]^{\frac{1}{u}}$$

$$\leqslant C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\sum_{\ell=-\infty}^{-1}\sum_{j=-\infty}^{\ell} 2^{\alpha(0)uj}\|f_j\|^u_{L^{q_1(\cdot)}(w)}2^{bu(j-\ell)/2}\right]^{\frac{1}{u}}$$

$$\leqslant C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\sum_{j=-\infty}^{-1} 2^{\alpha(0)uj}\|f_j\|^u_{L^{q_1(\cdot)}(w)}\sum_{\ell=j}^{-1}2^{bu(j-\ell)/2}\right]^{\frac{1}{u}}$$

$$\leqslant C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{j=-\infty}^{k_0} \left\| 2^{\alpha(\cdot)j} f_j \right\|^u_{L^{q_1(\cdot)}(w)} \right)^{\frac{1}{u}}$$
$$\leqslant C\|f\|_{M\dot{K}^{\alpha(\cdot),u}_{\lambda,q_1(\cdot)}(w)}.$$

Now for E_2, using Minkowski's inequality we have

$$E_2 \leqslant \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{\ell=0}^{k_0} 2^{\ell\alpha_\infty u} \left(\sum_{j=-\infty}^{\ell} \|\chi_\ell (1+|x|)^{-\lambda(x)} \mathscr{H}(f_j)\|_{L^{q_2(\cdot)}(w)} \right)^u \right)^{\frac{1}{u}}$$
$$\leqslant \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{\ell=0}^{k_0} 2^{\ell\alpha_\infty u} \left(\sum_{j=-\infty}^{-1} \|\chi_\ell (1+|x|)^{-\lambda(x)} \mathscr{H}(f_j)\|_{L^{q_2(\cdot)}(w)} \right)^u \right)^{\frac{1}{u}}$$
$$+ \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{\ell=0}^{k_0} 2^{\ell\alpha_\infty u} \left(\sum_{j=0}^{\ell} \|\chi_\ell (1+|x|)^{-\lambda(x)} \mathscr{H}(f_j)\|_{L^{q_2(\cdot)}(w)} \right)^u \right)^{\frac{1}{u}}$$
$$:= A_1 + A_2.$$

As we can easily find the estimate for A_2 in a similar manner to E_1. We will replace $q_1'(0)$ with $q_{1\infty}'$ and by virtue of the fact $b := \frac{n}{q_{1\infty}'} - \alpha_\infty > 0$ to get our desired results. For A_1 we have

$$2^{-\ell n}\|\chi_\ell\|_{L^{q_1(\cdot)}(w)}\|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \leqslant C2^{-\ell n} 2^{\frac{\ell n}{q_{1\infty}}} 2^{\frac{jn}{q_1'(0)}} \leqslant C2^{\frac{-\ell n}{q_{1\infty}}} 2^{\frac{jn}{q_1'(0)}}. \tag{3.3}$$

As $\alpha_\infty - \frac{n}{q_{1\infty}'} < 0$ we have

$$A_1 \leqslant \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{\ell=0}^{k_0} 2^{\ell\alpha_\infty u} \left(\sum_{j=-\infty}^{-1} \|\chi_\ell (1+|x|)^{-\lambda(x)} \mathscr{H}(f_j)\|_{L^{q_2(\cdot)}(w)} \right)^u \right)^{\frac{1}{u}}$$
$$\leqslant C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left[\sum_{\ell=0}^{k_0} 2^{\ell\alpha_\infty u} \left(\sum_{j=-\infty}^{-1} 2^{-\ell n} 2^{\frac{\ell n}{q_{1\infty}}} 2^{\frac{jn}{q_1'(0)}} \|f_j\|_{L^{q_1(\cdot)}(w)} \right)^u \right]^{\frac{1}{u}}$$
$$\leqslant C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left[\sum_{\ell=0}^{k_0} 2^{\frac{\ell\alpha_\infty u - \ell n u}{q_{1\infty}'}} \left(\sum_{j=-\infty}^{-1} 2^{\frac{jn}{q_1'(0)}} \|f_j\|_{L^{q_1(\cdot)}(w)} \right)^u \right]^{\frac{1}{u}}$$
$$\leqslant C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\left(\sum_{j=-\infty}^{-1} 2^{\frac{jn}{q_1'(0)}} \|f_j\|_{L^{q_1(\cdot)}(w)} \right)^u \right)^{\frac{1}{u}}$$
$$\leqslant C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\left(\sum_{j=-\infty}^{-1} 2^{\alpha(0)j} \|f_j\|_{L^{q_1(\cdot)}(w)} 2^{\frac{jn}{q_1'(0)} - \alpha(0)j} \right)^u \right)^{\frac{1}{u}}.$$

Now by applying Hölder's inequality and using the fact that $\frac{n}{q'(0)} - \alpha(0) > 0$ we have

$$\begin{aligned} A_1 &\leqslant C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left[\left(\sum_{j=-\infty}^{-1} 2^{\alpha(0)ju} \|f_j\|^u_{L^{q_1(\cdot)}(w)} \right) \left(\sum_{j=-\infty}^{-1} 2^{(\frac{jn}{q'(0)} - \alpha(0)j)u'} \right)^{\frac{u}{u'}} \right]^{\frac{1}{u}} \\ &\leqslant C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\left(\sum_{j=-\infty}^{k_0} \|2^{\alpha(\cdot)j} f_j\|^u_{L^{q_1(\cdot)}(w)} \right) \right)^{\frac{1}{u}} \\ &\leqslant C\|f\|_{M\dot{K}^{\alpha(\cdot),u}_{\lambda,q_1(\cdot)}(w)}. \end{aligned}$$

Combining these estimates we get

$$\|(1+|x|)^{-\lambda(x)}\mathscr{H}(f)\|_{M\dot{K}^{\alpha(\cdot),u}_{\lambda,q_2(\cdot)}(w)} \leqslant C\|f\|_{M\dot{K}^{\alpha(\cdot),u}_{\lambda,q_1(\cdot)}(w)}. \quad \square$$

THEOREM 2. *Let* $1 < u < \infty$, $1/q_1(x) - 1/q_2(x) = \zeta(x)/n$, $0 < \zeta(x) < n$, *and* $a, q_2 \in \mathscr{P}_{0,\infty}(\mathbb{R}^n)$, *such that*

$$\frac{-n}{q_{1\infty}} < \alpha_\infty < \frac{n}{q'_{1\infty}}, \quad \frac{-n}{q_1(0)} < \alpha(0) < \frac{n}{q'_1(0)}.$$

Then

$$\|(1+|x|)^{-\lambda(x)}\mathscr{H}^*(f)\|_{M\dot{K}^{\alpha(\cdot),u}_{\lambda,q_2(\cdot)}(w)} \leqslant C\|f\|_{M\dot{K}^{\alpha(\cdot),u}_{\lambda,q_1(\cdot)}(w)},$$

where $q(\cdot)$ *is the variable Sobolev exponent defined by*

$$\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\zeta(x)}{n},$$

and

$$\lambda(x) = C_\infty \zeta(x) \left(1 - \frac{\zeta(x)}{n}\right) \leqslant \frac{n}{4} C_\infty, \tag{3.4}$$

C_∞ *is being the decay constant for* $p(\cdot)$.

Proof. Let $f \in M\dot{K}^{\alpha(\cdot),u}_{\lambda,q_2(\cdot)(w)}$, and $f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x)$, we have

$$\begin{aligned} &|(1+|x|)^{-\lambda(x)}\mathscr{H}^*(f)(x).\chi_\ell(x)| \\ &\leqslant \int_{\mathbb{R}^n\setminus R_\ell} \frac{1}{|x|^{n-\zeta(x)}} |f(x)|(1+|x|)^{-\lambda(x)}\chi_\ell(x)dx \\ &\leqslant C \sum_{j=\ell+1}^{\infty} \|f_j\|_{L^{q_1(\cdot)}(w)} \|(1+|x|)^{-\lambda(x)}| \cdot |^{\zeta(x)-n}\chi_j\|_{L^{q'_1(\cdot)}(w)}\chi_\ell(x). \end{aligned}$$

It is known, see e.g. [32] that

$$\begin{aligned} I^{\zeta(\cdot)}(\chi_j)(x) &\geqslant I^{\zeta(\cdot)}(\chi_j)(x).(\chi_j)(x) \\ &= \int_{R_j} \frac{1}{|x-y|^{\zeta(x)-n}} dy.\chi_j(x) \\ &\geqslant C|x|^{\zeta(x)}\chi_j(x). \end{aligned}$$

Consequently, we have

$$\begin{aligned} &\|\chi_\ell(1+|x|)^{-\lambda(x)}\mathscr{H}^*(f)\|_{L^{q_2(\cdot)}(w)} \\ &\leqslant C\sum_{j=\ell+1}^{\infty}\|f_j\|_{L^{q_1(\cdot)}(w)}\|(1+|x|)^{-\lambda(x)}|\cdot|^{\zeta(x)-n}\chi_j\|_{L^{q_1'(\cdot)}(w)}\|\chi_\ell\|_{L^{q_2(\cdot)}(w)} \\ &\leqslant C\sum_{j=\ell+1}^{\infty}2^{-jn}\|f_j\|_{L^{q_1(\cdot)}(w)}\|(1+|x|)^{-\lambda(x)}|x|^{\zeta(x)}\chi_j\|_{L^{q_1'(\cdot)}(w)}\|\chi_\ell\|_{L^{q_2(\cdot)}(w)} \\ &\leqslant C2^{-jn}\sum_{j=\ell+1}^{\infty}\|f_j\|_{L^{q_1(\cdot)}(w)}\|\chi_j\|_{L^{q_2'(\cdot)}(w)}\|\chi_\ell\|_{L^{q_2(\cdot)}(w)}. \end{aligned}$$

$$\begin{aligned} &\|(1+|x|)^{-\lambda(x)}\mathscr{H}^*(f)\|_{M\dot{K}^{\alpha(\cdot),u}_{\lambda,q_2(\cdot)}(w)} \\ &= \sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\sum_{\ell=-\infty}^{k_0}\|2^{\ell\alpha(\cdot)}\chi_\ell(1+|x|)^{-\lambda(x)}\mathscr{H}^*(f)\|^u_{L^{q_2(\cdot)}(w)}\right)^{\frac{1}{u}} \\ &\leqslant \sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\sum_{\ell=-\infty}^{-1}2^{\ell\alpha(0)u}\left(\sum_{j=\ell+1}^{\infty}2^{-jn}\|f_j\|_{L^{q_1(\cdot)}(w)}\|\chi_j\|_{L^{q_2'(\cdot)}(w)}\|\chi_\ell\|_{L^{q_2(\cdot)}(w)}\right)^u\right)^{\frac{1}{u}} \\ &\quad+\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\sum_{\ell=0}^{k_0}2^{\ell\alpha_\infty u}\left(\sum_{j=\ell+1}^{\infty}2^{-jn}\|f_j\|_{L^{q_1(\cdot)}(w)}\|\chi_j\|_{L^{q_2'(\cdot)}(w)}\|\chi_\ell\|_{L^{q_2(\cdot)}(w)}\right)^u\right)^{\frac{1}{u}} \\ &=: E_1+E_2. \end{aligned}$$

We will find estimate of E_2,

$$2^{-jn}\|\chi_\ell\|_{L^{q_2(\cdot)}(w)}\|\chi_j\|_{L^{q_2'(\cdot)}(w} \leqslant C2^{-jn}2^{\frac{\ell n}{q_{2\infty}}}2^{\frac{jn}{q_{2\infty}}} \leqslant C2^{\frac{(\ell-j)n}{q_{2\infty}}}. \tag{3.5}$$

$$\begin{aligned} E_2 &\leqslant C\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\sum_{\ell=0}^{\infty}2^{\ell\alpha_\infty u}\left(\sum_{j=\ell+1}^{\infty}\|\chi_\ell(1+|x|)^{-\lambda(x)}\mathscr{H}^*(f_j)\|_{L^{q_2(\cdot)}(w)}\right)^u\right)^{\frac{1}{u}} \\ &\leqslant C\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\sum_{\ell=0}^{k_0}\left(\sum_{j=\ell+1}^{\infty}2^{\alpha_\infty j}\|f_j\|_{L^{q_1(\cdot)}(w)}2^{d(\ell-j)}\right)^u\right)^{\frac{1}{u}}, \end{aligned}$$

where $d = \frac{n}{q_{2\infty}} + \alpha_\infty > 0$. Then by virtue of the well known Hölder's theorem yields

$$\begin{aligned}
E_2 &\leqslant C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left[\sum_{\ell=0}^{k_0} \left(\sum_{j=\ell+1}^{\infty} 2^{\alpha_\infty u j} \|f_j\|^u_{L^{q_1(\cdot)}(w)} 2^{du(\ell-j)/2} \right) \right. \\
&\qquad \left. \times \left(\sum_{j=\ell+1}^{\infty} 2^{du'(\ell-j)/2} \right)^{\frac{u}{u'}} \right]^{\frac{1}{u}} \\
&\leqslant C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left[\sum_{\ell=0}^{k_0} \sum_{j=\ell+1}^{\infty} 2^{\alpha_\infty u j} \|f_j\|^u_{L^{q_1(\cdot)}(w)} 2^{du(\ell-j)/2} \right]^{\frac{1}{u}} \\
&\leqslant C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{j=0}^{\infty} 2^{\alpha_\infty u j} \|f_j\|^u_{L^{q_1(\cdot)}(w)} \sum_{\ell=0}^{j-1} 2^{du(\ell-j)/2} \right)^{\frac{1}{u}} \\
&\leqslant C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{j=0}^{\infty} \sum_{l=-\infty}^{j} 2^{\alpha_\infty u l} \|f_l\|^u_{L^{q_1(\cdot)}(w)} \sum_{\ell=0}^{j-1} 2^{du(\ell-j)/2} \right)^{\frac{1}{u}} \\
&\leqslant C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{j=0}^{\infty} \sum_{\ell=0}^{j-1} 2^{du(\ell-j)/2} \right)^{\frac{1}{u}} \|f\|_{M\dot{K}^{\alpha(\cdot),u}_{\lambda,q_1(\cdot)}(w)} \\
&\leqslant C \|f\|_{M\dot{K}^{\alpha(\cdot),u}_{\lambda,q_1(\cdot)}(w)}.
\end{aligned}$$

For E_1 by using Minkowski's inequality

$$\begin{aligned}
E_1 &\leqslant \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{\ell=-\infty}^{-1} 2^{\ell\alpha(0)u} \left(\sum_{j=\ell+1}^{\infty} \|\chi_\ell (1+|x|)^{-\lambda(x)} \mathscr{H}^*(f_j)\|_{L^{q_2(\cdot)}(w)} \right)^u \right)^{\frac{1}{u}} \\
E_1 &\leqslant \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{\ell=-\infty}^{-1} 2^{\ell\alpha(0)u} \left(\sum_{j=\ell+1}^{-1} \|\chi_\ell (1+|x|)^{-\lambda(x)} \mathscr{H}^*(f_j)\|_{L^{q_2(\cdot)}(w)} \right)^u \right)^{\frac{1}{u}} \\
&\quad + \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{\ell=-\infty}^{-1} 2^{\ell\alpha(0)u} \left(\sum_{j=0}^{\infty} \|\chi_\ell (1+|x|)^{-\lambda(x)} \mathscr{H}^*(f_j)\|_{L^{q_2(\cdot)}(w)} \right)^u \right)^{\frac{1}{u}} \\
&:= B_1 + B_2.
\end{aligned}$$

The estimate of B_1 can be obtained in a similar way to E_2 by replacing $q_{2\infty}$ with $q_2(0)$ and using the inequality $\frac{n}{q_2(0)} + \alpha(0) > 0$, $\frac{n}{q_{2\infty}} + \alpha_\infty > 0$. For B_2 we have

$$2^{-jn} \|\chi_\ell\|_{L^{q_1(\cdot)}(w)} \|\chi_j\|_{L^{q_2'(\cdot)}(w} \leqslant C 2^{-jn} 2^{\frac{\ell n}{q_2(0)}} 2^{\frac{-jn}{q_{2\infty}}} \leqslant C 2^{\frac{\ell n}{q_2(0)}} 2^{\frac{-jn}{q_{2\infty}}}, \tag{3.6}$$

$$
\begin{aligned}
B_2 &\leqslant C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{\ell=-\infty}^{-1} 2^{\ell\alpha(0)u}\left(\sum_{j=0}^{\infty}\|\chi_\ell(1+|x|)^{-\lambda(x)}\mathscr{H}^*(f_j)\|_{L^{q_2(\cdot)}(w)}\right)^u\right)^{\frac{1}{u}}\\
&\leqslant C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{\ell=-\infty}^{-1} 2^{\ell\alpha(0)u}\left(\sum_{j=0}^{\infty}2^{-jn}2^{\frac{\ell n}{q_2(0)}}2^{\frac{jn}{q_{2\infty}}}\|f_j\|_{L^{q_1(\cdot)}(w)}\right)^u\right)^{\frac{1}{u}}\\
&\leqslant C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{\ell=-\infty}^{-1} 2^{\ell\alpha(0)u}\left(\sum_{j=0}^{\infty}2^{\frac{\ell n}{q_2(0)}}2^{\frac{-jn}{q_{2\infty}}}\|f_j\|_{L^{q_1(\cdot)}(w)}\right)^u\right)^{\frac{1}{u}}\\
&\leqslant C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{\ell=-\infty}^{-1} 2^{\ell(\alpha(0)+n)/q_2(0)u}\left(\sum_{j=0}^{\infty}2^{\frac{-jn}{q_{2\infty}}}\|f_j\|_{L^{q_1(\cdot)}(w)}\right)^u\right)^{\frac{1}{u}}\\
&\leqslant C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\left(\sum_{j=0}^{\infty}2^{\alpha_\infty j}\|f_j\|_{L^{q_1(\cdot)}(w)}2^{j(n/q_{2\infty}+\alpha_\infty)}\right)^u\right)^{\frac{1}{u}}\\
&\leqslant C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\left(\sum_{j=0}^{\infty}\sum_{l=-\infty}^{j}2^{\alpha_\infty l}\|f_l\|_{L^{q_1(\cdot)}(w)}2^{j(n/q_{2\infty}+\alpha_\infty)}\right)^u\right)^{\frac{1}{u}}\\
&\leqslant C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\left(\sum_{j=0}^{\infty}2^{j(n/q_{2\infty}+\alpha_\infty)}\right)^u\right)^{\frac{1}{u}}\|f\|_{M\dot{K}^{\alpha(\cdot),u}_{\lambda,q_1(\cdot)}(w)}\\
&\leqslant C\|f\|_{M\dot{K}^{\alpha(\cdot),u}_{\lambda,q_1(\cdot)}(w)}.
\end{aligned}
$$

Combining the estimates for E_1 and E_2 yields

$$\|(1+|x|)^{-\lambda(x)}\mathscr{H}^*(f)\|_{M\dot{K}^{\alpha(\cdot),u}_{\lambda,q_2(\cdot)}(w)} \leqslant C\|f\|_{M\dot{K}^{\alpha(\cdot),u}_{\lambda,q_1(\cdot)}(w)},$$

which completes the proof. □

Data availability. No data were used to support this study.

Conflicts of interest. The authors declare no conflict of interest.

Funding. This work did not receive any external funding.

Authors contribution. Contributions from all authors were equal and significant. The original manuscript was read and approved by all authors.

REFERENCES

[1] L. Diening and A. Nekvinda, *Muckenhoupt weights in variable exponent spaces*, Math. Inequal. Appl. **11** (2008), no. 2, 235–245.

[2] A. Hussain, I. Khan and A. Mohamed, *Variable Herz-Morrey estimates for rough fractional Hausdorff operator*, J. Inequal. Appl. **33** (2024).

[3] M. Izuki, *Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization*, Analysis Mathematica **36** (2010), no. 1, 33–50.

[4] M. Izuki, *Boundedness of vector-valued sublinear operators on Herz–Morrey spaces with variable exponent*, Mathematical Sciences Research Journal **13** (2009), no. 10, 243–253.

[5] M. Izuki and T. Noi, *An intrinsic square function on weighted Herz spaces with variable exponents*, J. Math. Inequal. **11** (2017), no. 3, 799–816.

[6] M. Izuki and T. Noi, *Boundedness of fractional integrals on weighted Herz spaces with variable exponent*, J. Inequal. Appl. (2016), Article ID 1142, 1–15.

[7] V. Kokilashvili, A. Meskhi, H. Rafeiro and S. Samko, *Integral Operators in Non-Standard Function Spaces. Vol. 1: Variable Exponent Lebesgue and Amalgam Spaces*, Oper. Theory Adv. Appl. **248**, Birkhäuser/Springer, Cham, 2016.

[8] V. Kokilashvili, A. Meskhi, H. Rafeiro and S. Samko, *Integral Operators in Non-Standard Function Spaces. Vol. 2: Variable Exponent Hölder, Morrey–Campanato and Grand Spaces*, Oper. Theory Adv. Appl. **249**, Birkhäuser/Springer, Cham, 2016.

[9] V. Kokilashvili and S. Samko, *On Sobolev theorem for Riesz-type potentials in the Lebesgue spaces with variable exponent*, Z. Anal. Anwend. **22** (2003), 899–910.

[10] A. Meskhi, *Maximal functions, potentials and singular integrals in grand Morrey spaces*, Complex Var. Elliptic Equ. **55** (2010), 1007–1019.

[11] B. Sultan, *Atomic decomposition of anisotropic grand variable weighted Herz spaces and applications*, J. Pseudo-Differ. Oper. Appl. **16** (2025), 53.

[12] B. Sultan, *Atomic decomposition on grand Herz–Hardy spaces on locally compact Vilenkin groups*, Filomat **40** (2026), no. 4, 1301–1309.

[13] B. Sultan, F. Azmi, M. Sultan, M. Mehmood and N. Mlaiki, *Boundedness of Riesz potential operator on grand Herz–Morrey spaces*, Axioms **11** (2022), no. 11, 583.

[14] B. Sultan, F. Azmi, M. Sultan, T. Mahmood, N. Mlaiki and N. Souayah, *Boundedness of fractional integrals on grand weighted Herz–Morrey spaces with variable exponent*, Fractal Fract. **6** (2022), no. 11, 660–670.

[15] B. Sultan and A. Hussain, *Sobolev embeddings in grand variable Herz-Morrey Besov spaces*, Bound. Value Probl. **2026** (2026), Article 11.

[16] B. Sultan, A. Hussain and M. Sultan, *Characterization of generalized Campanato spaces with variable exponents via fractional integrals*, J. Pseudo-Differ. Oper. Appl. **16** (2025), 22.

[17] B. Sultan and M. Sultan, *Boundedness of commutators of rough Hardy operators on grand variable Herz spaces*, Forum Math. **36** (2024), no. 3, 717–733.

[18] B. Sultan and M. Sultan, *Boundedness of higher order commutators of fractional variable rough Hardy operators on grand variable Herz spaces*, J. Classical Anal. **26** (2025), no. 1, 1–19.

[19] B. Sultan and M. Sultan, *Boundedness of higher order commutators of Hardy operators on grand Herz–Morrey spaces*, Bull. Sci. Math. **190** (2024), 103373.

[20] B. Sultan and M. Sultan, *Sobolev-type theorem for commutators of Hardy operators in grand Herz spaces*, Ukr. Math. J. (2024).

[21] B. Sultan, M. Sultan and A. Hussain, *Boundedness of the Bochner–Riesz operators on the weighted Herz–Morrey type Hardy spaces*, Complex Anal. Oper. Theory **19** (2025), 49.

[22] B. Sultan, M. Sultan and I. Khan, *On Sobolev theorem for higher commutators of fractional integrals in grand variable Herz spaces*, Commun. Nonlinear Sci. Numer. Simul. **126** (2023).

[23] B. Sultan, M. Sultan, M. Mehmood, F. Azmi, M. A. Alghafli and N. Mlaiki, *Boundedness of fractional integrals on grand weighted Herz spaces with variable exponent*, AIMS Math. **8** (2023), no. 1, 752–764.

[24] B. Sultan, M. Sultan, Q. Q. Zhang, N. Mlaiki, *Boundedness of Hardy operators on grand variable weighted Herz spaces*, AIMS Math. **8** (2023), no. 10, 24515–24527.

[25] M. Sultan and B. Sultan, *A note on the boundedness of higher order commutators on fractional integrals in grand variable Herz–Morrey spaces*, Kragujevac J. Math. **50** (2026), no. 7, 1063–1080.

[26] M. SULTAN AND B. SULTAN, *A note on the boundedness of the Marcinkiewicz integral operator on continual Herz–Morrey spaces*, Filomat **39** (2025), no. 6, 2017–2027.
[27] M. SULTAN AND B. SULTAN, *Boundedness of sublinear operators on grand central Orlicz–Morrey spaces*, Bull. Sci. Math. **205** (2025), 103704.
[28] M. SULTAN AND B. SULTAN, *λ -Central Musielak–Orlicz–Morrey spaces*, Arab. J. Math. **14** (2025), 357–363.
[29] M. SULTAN, B. SULTAN, A. ALOQAILY AND N. MLAIKI, *Boundedness of some operators on grand Herz spaces with variable exponent*, AIMS Math. **8** (2023), no. 6, 12964–12985.
[30] M. SULTAN, B. SULTAN AND R. E. CASTILLO, *Weighted composition operator on Gamma spaces with variable exponent*, J. Pseudo-Differ. Oper. Appl. **15** (2024), 46.
[31] L. WANG, *Parametrized Littlewood–Paley operators on grand variable Herz spaces*, Ann. Funct. Anal. **13** (2022), 72.
[32] J. L. WU AND W. J. ZHAO, *Boundedness for fractional Hardy-type operator on variable-exponent Herz–Morrey spaces*, Kyoto J. Math. **56** (2016), no. 4, 831–845.
[33] J. YOUNAS, A. HUSSAIN, H. ALHAZMI, A. F. ALJOHANI, I. KHAN, BMO *estimates for commutators of the rough fractional Hausdorff operator on grand-variable-Herz-Morrey spaces*, AIMS Math. **9** (2024), no. 9, 23434–23448.

(Received February 2, 2024)

Mehvish Sultan
Department of Mathematics
Quaid-I-Azam University
Islamabad 45320, Pakistan
e-mail: mehvishsultanbaz@gmail.com

Babar Sultan
Department of Mathematics
Quaid-I-Azam University
Islamabad 45320, Pakistan
e-mail: babarsultan40@yahoo.com

René Erlin Castillo
Departamento de Matemáticas
Universidad Nacional de Colombia
Carrera 30, calle 45, Bogotá, Colombia
e-mail: recastillo@unal.edu.co