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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR NONLINEAR FRACTIONAL CAPUTO–LANGEVIN EQUATION WITH ANTI–PERIODIC BOUNDARY VALUE PROBLEM

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Abstract. In this paper, we consider a nonlinear fractional Caputo-Langevin equation with anti-periodic boundary value problem. Existence results and uniqueness are obtained by using Schauder fixed point theorem and Banach contraction mapping principle. As applications, two examples are offered to illustrate our main results.

1. Introduction

Fractional-order differential equations have attracted the attention of many researchers because of their applications in many scientific and engineering disciplines. Quantum calculus was first studied by Jackson [10, 11] which roughly speaking, is ordinary calculus without limits.

Motivated by these applications of q-calculus, many researchers have developed the theory of quantum calculus based on two parameter p and q. In 1991, Chakrabarti and Jagannathan first investigated the (p,q)-calculus (post quantum calculus) in quantum algebras [4] in order to generalize or unify several forms of q-oscillator algebras well-known in the physics literature. One can find some useful information about the (p,q)-calculus in [3, 12, 16, 17, 25].

On the other hand, the Langevin equation (first formulated by Langevin in 1908) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [5]. For some new developments on the fractional Langevin equation, see, for example, [2, 6, 14, 24].

In [21], Sitho, et al. studied the existence of solutions for quantum difference Langevin equation with multi-quantum numbers q-derivatives nonlocal conditions of the form

$$\begin{cases} D_q(D_q+\lambda)x(t) = f(t,x(t)), & t \in [0,T], \\ x(0) = \alpha D_p x(0), & \sum_{i=1}^{m} \beta_i D_{r_i} x(\xi_i) = \gamma, \end{cases}$$

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where quantum numbers $0<p,q,r_i<1$, $\lambda,\alpha,\beta_i,\gamma\in\mathbb{R}$, $i=1,2,\ldots,m$ are given constants, $f\in C([0,T]\times\mathbb{R},\mathbb{R})$ and $0<\xi_1<\ldots<\xi_m<T$. Existence and uniqueness results are proved by using fixed point theorems.

In [1], Ahmad, et al. introduce a q-variant of the Langevin equation with two different fractional orders supplemented with separated boundary conditions. Precisely, they consider the following problem

$$\begin{cases} {}^cD_q^{\alpha}({}^cD_q^{\beta}+\lambda)u(t)=pf(t,u(t))+kI_q^{\xi}g(t,u(t)), & t\in J=[0,1], \quad 0<q<1,\\ a_1u(0)-b_1(t^{(1-\beta)}D_qu(0))|_{t=0}=\delta_1u(\eta_1),\\ a_2u(1)+b_2D_qu(1)=\delta_2u(\eta_2), \end{cases}$$

where $0<\beta,\alpha<1$, ${}^cD_q^{\alpha}$ and ${}^cD_q^{\beta}$ denote the fractional q-derivative of the Caputo type, I_q^{ξ} denotes Riemann-Liouville integral with $0<\xi<1$, f,g are given continuous functions, λ and p,k are real constants and a_i, $b_i,\delta_i\in\mathbb{R}$, $0<\eta_i<1$, $i=1,2$. Their results are based on some classical fixed point theorems.

In 2021, Qin, et al. [18] studied positive solutions for fractional (p,q)-difference boundary value problems given by the following

$$\begin{cases} {}^cD_{p,q}^{\alpha}u(t)+f(p^{\alpha}t,u(p^{\alpha}t))=0, & t\in[0,1], \ 1<\alpha\leqslant 2,\\ u(0)=u(1)=0, \end{cases}$$

where $f\in C([0,1]\times\mathbb{R},\mathbb{R})$ is a non-negative continuous function, $0<q<p\leqslant 1$, $D_{p,q}^{\alpha}$ is a Riemann-Liouville-type fractional (p,q)-difference operator.

Moreover, Neang, et al. [15] considered the nonlocal boundary value problem of nonlinear fractional (p,q)-difference equations with taking care of solutions of existence and uniqueness results obtained by

$$\begin{cases} {}^cD_{p,q}^{\alpha}u(t)=f(t,u(p^{\alpha}t)), & t\in[0,T/p^{\alpha}], \ 1<\alpha\leqslant 2,\\ \beta_1u(0)+\gamma_1D_{p,q}u(0)=\xi_1u(\eta_1),\\ \beta_2u(T/p)+\gamma_2D_{p,q}u(T/p)=\xi_2u(\eta_2) \end{cases}$$

where $f\in C([0,T/p^{\alpha}]\times\mathbb{R},\mathbb{R})$, β_i, γ_i, $\xi_i,(i=1,2)$ are constants, ${}^cD_{p,q}^{\alpha}$ denoted by Caputo fractional (p,q) type, while $D_{p,q}$ denoted by first-order (p,q)-derivative.

Inspired by the above literature, in this paper, we investigate the existence and uniqueness of solutions for the following fractional (p,q)-differential equation with anti-periodic boundary conditions. Let $J=[0,T/p^{\alpha+\beta}]\subset\mathbb{R}$, where $T>0$. Here, J denotes a compact real interval serving as the domain of the independent variable t. Although the (p,q)-difference and (p,q)-integral operators involve a discrete geometric structure through their corresponding scaled points, the domain J itself is the usual continuous real interval. We assume that $0<\alpha,\beta\leqslant 1$, $1<\alpha+\beta\leqslant 2$, $0<q<p\leqslant 1$. Then, the problem under consideration is

$$\begin{cases} {}^cD_{p,q}^{\alpha}({}^cD_{p,q}^{\beta}+\lambda)x(t)=f(t,x(p^{\alpha}t)), & t\in J=[0,T/p^{\alpha+\beta}],\\ x(0)+x(T/p^{\alpha+\beta})=0,\\ {}^cD_{p,q}^{\beta}x(0)+{}^cD_{p,q}^{\beta}x(T/p^{\alpha+\beta})=0, \end{cases} \tag{1.1}$$

Here, ${}^{c}D^{\alpha}_{p,q}$ and ${}^{c}D^{\beta}_{p,q}$ denote the Caputo-type fractional (p,q)-difference operators, and $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

The rest of this paper is organized as follows: In Section 2, we recall some useful definitions and properties. In Section 3, we give some sufficient conditions for the existence of solutions for boundary value problem (1.1). Two examples are also provided to illustrate our main results in Section 4.

2. Preliminaries

In this part, some fundamental results and definitions of the (p,q)-calculus, which can be found in [8, 19, 20] are given.

Throughout this paper, let $[a,b]$ be an interval with $-\infty < a < b < +\infty$, and $0 < q < p \leqslant 1$ be constants. We define

$$[k]_{p,q} = \frac{p^k - q^k}{p-q}, \qquad k \in \mathbb{N},$$

$$[k]_{p,q}! = \begin{cases} [k]_{p,q}[k-1]_{p,q}\cdots[1]_{p,q} = \prod_{i=0}^{k} \frac{p^i - q^i}{p-q}, & k \in \mathbb{N}, \\ 1, & k = 0. \end{cases}$$

The q-analogue of the power function $(a-b)_q^{(n)}$ with $n \in \mathbb{N}_0 := \{0,1,2,\ldots\}$ is given by

$$(a-b)_q^{(0)} = 1, \qquad (a-b)_q^{(n)} := \prod_{k=0}^{n-1}(a - bq^k), \quad a,b \in \mathbb{R}.$$

The (p,q)-analogue of the power function $(a-b)_{p,q}^{(n)}$ with $n \in \mathbb{N}_0 := \{0,1,2,\ldots\}$ is defined by

$$(a-b)_{p,q}^{(0)} = 1, \qquad (a-b)_{p,q}^{(n)} := \prod_{k=0}^{n-1}(ap^k - bq^k), \quad a,b \in \mathbb{R}.$$

DEFINITION 2.1. [19] Let $g : [a,b] \to \mathbb{R}$. The (p,q)-derivative of function g is defined by

$$D_{p,q}g(t) = \frac{g(pt) - g(qt)}{(p-q)t}, \qquad t \neq 0, \tag{2.1}$$

and $D_{p,q}g(0) = \lim_{t\to 0} D_{p,q}g(t)$, provided that g is differentiable at 0.

DEFINITION 2.2. [19] Let $g : [a,b] \to \mathbb{R}$, and t be a real number. The generalized quantum integral with (p,q)-parameters is defined as

$$\int_0^t g(s)\, d_{p,q}s = (p-q)t \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} g\Big(\frac{q^n}{p^{n+1}}t\Big), \tag{2.2}$$

which converges for each fixed $t \in [a,b]$. Thus, although the evaluation points $q^n t/p^{n+1}$ form a discrete set, the variable t itself ranges over the whole interval $[a,b]$.

DEFINITION 2.3. The generalisation of the q-gamma function, which is called the (p,q)-gamma function, is given by

$$\Gamma_{p,q}(t) = \frac{(p-q)_{p,q}^{(t-1)}}{(p-q)^{t-1}} = \frac{p^{\binom{t-1}{2}}}{1-(q/p)^{t-1}} \prod_{k=0}^{\infty} \frac{1-(q/p)^{k+1}}{1-(q/p)^{k+t+1}}, \tag{2.3}$$

for $t \in \mathbb{R} \setminus \{0,-1,-2,\ldots\}$, and an equivalent definition of (2.3) is given in [20] as

$$\Gamma_{p,q}(t) = p^{t(t-1)/2} \int_0^{\infty} x^{t-1} E_{p,q}^{-qx} d_{p,q}x,$$

where

$$E_{p,q}^{-qx} = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{[n]_{p,q}!} (-qx)^n.$$

Note that $\Gamma_{p,q}(t+1) = [t]_{p,q}\Gamma_{p,q}(t)$ and $\Gamma_{p,q}(t) \leqslant p^{\binom{t-1}{2}}(1-(q/p)_{p,q}^{1-t})$.

For $s,t>0$, the definition of the (p,q)-beta function is

$$B_{p,q}(s,t) = \int_0^1 u^{s-1}(1-qu)_{p,q}^{(t-1)} d_{p,q}u, \tag{2.4}$$

and (2.4) can also be written as

$$B_{p,q}(s,t) = p^{(t-1)(2s+t-2)/2} \frac{\Gamma_{p,q}(s)\Gamma_{p,q}(t)}{\Gamma_{p,q}(s+t)}.$$

THEOREM 2.4. [19] *Assume that $f,g:[0,b]\to\mathbb{R}$ are (p,q)-differentiable functions on $[0,b/p]$. Then*

(i) $f+g:[0,b]\to\mathbb{R}$ is (p,q)-differentiable on $[0,b/p]$ and

$$D_{p,q}(f(t)+g(t)) = D_{p,q}f(t) + D_{p,q}g(t).$$

(ii) For any constant c, $cf:[0,b]\to\mathbb{R}$ is (p,q)-differentiable on $[0,b/p]$ and

$$D_{p,q}cf(t) = cD_{p,q}f(t).$$

(iii) $fg:[0,b]\to\mathbb{R}$ is (p,q)-differentiable on $[0,b/p]$ and

$$D_{p,q}(fg)(t) = f(pt)D_{p,q}g(t) + g(qt)D_{p,q}f(t),$$
$$D_{p,q}(fg)(t) = g(pt)D_{p,q}f(t) + f(qt)D_{p,q}g(t).$$

(iv) If $g(t)\neq 0$ then f/g is (p,q)-differentiable on $[0,b/p]$ with

$$D_{p,q}(f/g)(t) = \frac{g(qt)D_{p,q}f(t) + f(qt)D_{p,q}g(t)}{g(pt)g(qt)}.$$

THEOREM 2.5. [23] *Let $g:[0,b]\to\mathbb{R}$ be a continuous function. Then*

$$\begin{aligned}
&(i)\quad D_{p,q}\int_0^t g(s)\,d_{p,q}s = g(t),\\
&(ii)\quad \int_0^t D_{p,q}g(s)\,d_{p,q}s = g(t)-g(0),\\
&(iii)\quad \int_a^t D_{p,q}g(s)\,d_{p,q}s = g(t)-g(a),\quad \text{for } a\in(0,t).
\end{aligned}$$

DEFINITION 2.6. [22] Let $f:[0,b]\to\mathbb{R}$ be a continuous function and $\alpha\geqslant 0$. The fractional (p,q)-integral of fractional Riemann-Liouville type is given by $(I^0_{p,q}f)(t)=f(t)$ and

$$\begin{aligned}
\big(I^\alpha_{p,q}f\big)(t) &= \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}_{p,q} f\big(s/p^{\alpha-1}\big)\,d_{p,q}s\\
&= \frac{(p-q)t}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\big(t-\frac{q^{n+1}}{p^{n+1}}t\big)^{(\alpha-1)}_{p,q} f\big(\frac{q^n}{p^{\alpha+n}}t\big),
\end{aligned}$$

where $t\in[0,p^\alpha b]$.

DEFINITION 2.7. [22] The fractional (p,q)-derivative of Riemann-Liouville type of order $\alpha\geqslant 0$ of a continuous function f is defined by $(D^0_{p,q}f)(t)=f(t)$ and

$$\big(D^\alpha_{p,q}f\big)(t) = \big(D^{[\alpha]}_{p,q}I^{[\alpha]-\alpha}_{p,q}f\big)(t),$$

where $[\alpha]$ is the smallest integer greater than or equal to α.

DEFINITION 2.8. [22] The fractional (p,q)-derivative of Caputo type of order $\alpha\geqslant 0$ of a continuous function f is defined by $({}^cD^0_{p,q}f)(t)=f(t)$ and

$$\big({}^cD^\alpha_{p,q}f\big)(t) = \big(I^{[\alpha]-\alpha}_{p,q}D^{[\alpha]}_{p,q}f\big)(t),$$

where $[\alpha]$ is the smallest integer greater than or equal to α.

LEMMA 2.9. [22] *Let $f:[a,b]\to\mathbb{R}$ be a continuous function and $\alpha,\beta\geqslant 0$. Then, the following formulas hold:*

$$\begin{aligned}
&(i)\quad \big(I^\alpha_{p,q}I^\beta_{p,q}f\big)(t) = \big(I^{\alpha+\beta}_{p,q}f\big)(t),\\
&(ii)\quad \big(D^\alpha_{p,q}I^\alpha_{p,q}f\big)(t) = f(t).
\end{aligned}$$

LEMMA 2.10. [22] *Let $f:[a,b]\to\mathbb{R}$ be a continuous function, $\alpha\geqslant 0$, then, the following formula holds*

$$\big(I^\alpha_{p,q}\,{}^cD^\alpha_{p,q}f\big)(t) = f(t)-\sum_{k=0}^{[\alpha]-1}\frac{t^k}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(k+1)}\big(D^k_{p,q}f\big)(0).$$

THEOREM 2.11. [7] (Contraction Mapping Principle) *If (X,P) is a complete distance space and T is a compressed map from (X,P) to itself, then T has a unique fixed point in X.*

THEOREM 2.12. [26] (Arzelà-Ascoli Theorem) *Let X be a compact metric space. A closed subspace of $C(X;\mathbb{R})$ or $C(X;\mathbb{C})$ is compact if and only if it is bounded and equicontinuous.*

LEMMA 2.13. [9] *(Nonlinear alternative for single value maps) Let E be a Banach space, C a closed, convex subset of E and U an open subset of C with $u \in U$. Suppose that $\Theta : \bar{U} \to C$ is a continuous, compact function; that is, $\Theta(\bar{U})$ is a relatively compact subset of C. Then, either*

(i) Θ has a fixed point in $\bar{U}$, or

(ii) there is a $u \in \partial U$ (the boundary of U in C) and $\mu \in (0,1)$ with $u = \mu\Theta u$.

In order to define the solution of the boundary value problem (1.1), we need the following lemma.

LEMMA 2.14. *Let $0 < \alpha, \beta \leqslant 1$, $1 < \alpha + \beta \leqslant 2$, $x, h \in C(J,\mathbb{R})$. For any $t \in J$, x is a solution of the problem*

$$\begin{cases} {}^{c}D^{\alpha}_{p,q}({}^{c}D^{\beta}_{p,q} + \lambda)x(t) = g(t), & t \in J = [0, T/p^{\alpha+\beta}], \\ x(0) + x(T/p^{\alpha+\beta}) = 0, \\ {}^{c}D^{\beta}_{p,q}x(0) + {}^{c}D^{\beta}_{p,q}x(T/p^{\alpha+\beta}) = 0, \end{cases} \tag{2.5}$$

if and only if x satisfies the following integral equation

$$\begin{aligned} x(t) = {} & \int_0^t \frac{(t-qs)^{(\alpha+\beta-1)}_{p,q}}{p^{\binom{\alpha+\beta}{2}}\Gamma_{p,q}(\alpha+\beta)} g(s/p^{\alpha+\beta-1})\, d_{p,q}s - \lambda \int_0^t \frac{(t-qs)^{(\beta-1)}_{p,q}}{p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta)} x(s/p^{\beta-1})\, d_{p,q}s \\ & - \frac{1}{2}\frac{t^{\beta}}{p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)} \int_0^{T/p^{\alpha+\beta}} \frac{(T/p^{\alpha+\beta} - qs)^{(\alpha-1)}_{p,q}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} g(s/p^{\alpha-1})\, d_{p,q}s \\ & - \frac{1}{2}\int_0^{T/p^{\alpha+\beta}} \frac{(T/p^{\alpha+\beta} - qs)^{(\alpha+\beta-1)}_{p,q}}{p^{\binom{\alpha+\beta}{2}}\Gamma_{p,q}(\alpha+\beta)} g(s/p^{\alpha+\beta-1})\, d_{p,q}s \\ & + \frac{\lambda}{2}\int_0^{T/p^{\alpha+\beta}} \frac{(T/p^{\alpha+\beta} - qs)^{(\beta-1)}_{p,q}}{p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta)} x(s/p^{\beta-1})\, d_{p,q}s \\ & + \frac{1}{4}\frac{T^{\beta}}{p^{(\alpha+\beta)\beta}p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)} \int_0^{T/p^{\alpha+\beta}} \frac{(T/p^{\alpha+\beta} - qs)^{(\alpha-1)}_{p,q}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} g(s/p^{\alpha-1})\, d_{p,q}s, \end{aligned} \tag{2.6}$$

Proof. We apply (p,q)-integral operator of order α on the first equation in (2.5), and use Lemma 2.10, to obtain

$$ {}^{c}D^{\beta}_{p,q}x(t) = I^{\beta}_{p,q}g(t) - \lambda x(t) + c_0. \tag{2.7}$$

We obtain by taking the fractional (p,q)-integrating to both side of (2.7) as

$$ x(t) = I^{\alpha+\beta}_{p,q}g(t) - \lambda I^{\beta}_{p,q}x(t) + \frac{t^{\beta}}{p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)}c_0 + c_1, \tag{2.8}$$

where c_0 and c_1 are constants and $t \in [0, T/p^{\alpha+\beta}]$. Applying the boundary conditions of (2.5) to (2.7) and (2.8), we solve the resulting system for c_0 and c_1 to find

$$\begin{aligned}
c_0 &= -\frac{1}{2}\int_0^{T/p^{\alpha+\beta}} \frac{(T/p^{\alpha+\beta} - qs)^{(\alpha-1)}_{p,q}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} g\left(s/p^{\alpha-1}\right) d_{p,q}s, \\
c_1 &= -\frac{1}{2}\int_0^{T/p^{\alpha+\beta}} \frac{(T/p^{\alpha+\beta} - qs)^{(\alpha+\beta-1)}_{p,q}}{p^{\binom{\alpha+\beta}{2}}\Gamma_{p,q}(\alpha+\beta)} g\left(s/p^{\alpha+\beta-1}\right) d_{p,q}s \\
&\quad + \frac{\lambda}{2}\int_0^{T/p^{\alpha+\beta}} \frac{(T/p^{\alpha+\beta} - qs)^{(\beta-1)}_{p,q}}{p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta)} x\left(s/p^{\beta-1}\right) d_{p,q}s \\
&\quad + \frac{1}{4}\frac{T^{\beta}}{p^{(\alpha+\beta)\beta}p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)}\int_0^{T/p^{\alpha+\beta}} \frac{(T/p^{\alpha+\beta} - qs)^{(\alpha-1)}_{p,q}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} g\left(s/p^{\alpha-1}\right) d_{p,q}s.
\end{aligned}$$

Substituting these values of c_0 and c_1 into (2.8) yields (2.6); conversely, suppose x is a solution of (2.6). Differentiating (2.6) recovers the differential equation in (2.5), which completes the proof. □

3. Main results

In this section, we discuss the existence of solution for boundary value problems (1.1) by the Schauder fixed point theorem and the Banach contraction principle.

Let $X = C(J,\mathbb{R})$. Obviously, the space X is a Banach space if it is endowed with the norm as follows

$$\|x\| = \max_{t\in J}|x(t)|.$$

Define an operator $\Theta : X \to X$ by

$$\begin{aligned}
\Theta x(t) &= \int_0^t \frac{(t-qs)^{(\alpha+\beta-1)}_{p,q}}{p^{\binom{\alpha+\beta}{2}}\Gamma_{p,q}(\alpha+\beta)} f(s, x(p^{\alpha+\beta-1}s))\, d_{p,q}s - \lambda\int_0^t \frac{(t-qs)^{(\beta-1)}_{p,q}}{p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta)} x(p^{\beta-1}s)\, d_{p,q}s \\
&\quad - \frac{1}{2}\frac{t^{\beta}}{p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)}\int_0^{T/p^{\alpha+\beta}} \frac{(T/p^{\alpha+\beta} - qs)^{(\alpha-1)}_{p,q}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} f(s, x(p^{\alpha-1}s))\, d_{p,q}s
\end{aligned}$$

$$-\frac{1}{2}\int_0^{T/p^{\alpha+\beta}} \frac{(T/p^{\alpha+\beta}-qs)_{p,q}^{(\alpha+\beta-1)}}{p^{\binom{\alpha+\beta}{2}}\Gamma_{p,q}(\alpha+\beta)} f(s,x(p^{\alpha+\beta-1}s))\,d_{p,q}s$$
$$+\frac{\lambda}{2}\int_0^{T/p^{\alpha+\beta}} \frac{(T/p^{\alpha+\beta}-qs)_{p,q}^{(\beta-1)}}{p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta)} x(p^{\beta-1}s)\,d_{p,q}s$$
$$+\frac{1}{4}\frac{T^\beta}{p^{(\alpha+\beta)\beta}p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)}\int_0^{T/p^{\alpha+\beta}} \frac{(T/p^{\alpha+\beta}-qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} f(s,x(p^{\alpha-1}s))\,d_{p,q}s.$$

To prove the main result, we need the following assumptions

(H_1) $f: J\times\mathbb{R}\to\mathbb{R}$ be a continuous function.

(H_2) There exists a (p,q)-integrable function $L_1: J\to\mathbb{R}$ and $M>0$, such that

$$|f(t,x)|\leqslant L_1(t), \qquad \forall t\in J, \quad x\in[-M,M].$$

LEMMA 3.1. *Assume that H_1, H_2 hold. Then $\Theta: X\to X$ is completely continuous.*

Proof. The proof is similar to that of Lemma 3.2 in Zhou and Qiao [27]. □

THEOREM 3.2. *Assume that H_1, H_2 hold. Furthermore, suppose there exists a number $K>0$ such that*

$$\frac{1-a}{b}K>1, \tag{3.1}$$

where a and b are defined as

$$a=\frac{3|\lambda|}{2}\frac{T^\beta}{p^{(\beta+\alpha)\beta}p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)},$$
$$b=\frac{3}{2}I_{p,q}^{\beta+\alpha}L_1(T/p^{\alpha+\beta})+\frac{3}{4}\frac{T^\beta}{p^{(\beta+\alpha)\beta}p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)}I_{p,q}^{\alpha}L_1(T/p^{\alpha+\beta}).$$

Then, the fractional differential equation (1.1) has at least one solution.

Proof. According to Lemma 3.1, operator Θ is completely continuous. Now, the operator $\Theta: X\to X$ satisfies all the conditions of Lemma 2.13, therefore, by its conclusion, either condition (i) or condition (ii) holds. Now, we show that the conclusion (ii) is not possible. Let $U=\{x\in C(J,\mathbb{R}): \|x\|\leqslant K\}$ with $aK+b<K$. It can be shown that

$$\begin{aligned}|(\Theta x)(t)| &\leqslant \frac{3|\lambda|}{2}\frac{T^\beta}{p^{(\beta+\alpha)\beta}p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)}\|x\| \\ &\quad+\frac{3}{2}I_{p,q}^{\beta+\alpha}L_1(T/p^{\alpha+\beta})+\frac{3}{4}\frac{T^\beta}{p^{(\beta+\alpha)\beta}p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)}I_{p,q}^{\alpha}L_1(T/p^{\alpha+\beta}) \\ &\leqslant aK+b<K.\end{aligned}$$

Suppose there exists $x \in \partial U$ and $\mu \in (0,1)$ such that $x = \mu \Theta x$. For such a choice of x and μ, we have

$$K = \|x\| = \mu \|\Theta x\| \leqslant \|\Theta x\| \leqslant aK + b < K.$$

This leads to a contradiction. Accordingly, by Lemma 2.13, we have that Θ has a fixed point $x \in U$, which is a solution of the problem (1.1). Therefore, the proof is completed. □

THEOREM 3.3. *Suppose for $L \in C(J, \mathbb{R})$, f satisfies the condition*

$$|f(t,x) - f(t,y)| \leqslant L(t)|x-y|, \qquad \forall t \in J, \quad x, y \in \mathbb{R},$$

and

$$\phi = \frac{3}{2} I_{p,q}^{\alpha+\beta} L(T/p^{\alpha+\beta}) + \frac{3|\lambda|}{2} \frac{T^{\beta}}{p^{(\alpha+\beta)\beta} p^{\binom{\beta}{2}} \Gamma_{p,q}(\beta+1)} + \frac{3}{4} \frac{T^{\beta}}{p^{(\alpha+\beta)\beta} p^{\binom{\beta}{2}} \Gamma_{p,q}(\beta+1)} I_{p,q}^{\alpha} L(T/p^{\alpha+\beta}) < 1.$$

Then the fractional differential equation (1.1) has a unique solution.

Proof. Define $Q = \max_{t \in J} |f(t,0)|$ and select $r \geqslant (Q\Omega)(1-\phi)^{-1}$, where

$$\Omega = \frac{3}{2} \frac{T^{\alpha+\beta}}{p^{(\alpha+\beta)^2} p^{\binom{\alpha+\beta}{2}} \Gamma_{p,q}(\alpha+\beta+1)} + \frac{3}{4} \frac{T^{\alpha+\beta}}{p^{(\alpha+\beta)^2} p^{\binom{\beta}{2}} p^{\binom{\alpha}{2}} \Gamma_{p,q}(\beta+1) \Gamma_{p,q}(\alpha+1)}.$$

Define a closed ball as $B_r = \{x \in X : \|x\| \leqslant r\}$, then for $x \in B_r$, we have

$$\begin{aligned}
|\Theta x(t)| \leqslant & \int_0^{T/p^{\alpha+\beta}} \frac{(T/p^{\alpha+\beta} - qs)_{p,q}^{(\alpha+\beta-1)}}{p^{\binom{\alpha+\beta}{2}} \Gamma_{p,q}(\alpha+\beta)} |f(s, x(p^{\alpha+\beta-1}s))| \, d_{p,q}s \\
& + |\lambda| \int_0^{T/p^{\alpha+\beta}} \frac{(T/p^{\alpha+\beta} - qs)_{p,q}^{(\beta-1)}}{p^{\binom{\beta}{2}} \Gamma_{p,q}(\beta)} |x(p^{\beta-1}s)| \, d_{p,q}s \\
& + \frac{1}{2} \frac{T^{\beta}}{p^{(\alpha+\beta)\beta} p^{\binom{\beta}{2}} \Gamma_{p,q}(\beta+1)} \int_0^{T/p^{\alpha+\beta}} \frac{(T/p^{\alpha+\beta} - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \\
& \times |f(s, x(p^{\alpha-1}s))| \, d_{p,q}s \\
& + \frac{1}{2} \int_0^{T/p^{\alpha+\beta}} \frac{(T/p^{\alpha+\beta} - qs)_{p,q}^{(\alpha+\beta-1)}}{p^{\binom{\alpha+\beta}{2}} \Gamma_{p,q}(\alpha+\beta)} |f(s, x(p^{\alpha+\beta-1}s))| \, d_{p,q}s \\
& + \frac{|\lambda|}{2} \int_0^{T/p^{\alpha+\beta}} \frac{(T/p^{\alpha+\beta} - qs)_{p,q}^{(\beta-1)}}{p^{\binom{\beta}{2}} \Gamma_{p,q}(\beta)} |x(p^{\beta-1}s)| \, d_{p,q}s
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}\frac{T^{\beta}}{p^{(\alpha+\beta)\beta}p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)}\int_0^{T/p^{\alpha+\beta}}\frac{(T/p^{\alpha+\beta}-qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\\
&\times|f(s,x(p^{\alpha-1}s))|\,d_{p,q}s\\
&\leqslant\frac{3}{2}\int_0^{T/p^{\alpha+\beta}}\frac{(T/p^{\alpha+\beta}-qs)_{p,q}^{(\alpha+\beta-1)}}{p^{\binom{\alpha+\beta}{2}}\Gamma_{p,q}(\alpha+\beta)}\\
&\times\big(|f(s,x(p^{\alpha+\beta-1}s))-f(s,0)|+|f(s,0)|\big)\,d_{p,q}s\\
&+\frac{3}{2}|\lambda|\int_0^{T/p^{\alpha+\beta}}\frac{(T/p^{\alpha+\beta}-qs)_{p,q}^{(\beta-1)}}{p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta)}|x(p^{\beta-1}s)|\,d_{p,q}s\\
&+\frac{3}{4}\frac{T^{\beta}}{p^{(\alpha+\beta)\beta}p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)}\int_0^{T/p^{\alpha+\beta}}\frac{(T/p^{\alpha+\beta}-qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\\
&\quad\times\big(|f(s,x(p^{\alpha-1}s))-f(s,0)|+|f(s,0)|\big)\,d_{p,q}s\\
&\leqslant\frac{3}{2}\int_0^{T/p^{\alpha+\beta}}\frac{(T/p^{\alpha+\beta}-qs)_{p,q}^{(\alpha+\beta-1)}}{p^{\binom{\alpha+\beta}{2}}\Gamma_{p,q}(\alpha+\beta)}\big(L(s/p^{\alpha+\beta-1})r+Q\big)\,d_{p,q}s\\
&+\frac{3}{2}|\lambda|r\int_0^{T/p^{\alpha+\beta}}\frac{(T/p^{\alpha+\beta}-qs)_{p,q}^{(\beta-1)}}{p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta)}\,d_{p,q}s\\
&+\frac{3}{4}\frac{T^{\beta}}{p^{(\alpha+\beta)\beta}p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)}\int_0^{T/p^{\alpha+\beta}}\frac{(T/p^{\alpha+\beta}-qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\\
&\times\big(L(s/p^{\alpha-1})r+Q\big)\,d_{p,q}s\\
&\leqslant r\bigg[\frac{3}{2}\int_0^{T/p^{\alpha+\beta}}\frac{(T/p^{\alpha+\beta}-qs)_{p,q}^{(\alpha+\beta-1)}}{p^{\binom{\alpha+\beta}{2}}\Gamma_{p,q}(\alpha+\beta)}L(s/p^{\alpha+\beta-1})\,d_{p,q}s\\
&+\frac{3}{2}|\lambda|\int_0^{T/p^{\alpha+\beta}}\frac{(T/p^{\alpha+\beta}-qs)_{p,q}^{(\beta-1)}}{p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta)}\,d_{p,q}s\\
&+\frac{3}{4}\frac{T^{\beta}}{p^{(\alpha+\beta)\beta}p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)}\\
&\times\int_0^{T/p^{\alpha+\beta}}\frac{(T/p^{\alpha+\beta}-qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}L(s/p^{\alpha-1})\,d_{p,q}s\bigg]\\
&+Q\bigg[\frac{3}{4}\frac{T^{\beta}}{p^{(\alpha+\beta)\beta}p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)}\int_0^{T/p^{\alpha+\beta}}\frac{(T/p^{\alpha+\beta}-qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\,d_{p,q}s\\
&+\frac{3}{2}\int_0^{T/p^{\alpha+\beta}}\frac{(T/p^{\alpha+\beta}-qs)_{p,q}^{(\alpha+\beta-1)}}{p^{\binom{\alpha+\beta}{2}}\Gamma_{p,q}(\alpha+\beta)}\,d_{p,q}s\bigg]
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant r\left[\frac{3}{2}I_{p,q}^{\alpha+\beta}L(T/p^{\alpha+\beta})+\frac{3|\lambda|}{2}\frac{T^{\beta}}{p^{(\alpha+\beta)\beta}p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)}\right.\\
&\quad\left.+\frac{3}{4}\frac{T^{\beta}}{p^{(\alpha+\beta)\beta}p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)}I_{p,q}^{\alpha}L(T/p^{\alpha+\beta})\right]\\
&\quad+Q\left[\frac{3}{2}\frac{T^{\alpha+\beta}}{p^{(\alpha+\beta)^2}p^{\binom{\alpha+\beta}{2}}\Gamma_{p,q}(\alpha+\beta+1)}\right.\\
&\quad\left.+\frac{3}{4}\frac{T^{\alpha+\beta}}{p^{(\alpha+\beta)^2}p^{\binom{\beta}{2}}p^{\binom{\alpha}{2}}\Gamma_{p,q}(\beta+1)\Gamma_{p,q}(\alpha+1)}\right]\\
&=r\phi+Q\Omega\leqslant r.
\end{aligned}
$$

Which implies that $|\Theta x(t)|\leqslant r$, that is, $\Theta(B_r)\subset B_r$. In what follows, for $x,y\in X$, and $t\in J$, we can get that

$$
\begin{aligned}
&|\Theta x(t)-\Theta y(t)|\\
&\leqslant\int_0^t\frac{(t-qs)_{p,q}^{(\alpha+\beta-1)}}{p^{\binom{\alpha+\beta}{2}}\Gamma_{p,q}(\alpha+\beta)}|f(s,x(p^{\alpha+\beta-1}s))-f(s,y(p^{\alpha+\beta-1}s))|\,d_{p,q}s\\
&\quad+|\lambda|\int_0^t\frac{(t-qs)_{p,q}^{(\beta-1)}}{p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta)}|x(p^{\beta-1}s)-y(p^{\beta-1}s)|\,d_{p,q}s\\
&\quad+\frac{1}{2}\frac{t^{\beta}}{p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)}\int_0^{T/p^{\alpha+\beta}}\frac{(T/p^{\alpha+\beta}-qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\\
&\qquad\times|f(s,x(p^{\alpha-1}s))-f(s,y(p^{\alpha-1}s))|\,d_{p,q}s\\
&\quad+\frac{1}{2}\int_0^{T/p^{\alpha+\beta}}\frac{(T/p^{\alpha+\beta}-qs)_{p,q}^{(\alpha+\beta-1)}}{p^{\binom{\alpha+\beta}{2}}\Gamma_{p,q}(\alpha+\beta)}\\
&\qquad\times|f(s,x(p^{\alpha+\beta-1}s))-f(s,y(p^{\alpha+\beta-1}s))|\,d_{p,q}s\\
&\quad+\frac{|\lambda|}{2}\int_0^{T/p^{\alpha+\beta}}\frac{(T/p^{\alpha+\beta}-qs)_{p,q}^{(\beta-1)}}{p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta)}|x(p^{\beta-1}s)-y(p^{\beta-1}s)|\,d_{p,q}s\\
&\quad+\frac{1}{4}\frac{T^{\beta}}{p^{(\alpha+\beta)\beta}p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)}\int_0^{T/p^{\alpha+\beta}}\frac{(T/p^{\alpha+\beta}-qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\\
&\qquad\times|f(s,x(p^{\alpha-1}s))-f(s,y(p^{\alpha-1}s))|\,d_{p,q}s\\
&\leqslant\|x-y\|\left[\frac{3}{2}\int_0^{T/p^{\alpha+\beta}}\frac{(t-qs)_{p,q}^{(\alpha+\beta-1)}}{p^{\binom{\alpha+\beta}{2}}\Gamma_{p,q}(\alpha+\beta)}L(s/p^{\alpha+\beta-1})\,d_{p,q}s\right.\\
&\quad\left.+\frac{3|\lambda|}{2}\int_0^{T/p^{\alpha+\beta}}\frac{(t-qs)_{p,q}^{(\beta-1)}}{p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta)}\,d_{p,q}s\right.
\end{aligned}
$$

$$+\frac{3}{4}\frac{T^{\beta}}{p^{(\alpha+\beta)\beta}p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)}\int_0^{T/p^{\alpha+\beta}}\frac{(T/p^{\alpha+\beta}-qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}L(s/p^{\alpha-1})\,d_{p,q}s\Bigg]$$

$$\leqslant\left[\frac{3}{2}I_{p,q}^{\alpha+\beta}L(T/p^{\alpha+\beta})+\frac{3|\lambda|}{2}\frac{T^{\beta}}{p^{(\alpha+\beta)\beta}p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)}\right.$$

$$\left.+\frac{3}{4}\frac{T^{\beta}}{p^{(\alpha+\beta)\beta}p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)}I_{p,q}^{\alpha}L(T/p^{\alpha+\beta})\right]\|x-y\|$$

$$\leqslant\phi\|x-y\|.$$

This implies that Θ is a contraction. Thus, by the Banach contraction mapping principle, we obtain that Θ has a unique fixed point in J, that is, (1.1) has a unique solution. The proof is completed. □

4. Examples

In this section, we present two examples, which indicate how our theorems can be applied to concrete problems.

EXAMPLE 4.1. Consider problem (1.1) with

$$\alpha=7/10,\quad \beta=4/5,\quad p=0.9,\quad q=0.05,\quad T=1,\quad \lambda=1/6,\quad J=[0,1.17].$$

Case 1.

$$f(t,x(p^{\alpha}t))=\frac{1}{20}e^{t}\left|\frac{x^2(p^{\alpha}t)}{1+x^2(p^{\alpha}t)}\right|.$$

Clearly,

$$|f(t,x(p^{\alpha}t))|\leqslant\frac{1}{20}e^{t}=L_1(t),$$

$$a=\frac{3|\lambda|}{2}\frac{T^{\beta}}{p^{(\beta+\alpha)\beta}p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)}\simeq 0.266,$$

$$b=\frac{3}{2}I_{p,q}^{\beta+\alpha}L_1(T/p^{\alpha+\beta})+\frac{3}{4}\frac{T^{\beta}}{p^{(\beta+\alpha)\beta}p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)}I_{p,q}^{\alpha}L_1(T/p^{\alpha+\beta})$$

$$\simeq 0.037,$$

with condition (3.1), it follows that $K>0.05$. Thus, all the assumptions of Theorem 3.2 hold true. Therefore, the problem (1.1) has at least one solution on $[0,1.17]$.

Case 2.

$$f(t,x(p^{\alpha}t))=\frac{1}{15}e^{\sin t}\frac{\arctan(x(p^{\alpha}t))}{1+x(p^{\alpha}t)^2}.$$

Clearly,

$$|\arctan x|\leqslant\pi/2\quad\text{and}\quad\frac{1}{1+x^2}\leqslant 1,$$

we obtain

$$\begin{aligned}
|f(t,x(p^{\alpha}t))| &\leqslant \frac{\pi}{30}e^{\sin t} = L_1(t),\\
a &= \frac{3|\lambda|}{2}\frac{T^{\beta}}{p^{(\beta+\alpha)\beta}p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)} \simeq 0.267,\\
b &= \frac{3}{2}I_{p,q}^{\beta+\alpha}L_1(T/p^{\alpha+\beta}) + \frac{3}{4}\frac{T^{\beta}}{p^{(\beta+\alpha)\beta}p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)}I_{p,q}^{\alpha}L_1(T/p^{\alpha+\beta})\\
&\simeq 0.065,
\end{aligned}$$

with condition (3.1), it follows that $K > 0.09$. Thus, all the assumptions of Theorem 3.2 hold true. Therefore, the problem (1.1) has at least one solution on $[0, 1.17]$.

EXAMPLE 4.2. Consider the fractional differential equation (1.1) with the following parameters:

$$\alpha = 0.7, \quad \beta = 0.8, \quad p = 0.9, \quad q = 0.05, \quad T = 1, \quad \lambda = 1/6,$$

and $J = [0, T/p^{\alpha+\beta}] \simeq [0, 1.17]$. Let the nonlinearity f be defined by

$$f(t,x(p^{\alpha}t)) = \frac{1}{50}\cos t + \frac{1}{100}x(p^{\alpha}t) + \frac{1}{100}\sin(x(p^{\alpha}t)).$$

Note that f is nonlinear and is unbounded in the variable x. Furthermore, f satisfies the Lipschitz condition with $L(t) = 1/50$ for all $t \in J$, since

$$\begin{aligned}
|f(t,x) - f(t,y)| &\leqslant \frac{1}{100}|x-y| + \frac{1}{100}|\sin x - \sin y|\\
&\leqslant \frac{1}{100}|x-y| + \frac{1}{100}|x-y| = \frac{1}{50}|x-y|.
\end{aligned}$$

By applying the definitions of the (p,q)-fractional integral and the parameters above, we compute

$$\begin{aligned}
\phi &= \frac{3}{2}I_{p,q}^{\alpha+\beta}L(T/p^{\alpha+\beta}) + \frac{3|\lambda|}{2}\frac{T^{\beta}}{p^{(\alpha+\beta)\beta}p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)}\\
&\quad + \frac{3}{4}\frac{T^{\beta}}{p^{(\alpha+\beta)\beta}p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta+1)}I_{p,q}^{\alpha}L(T/p^{\alpha+\beta})\\
&\simeq 0.024 + 0.296 + 0.022 = 0.342 < 1.
\end{aligned}$$

Since $\phi < 1$, all the conditions of Theorem 3.3 are satisfied. Consequently, the problem (1.1) possesses a unique solution on J, even though the function f is unbounded in x.

Declarations

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REFERENCES

[1] B. AHMAD, J. J. NIETO, A. ALSAEDIA AND H. AL-HUTAMIA, *Existence of solutions for nonlinear fractional q-difference integral equations with two fractional orders and nonlocal four-point boundary conditions*, (2014).
[2] B. AHMAD, J. J. NIETO, A. ALSAEDI AND M. EL-SHAHED, *A study of nonlinear Langevin equation involving two fractional orders in different intervals*, Nonlinear Anal. Real World Appl. **13** (2012).
[3] A. ARAL AND V. GUPTA, *Applications of the (p,q)-gamma function to Szász-Durrmeyer operators*, Publ. Inst. Math. (2017).
[4] R. CHAKRABARTI AND R. JAGANNATHAN, *A (p,q)-oscillator realization of two-parameter quantum algebras*, J. Phys. A Math. Gen. **24** (1991), 5683–5701.
[5] W. T. COFFEY, Y. P. KALMYKOV AND J. T. WALDRON, *The Langevin equation: with applications to stochastic problems in physics, chemistry and electrical engineering*, 2nd ed., World Scientific, Singapore (2004).
[6] S. I. DENISOV, H. KANTZ AND P. HÄNGGI, *Langevin equation with super-heavy-tailed noise*, J. Phys. A **43** (2010).
[7] F. T. FEN AND I. Y. KARACA, *Nonlinear four-point impulsive fractional differential equations with p-Laplacian operator*, Discontinuity, Nonlinearity and Complexity (2015).
[8] R. FERREIRA, *Nontrivial solutions for fractional q-difference boundary value problems*, Electron. J. Qual. Theory Differ. Equ. (2010).
[9] A. GRANAS AND J. DUGUNDJI, *Fixed point Theory and Applications*, Springer, New York, NY, USA (2003).
[10] F. H. JACKSON, *On q-definite integrals*, Quart. J. Pure Appl. Math. **41** (1910).
[11] F. H. JACKSON, *On q-functions and a certain difference operator*, Trans. Roy. Soc. Edin. **46** (1908).
[12] N. KAMSRISUK, C. PROMSAKON, S. K. NTOUYAS AND J. TARIBOON, *Nonlocal boundary value problems for (p,q)-difference equations*, Differ. Equ. Appl. (2018).
[13] J. LERAY AND J. SCHAUDER, *Topologie et équations fonctionnelles*, Ann. Sci. École Norm. Sup. (3) **51** (1934).
[14] S. C. LIM, M. LI AND L. P. TEO, *Langevin equation with two fractional orders*, Phys. Lett. A **372** (2008).
[15] P. NEANG, K. NONLAOPON, J. TARIBOON, S. K. NTOUYAS AND B. AHMAD, *Nonlocal boundary value problems of nonlinear fractional (p,q)-difference equations*, Fractal Fract. (2021).
[16] J. PRABSEANG, K. NONLAOPON AND J. TARIBOON, *(p,q)-Hermite-Hadamard inequalities for double integral and (p,q)-differentiable convex functions*, Axioms (2019).
[17] C. PROMSAKON, N. KAMSRISUK, S. K. NTOUYAS AND J. TARIBOON, *On the second-order (p,q)-difference equation with separated boundary conditions*, Adv. Math. Phys. (2018).
[18] Z. QIN AND S. SUN, *Positive solutions for fractional (p,q)-difference boundary value problems*, J. Appl. Math. Comput. (2021).
[19] P. N. SADJANG, *On the fundamental theorem of (p,q)-calculus and some (p,q)-Taylor formulas*, Results Math. (2018).

[20] P. N. SADJANG, *On the (p,q)-gamma and the (p,q)-beta functions*, arXiv:1506.07394, https://doi.org/10.48550/arXiv.1506.07394 (2015).
[21] S. SITHO, S. LAOPRASITTICHOK, S. K. NTOUYAS AND J. TARIBOON, *Quantum difference Langevin equation with multi-quantum numbers q-derivative nonlocal conditions*, J. Nonlinear Sci. Appl. **9** (2016).
[22] J. SOONTHARANON AND T. SITTHIWIRATTHAM, *On fractional (p,q)-calculus*, Adv. Differ. Equ. (2020).
[23] M. TUNÇ AND E. GÖV, *(p,q)-integral inequalities*, RGMIA Res. Rep. Coll. (2016).
[24] M. URANAGASE AND T. MUNAKATA, *Generalized Langevin equation revisited: mechanical random force and self-consistent structure*, J. Phys. **43** (2010).
[25] T. USMAN, M. SAIF AND J. CHOI, *Certain identities associated with (p,q)-binomial coefficients and (p,q)-Stirling polynomials of the second kind*, Symmetry (2020).
[26] E. ZEIDLER, *Nonlinear Functional Analysis and its Applications I: Fixed Point Theorems*, Springer-Verlag (1985).
[27] Z. ZHOU AND Y. QIAO, *Solutions for a class of fractional Langevin equations with integral and anti-periodic boundary conditions*, Bound. Value Probl. (2018), 2018:152, https://doi.org/10.1186/s13661-018-1070-3.

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