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GENERALIZED BOUNDARY–LAYER FLOW DUE TO A SHRINKING PERMEABLE SHEET

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Abstract. This article analyzes a model proposed by Merkin and Pop [J. Eng. Math. **121** (2020)], of boundary-layer flow generated by a shrinking permeable sheet. The model is governed by two physical parameters, the transpiration rate, S, and the coupling between the fluid and the moving surface, c. We prove several results regarding the existence or nonexistence of solutions in the c-S parameter space. In regions where solutions are shown to exist, we prove that there is a continuum of infinitely many solutions. This is in contrast to previous numerical studies, which report either a unique solution or dual solutions.

1. Introduction

This work considers a generalized model of boundary layer flow induced by a stretching or shrinking surface. The original problem, considered by Crane [2], involved an impermeable horizontal surface which is stretching with a velocity given by $U(x) \propto x$. Since then the problem has been generalized to include surfaces with power law [8], $U(x) \propto x^{\alpha}$ and exponential [4], $U(x) \propto e^{x}$ stretching. In addition, the effects of a permeable surface have also been explored [3], [5].

In all of these studies the kinematics of the surface are prescribed. However, in 2012 Al-Housseiny and Stone [1] proposed a model that included the mechanics of the sheet, wherein the dynamics of both the fluid and the surface are coupled. Consideration of a similarity solution to the governing partial differential equations results in the following nonlinear boundary value problem:

$$g'''(y) + (c+1)g(y)g''(y) - g'(y)^2 = 0, \quad 0 < y < \infty, \tag{1.1}$$

subject to

$$g(0) = 0, \quad g'(0) = 1, \quad g'(\infty) = 0. \tag{1.2}$$

Al-Housseiny and Stone [1] investigate this problem in the case of an elastic sheet that obeys linear elasticity ($c = -3$) and the case of a sheet that deforms as a viscous Newtonian fluid ($c = -1$). Recently, Merkin and Pop [6] investigated a generalization

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of this problem by considering arbitrary values of c (both positive and negative) and by incorporating the effects of a permeable surface. They also expanded the analysis to shrinking, as well as stretching, surfaces. This leads to the more general boundary conditions:

$$g(0) = S, \quad g'(0) = \lambda, \quad g'(\infty) = 0. \tag{1.3}$$

Here $\lambda = 1$ corresponds to a stretching surface and $\lambda = -1$ represents a shrinking surface. Permeability of the surface is given by $S > 0$ for fluid withdrawal and $S < 0$ for fluid injection. The parameter c represents the coupling between the fluid flow and the dynamics of the moving surface.

Through numerical integration and asymptotic analysis, Merkin and Pop [6] draw several conclusions regarding the nature of the solutions of (1.1, 1.3) as a function of the parameters c, S, and λ. The purpose of this article is to analytically investigate the existence or nonexistence of solutions to the BVP (1.1, 1.3) as a function of the model parameters and compare the results to the numerical solutions obtained in [6]. We are able to prove existence/nonexistence results for a large portion of parameter space. These results are summarized graphically in Figure 4. This figure also includes the numerical results reported in [6]. We also prove several results regarding the multiplicity of solutions that expand significantly on the numerical results reported in [6]. As the case of a stretching sheet ($\lambda = 1$) is fairly straightforward mathematically, we will concentrate on the more interesting case of a shrinking sheet, and *hereafter set* $\lambda = -1$ *in* (1.3).

Merkin and Pop [6] investigate the BVP (1.1, 1.3) numerically for several values of $c > -1$. They find no solutions for an impermeable sheet ($S = 0$), nor in the case of fluid injection ($S < 0$). Further, they conjecture the existence of a value $S_{crit}(c) > 0$ for which dual solutions exist for $S > S_{crit}(c)$, a unique solution exists for $S = S_{crit}(c)$, and no solution exists for $S < S_{crit}(c)$. We partially verify this conjecture by proving the existence of a bound $S_b(c) > 0$, $c > -1$ such that for $S \leqslant S_b(c)$, no solution to the BVP (1.1, 1.3) exists (Theorems 6, 7, 8, and 9). Further, for $-1 < c \leqslant 0$ and $S > 2/(c+1)$, we prove the existence of not just two solutions (as reported in [6]), but in fact infinitely many solutions (Theorem 4). We also prove that for $-3/2 \leqslant c < -1$ and $S \geqslant 0$, no solution to the BVP (1.1, 1.3) exists (Theorem 10).

For $S = 0$ Merkin and Pop [6] find a unique solution for all $c < -3/2$ and present a closed form solution for $S = 0$ and $c = -2$. They also numerically explore several positive values of S ($S = 0.5, 1.0$ and 2.0) and find a unique solution for all $c < c_{crit}(S)$ where $-2 < c_{crit}(S) < -3/2$. However, it turns out that in this case as well, there is in fact a whole continuum of solutions. We prove that for $c \leqslant -2$ and $S > 0$ infinitely many solutions to (1.1, 1.3) exist (Theorem 2). Further, we prove that infinitely many solutions exist in the range $c \leqslant -2$ and $S \leqslant 0$ (Theorem 1) and $-2 < c \leqslant -3/2$ and $S < 0$ (Theorem 3).

The main shortcoming of this work is the absence of an existence result for the case $c > 0$ and $S \geqslant S_{crit}(c)$. However, for this case we are able to obtain bounds on the skin friction coefficient $g''(0)$ that any possible solutions must satisfy (Theorem 5). These bounds greatly aid in the search for numerically integrated solutions.

The existence proofs are presented in Section 2 and the nonexistence results in

Section 3. In Section 4 we list open problems and present numerical results for the problem. We also discuss the aspects of the problem that make numerical integration difficult.

2. Existence results

To investigate the solutions of the boundary value problem (1.1, 1.3) with $\lambda = -1$ for the unknown function $g(y)$, we will consider a family of related initial value problems, namely: Find $f_\alpha(y)$ such that

$$f_\alpha'''(y) + (c+1) f_\alpha(y) f_\alpha''(y) - f_\alpha'(y)^2 = 0, \quad 0 < y < \infty, \tag{2.1}$$

subject to

$$f_\alpha(0) = S, \tag{2.2}$$

$$f_\alpha'(0) = -1, \tag{2.3}$$

$$f_\alpha''(0) = \alpha, \tag{2.4}$$

where α is a free parameter. Hereafter, the equations (2.1–2.4) will be referred to as IVP. The subscript α in the notation $f_\alpha(y)$ emphasizes the dependence of this family of solutions to the IVP (2.1–2.4) on this free parameter α. To uncover solutions $g(y)$ to the BVP (1.1, 1.3), we will use a shooting argument involving the solutions $f_\alpha(y)$ to the IVP (2.1–2.4) for various values of α. For any value of α, the IVP (2.1–2.4) is guaranteed to have a unique solution on some initial interval of $y \geqslant 0$. Let $[0, \overline{y})$, where $\overline{y} = \overline{y}(\alpha)$, denote the maximal interval of existence for the solution $f_\alpha(y)$ to the IVP. The question then becomes: Can a value, or values, of α be found so that **(1)** $\overline{y}(\alpha) = \infty$ so that the interval of existence of $f_\alpha(y)$ extends to $0 \leqslant y < \infty$, and **(2)** $f_\alpha(y)$ has the desired behavior at infinity, namely $f_\alpha(\infty) = 0$. Such a value of α will produce a solution $f_\alpha(y)$ to the IVP (2.1–2.4) which corresponds to a solution $g(y)$ to the BVP (1.1, 1.3).

Thus in the remainder of the paper our main interest will be in whether a solution to the IVP (2.1–2.4) can be shown to satisfy $f_\alpha'(\infty) = 0$ or not. We will refer to $f_\alpha'(\infty) = 0$ as the existence case and the opposite as the nonexistence case. There is a one-to-one correspondence between values of α in the existence case for $f_\alpha(y)$ and solutions $g(y)$ to the BVP (1.1, 1.3).

We will decompose the c-S parameter plane into various regions of existence or nonexistence. In several of the regions, we will show that a continuum of solutions exists. Many of the proofs will involve roots of the function $f_\alpha(y)$ or one of its derivatives. Notationally, we let y_j denote a root of the function $f_\alpha^{(j)}(y)$. Thus, y_0 will denote a root of the function $f_\alpha(y)$ itself, y_1 a root of $f_\alpha'(y)$, etc.

THEOREM 1. *For $c \leqslant -2$ and $S \leqslant 0$, solutions $f_\alpha(y)$ of the IVP (2.1–2.4) with $\alpha \leqslant (c+1)S$ satisfy $f_\alpha'(\infty) = 0$, which is the right boundary condition of the BVP (1.1, 1.3). These values of $\alpha \leqslant (c+1)S$ therefore correspond to infinitely many solutions $g(y)$ to the BVP (1.1, 1.3).*

Proof. For the stated parameter values, we first list some properties that any solution to the IVP that satisfies $f'_\alpha(\infty) = 0$ must have. We then show that solutions to the IVP with these necessary properties do indeed exist. First, from the ODE (2.1) we see that $f'_\alpha(y)$ cannot have a maximum. Next, from $f'_\alpha(0) = -1$ we see that $f'_\alpha(y)$ is initially negative. These two facts imply that $f'_\alpha(y)$ can at no point increase through 0, since it could not subsequently attain a maximum, and therefore could not then satisfy $f'_\alpha(\infty) = 0$. Also note that if $f'_\alpha(y_1) = 0$ at some first point $y_1 > 0$, then we must have $f''_\alpha(y_1) > 0$, for if $f''_\alpha(y_1) = 0$, the ODE (2.1) then implies that $f'''_\alpha(y_1) = 0$ as well. But the unique solution to the ODE (2.1) with initial data $f'_\alpha(y_1) = 0$, $f''_\alpha(y_1) = 0$, and $f'''_\alpha(y_1) = 0$ is $f'_\alpha(y) \equiv 0$, which contradicts the fact that $f'_\alpha(0) = -1$. Thus $f'_\alpha(y) = 0$ must remain strictly negative. Finally, by assumption, $f_\alpha(0) = S \leqslant 0$, and since $f'_\alpha(y)$ is always negative, that forces $f'_\alpha(y)$ negative as well. Thus, we must therefore have

$$f_\alpha(y) < 0, \quad f'_\alpha(y) < 0, \quad \forall y > 0. \tag{2.5}$$

Further, in order for $f'_\alpha(y)$ to approach 0 from below, either $f'_\alpha(y)$ is increasing for all $y > 0$ or, since $f'_\alpha(y)$ cannot have a maximum, $f'_\alpha(y)$ initially decreases, attains a single minimum, and thereafter increases toward 0. Thus we must have either $f''_\alpha(y) > 0$ for all $y > 0$ or there exists a unique $y_2 \in (0,\infty)$ such that

$$f''_\alpha(y_2) = 0, \text{ with } f''_\alpha(y) < 0 \text{ on } (0,y_2) \text{ and } f''_\alpha(y) > 0 \text{ on } (y_2,\infty). \tag{2.6}$$

With these preliminaries in hand, the proof of the theorem will proceed along two cases defined below. In each case, we must first show that the solution of the IVP (2.1–2.4) exists on the entire half line $0 \leqslant y < \infty$. These cases will involve proving that certain properties hold for as long as the solution to the IVP (2.1–2.4) exists. We will then show that these properties necessarily result in existence for all of $0 \leqslant y < \infty$.

Case 1: $0 \leqslant f''_\alpha(0) = \alpha \leqslant (c+1)S$. We will first show that for values of α in this range we have $-1 < f'_\alpha(y) < 0$ for as long as the solution of the IVP exists. Note that as long as $-1 < f'_\alpha(y;\alpha) < 0$ we have that $f'_\alpha(y)$ is bounded above and below, and therefore on integration $f_\alpha(y)$ is also bounded above and below. Finally, integration of the ODE (2.1) from 0 to $y > 0$ results in

$$f''_\alpha(y) = \alpha - (c+1)S - (c+1)f_\alpha(y)f'_\alpha(y) + (c+2)\int_0^y f'_\alpha(t)^2\,dt, \tag{2.7}$$

from which we conclude, using the boundedness of $f_\alpha(y)$ and $f'_\alpha(y)$, that $f''_\alpha(y)$ is also bounded above and below. Thus the solution of the IVP $f_\alpha(y)$ can be extended for as long as $f'_\alpha(y)$ continues to satisfy $-1 < f'_\alpha(y) < 0$.

We next show that $f'_\alpha(y)$ cannot leave the range $-1 < f'_\alpha(y) < 0$. Case 1 includes the assumption that $f''_\alpha(0) = \alpha \geqslant 0$. If $f''_\alpha(0) > 0$, then $f'_\alpha(y)$ is initially increasing for $y > 0$. If $f''_\alpha(0) = 0$, then from (2.1–2.4) we have that $f'''_\alpha(0) = 1 > 0$, and again we conclude that $f'_\alpha(y)$ is initially increasing for $y > 0$. Since $f'_\alpha(y)$ cannot have a maximum, we have $f''_\alpha(y) \geqslant 0$ for $y \in (0,\overline{y})$ and thus $f'_\alpha(y)$ increases from -1 and we cannot thereafter have $f'_\alpha(y) = -1$ for any $y > 0$. Thus if $f'_\alpha(y)$ is to leave the range $-1 < f'_\alpha(y) < 0$ there must exist a first $y_1 \in (0,\overline{y})$ such that

$$f'_\alpha(y_1) = 0, \quad f''_\alpha(y_1) > 0. \tag{2.8}$$

(As shown earlier, to see that $f''_\alpha(y_1)$ must be positive, note that $f'_\alpha(y_1) = f''_\alpha(y_1) = 0$ implies that $f'_\alpha(y) \equiv 0$ for all $y > 0$, contradicting $f'_\alpha(0) = -1 \neq 0$.) Evaluating (2.7) at y_1 we have, since $\alpha \leqslant (c+1)S$,

$$f''_\alpha(y_1) = \alpha - (c+1)S + (c+2)\int_0^{y_1} f'_\alpha(t)^2\,dt \leqslant 0, \tag{2.9}$$

contradicting $f''_\alpha(y_1) > 0$. Thus $-1 < f'_\alpha(y) < 0$ on $(0,\overline{y})$ and from the reasoning of the previous paragraph, the solution to the IVP can be continued indefinitely so that $\overline{y} = \infty$.

Finally, we will show that for the stated range of α, the solutions to the IVP (2.1–2.4) satisfy $f'_\alpha(\infty) = 0$, corresponding to solutions to the BVP (1.1, 1.3). Since $f'_\alpha(y) < 0$ and $f''_\alpha(y) > 0$ for all $y > 0$ we have that $f'_\alpha(\infty) = L \leqslant 0$ exists. From straightforward consideration of the ODE (2.1) and its derivatives, we also conclude that $f''_\alpha(\infty) = f'''_\alpha(\infty) = 0$.

Now suppose that $f'_\alpha(\infty) = L < 0$. Then from (2.1) we have

$$\lim_{y\to\infty} f_\alpha(y) f''_\alpha(y) = \lim_{y\to\infty} \frac{f'_\alpha(y)^2}{c+1} - \lim_{y\to\infty} \frac{f'''_\alpha(y)}{c+1} = \frac{L^2}{c+1} = k_1 < 0. \tag{2.10}$$

Thus there exists a $\tilde{y} > 0$ such that

$$-1 < f'_\alpha(y) \quad \text{and} \quad f_\alpha(y) f''_\alpha(y) < \frac{k_1}{2} < 0, \quad \forall y > \tilde{y}. \tag{2.11}$$

Integrating the first inequality of (2.11) and rearranging the second we then have

$$f_\alpha(\tilde{y}) + \tilde{y} - y < f_\alpha(y) < 0 \quad \text{and} \quad f''_\alpha(y) > \frac{k_1}{2f_\alpha(y)} \quad \forall y > \tilde{y}. \tag{2.12}$$

Using the first inequality of (2.12) we can rewrite the second inequality of (2.12) as

$$f''_\alpha(y) > \frac{1}{k_2 + \frac{2(\tilde{y}-y)}{k_1}}, \quad \forall y > \tilde{y}, \tag{2.13}$$

where $k_2 = 2f_\alpha(\tilde{y})/k_1 > 0$. Integrating (2.13) we have

$$f'_\alpha(y) > f'_\alpha(\tilde{y}) + \frac{k_1}{2}\ln(k_2) - \frac{k_1}{2}\ln\left(k_2 + \frac{2(\tilde{y}-y)}{k_1}\right) \quad \forall y > \tilde{y}. \tag{2.14}$$

Since $k_1 < 0$, this implies that $f'_\alpha(y) \to +\infty$ as $y \to \infty$, contradicting $f'_\alpha(\infty) = L < 0$. Thus we must have $f'_\alpha(\infty) = L = 0$ for all $0 \leqslant f''_\alpha(0) = \alpha \leqslant (c+1)S$ giving infinitely many solutions, $f_\alpha(y)$, to the BVP.

Case 2: $f''_\alpha(0) = \alpha < 0$. To show that any value of $f''_\alpha(0) = \alpha < 0$ also results in $f'_\alpha(\infty) = 0$, we first need to show that $f'_\alpha(y)$ attains a minimum, i.e. there exists a $y_2 \in (0,\overline{y})$ such that $f''_\alpha(y_2) = 0$. Suppose not. Suppose that $f''_\alpha(y) < 0$ on $[0,\overline{y})$. Then $f_\alpha(y) \leqslant 0$ and $f'_\alpha(y) \leqslant -1$ on $[0,\overline{y})$ as well.

From the ODE (2.1) we then have

$$f'''_\alpha(y) = f'_\alpha(y)^2 - (c+1)f_\alpha(y)f''_\alpha(y) > 1 \quad \text{on } (0,\overline{y}). \tag{2.15}$$

Thus $\alpha < f''_\alpha(y) < 0$ is bounded above and below on $(0,\overline{y})$. On integration, $f'_\alpha(y)$ and $f_\alpha(y)$ are bounded above and below as well on $(0,\overline{y})$ and the solution can be extended to $y > \overline{y}$. Thus $\overline{y} = \infty$. However, $f'''_\alpha(y) > 1$ on $(0,\infty)$ implies that $f''_\alpha(y) \to +\infty$, contradicting $f''_\alpha(y) < 0$ on $[0,\overline{y})$. Thus we cannot have $f''_\alpha(y) < 0$ on $[0,\overline{y})$ and there exists a $y_2 \in (0,\overline{y})$ such that

$$f''_\alpha(y_2) = 0 \quad \text{and } f''_\alpha(y) > 0 \text{ on } (y_2,\overline{y}). \tag{2.16}$$

Arguing as in Case 1, we can show that $\overline{y} = \infty$ and $f'_\alpha(\infty;\alpha) = 0$ proving the theorem. □

Figure 1 illustrates some of the solutions from the continuum given in Theorem 1. For parameter values $c = -2$ and $S = 0$, any solution of the IVP with $f''_\alpha(0) = \alpha \leqslant (c+1)S = 0$ satisfies $f'_\alpha(\infty) = 0$ thus producing a corresponding solution to the BVP (1.1, 1.3). Figure 1 displays numerically integrated solutions for values of $f''_\alpha(0) = \alpha$ given by $0.0, -0.5, -1.0, -1.5, -2.0, -2.5, -3.0, -3.5, -4.0$. The more negative the value of α, the slower the convergence of $f'_\alpha(\infty)$ to 0. Although Theorem 1 only guarantees solutions for $\alpha \leqslant (c+1)S$, it appears from numerical integration that the actual maximal solution has a value of $(c+1)S < \alpha < (c+1)(S-1)$. (See Corollary 1 below.)

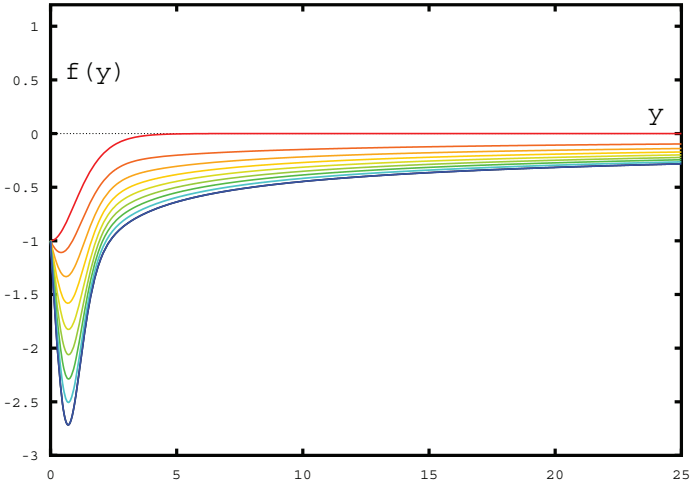


Figure 1: *Several solutions from the continuum of solutions guaranteed by Theorem 1. Parameter values:* $c = -2$, $S = 0$, *and* $f''_\alpha(0) = \alpha = 0.0, -0.5, -1.0, -1.5, -2.0, -2.5, -3.0, -3.5, -4.0$.

COROLLARY 1. *For $c \leqslant -2$ and $S \leqslant 0$, solutions of the IVP with $f''_\alpha(0) = \alpha \geqslant (c+1)(S-1)$ cannot satisfy $f'_\alpha(\infty) = 0$ and therefore do not correspond to solutions of the BVP (1.1, 1.3).*

Proof. For these values of α we will show that there exists a point $y_1 \in (0, \overline{y})$ such that $f'_\alpha(y_1) = 0$ and $f''_\alpha(y_1) > 0$. (Since $f'_\alpha(y)$ cannot have a maximum, once $f'_\alpha(y)$ becomes positive, it cannot satisfy the boundary condition $f'_\alpha(\infty) = 0$.) If such a point y_1 does not exist then $\overline{y} = \infty$ and $-1 < f'_\alpha(y) < 0$ on $(0, \infty)$. In particular, on the interval $y \in (0, 1]$, $-1 < f'_\alpha(y) < 0$ implies that $S - 1 < f_\alpha(y) < S$. Using these bounds in (2.7) gives

$$\begin{aligned} f''_\alpha(y) &= \alpha - (c+1)S - (c+1) f_\alpha(y) f'_\alpha(y) + (c+2) \int_0^y f'_\alpha(t)^2 \, dt \\ &> \alpha - (c+1)S + (c+2) \text{ on } (0,1]. \end{aligned} \tag{2.17}$$

Integrating (2.17) and evaluating at $y = 1$ implies that

$$f'_\alpha(1) > -1 + \alpha - (c+1)S + (c+2), \tag{2.18}$$

which is greater than or equal to 0 if $\alpha \geqslant (c+1)(S-1)$. Thus $f'_\alpha(y)$ must increase through 0 and since $f'_\alpha(y)$ cannot have a maximum, we cannot subsequently satisfy $f'_\alpha(\infty) = 0$. Thus values of $\alpha \geqslant (c+1)(S-1)$ do not correspond to solutions of the BVP (1.1, 1.3), proving the corollary. □

THEOREM 2. *Let $c \leqslant -2$ and $S > 0$. Then solutions of the IVP (2.1–2.4) with $f''_\alpha(0) = \alpha \leqslant (c+1)S < 0$ satisfy $f'_\alpha(\infty) = 0$. These values of α correspond to infinitely many solutions to the BVP (1.1, 1.3).*

Proof. Arguing as in the proof of Theorem 1, if $\alpha \leqslant (c+1)S < 0$, then $f'_\alpha(y) < 0$ on $(0, \overline{y})$ by (2.9). Since $f'_\alpha(y)$ starts of negative and decreasing, we first show that there exists a $y_2 \in (0, \overline{y})$ such that $f''_\alpha(y_2) = 0$. Suppose not. Then $f''_\alpha(y) < 0$ on $(0, \overline{y})$. Integrating

$$f'''_\alpha(y) + (c+1) f_\alpha(y) f''_\alpha(y) = f'_\alpha(y)^2 > 0 \text{ on } (0, \overline{y}), \tag{2.19}$$

we obtain

$$f''_\alpha(y) > \alpha \exp\left(-(c+1) \int_0^y f_\alpha(t) \, dt \right) \text{ on } (0, \overline{y}). \tag{2.20}$$

Thus

$$\alpha \exp\left(-(c+1) \int_0^y f_\alpha(t) \, dt \right) < f''_\alpha(y) < 0 \text{ on } (0, \overline{y}), \tag{2.21}$$

and we distinguish two cases. Case 1, $f_\alpha(y) > 0$ on $(0, \overline{y})$; Case 2, there exists a $y_0 \in (0, \overline{y})$ such that $f_\alpha(y_0) = 0$ with $f_\alpha(y) > 0$ on $(0, y_0)$ and $f_\alpha(y) < 0$ on $(y_0, \overline{y})$. We first show that in either case $f''_\alpha(y)$ is bounded on any finite interval $(0, \overline{y})$ and thus the solution can be continued to $\overline{y} = \infty$.

For Case 1, if $f_\alpha(y) > 0$ and $f'_\alpha(y) < 0$ on $(0,\overline{y})$, then $0 < f_\alpha(y) < S$. Thus $f''_\alpha(y)$ is bounded by (2.21) and $\overline{y} = \infty$. For Case 2, suppose there exists a y_0 with $f_\alpha(y_0) = 0$ with $f_\alpha(y) < 0$ for all $y > y_0$. From (2.1) we then have

$$f'''_\alpha(y_0) = f'_\alpha(y_0)^2 > 0. \tag{2.22}$$

We claim that $f'''_\alpha(y) > 0$ for all $y > y_0$, for if $f'''_\alpha(y_3) = 0$ at some $y_3 > y_0$, then from (2.1) we have

$$(c+1) f_\alpha(y_3) f''_\alpha(y_3) = f'_\alpha(y_3)^2, \tag{2.23}$$

which gives a contradiction since the left hand side of (2.23) is negative while the right is positive. Thus $f''_\alpha(y) < 0$ and $f'''_\alpha(y) > 0$ on $(y_0,\overline{y})$ and so $f''_\alpha(y)$ is bounded and so $\overline{y} = \infty$.

Thus, in either case, our assumption that $f''_\alpha(y) < 0$ on $(0,\overline{y})$ leads us to conclude that $\overline{y} = \infty$. We next show that we cannot have $f_\alpha(y) > 0$ on all of $(0,\infty)$. Since $f''_\alpha(y) < 0$ on $(0,\infty)$ we have that $f'_\alpha(y) < -1$ on $(0,\infty)$, which in turn implies that $f_\alpha(y)$ must become negative. Thus there exists a point y_0 such that $f_\alpha(y_0) = 0$ with $f_\alpha(y) > 0$ on $(0,y_0)$ and $f_\alpha(y) < 0$ on (y_0,∞). Thus

$$f'''_\alpha(y) = f'_\alpha(y)^2 - (c+1) f_\alpha(y) f''_\alpha(y) > 0 \text{ on } (y_0,\infty), \tag{2.24}$$

which in turn implies that $f''_\alpha(y) \to +\infty$, contradicting $f''_\alpha(y) < 0$ for all $y > 0$. Thus the assumption that $f''_\alpha(y) < 0$ on $(0,\overline{y})$ leads to a contradiction, thus there exists a $y_2 \in (0,\overline{y})$ such that $f''_\alpha(y_2) = 0$ with $f''_\alpha(y) > 0$ on $(y_2,\overline{y})$. Also, by (2.9) we have that $f'_\alpha(y) < 0$ on $(y_2,\overline{y})$.

Since $f'_\alpha(y) < 0$ and $f''_\alpha(y) > 0$ on $(y_2,\overline{y})$, $f'_\alpha(y)$ is bounded and then so is $f_\alpha(y)$. Thus $f''_\alpha(y)$ is also bounded by (2.7) and consequently the solution can be continued to $\overline{y} = \infty$. Also, $f'_\alpha(\infty) = L \leqslant 0$ exists, and to use the argument of Theorem 1 to show that $L = 0$, we first need to show that $f_\alpha(y)$ ultimately becomes negative. If not, then $f_\alpha(y) > 0$ on $(0,\infty)$ and from (2.1) we have

$$f'''_\alpha(y) = f'_\alpha(y)^2 - (c+1) f_\alpha(y) f''_\alpha(y) > 0 \text{ on } (y_2,\infty). \tag{2.25}$$

Choose $\hat{y} > y_2$. Then

$$f''_\alpha(y) > f''_\alpha(\hat{y}) = k > 0, \quad \text{on } (\hat{y},\infty). \tag{2.26}$$

Integrating (2.26) from $\hat{y}$ to $y > \hat{y}$ we have $f'_\alpha(y) > f'_\alpha(\hat{y}) + f''_\alpha(\hat{y})(y - \hat{y})$ for all $y > \hat{y}$, which implies that $f'_\alpha(y) \to +\infty$ as $y \to \infty$, contradicting $f'_\alpha(y) < 0$ for all $y > 0$. Thus there exists a point y_0 such that $f_\alpha(y_0) = 0$ with $f_\alpha(y) < 0$ for $y > y_0$. The argument of Theorem 1 now implies that $f'_\alpha(\infty) = L = 0$ and the theorem is proved. □

Numerically integrated solutions illustrating the results of Theorem 2 are given in Figure 2. For parameter values $c = -2$ and $S = 1$, Theorem 2 guarantees solutions to the BVP for all values of $f''_\alpha(0) = \alpha$ with $\alpha \leqslant (c+1)S = -1$. Figure 2 presents such solutions for values of $f''_\alpha(0) = \alpha = -1.0,\ -1.5,\ -2.0,\ -2.5,\ -3.0,\ -3.5,\ -4.0,\ -4.5,\ -5.0$. Again, it appears that the more negative the value of α, the slower the

convergence of $f'_\alpha(\infty)$ to 0. As in the case of Theorem 1, it appears that the maximal solution is given by a value of α larger than the bound given in Theorem 2. This maximal value of α appears to lie in the range $(c+1)S < \alpha < -(c+1)$. (See Corollary 2 below.)

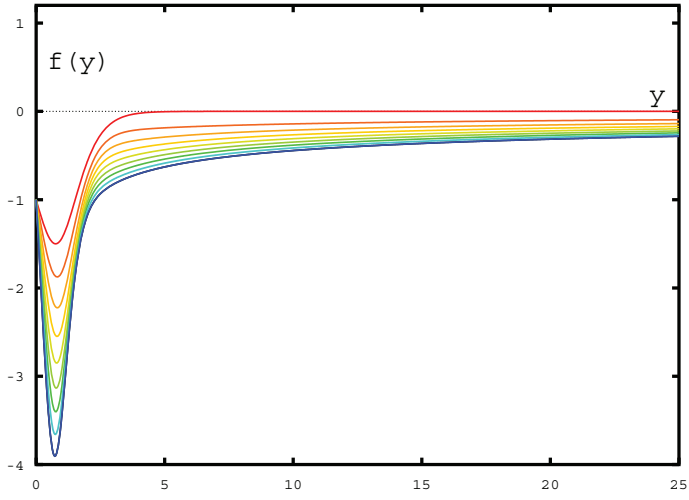


Figure 2: *Several solutions from the continuum of solutions guaranteed by Theorem 2. Parameter values:* $c = -2$, $S = 1$, *and* $f''_\alpha(0) = \alpha = -1.0,\ -1.5,\ -2.0,\ -2.5,\ -3.0,\ -3.5,\ -4.0,\ -4.5,\ -5.0$.

COROLLARY 2. *For* $c \leqslant -2$ *and* $S > 0$, *solutions to the IVP with* $f''_\alpha(0) = \alpha \geqslant -(c+1)$ *cannot satisfy* $f'_\alpha(\infty) = 0$.

Proof. We will assume a solution exists and derive a contradiction. If $f''_\alpha(0) = \alpha \geqslant -(c+1) > 0$, then $-1 < f'_\alpha(y) < 0$ on $(0,1) \subset (0,\infty)$ and the argument of Corollary 1 implies that $f_\alpha(1) \geqslant 0$, and thus $f'_\alpha(y)$ cannot satisfy $f'_\alpha(\infty) = 0$. □

THEOREM 3. *If* $-2 < c \leqslant -3/2$ *and* $S < 0$, *then any solution of the IVP* (2.1–2.4) *with* $f''_\alpha(0) = \alpha \leqslant (c+1)S + 1/(2S)$ *satisfies* $f'_\alpha(\infty) = 0$. *These solutions correspond to infinitely many solutions to the BVP* (1.1, 1.3).

Proof. First we show that for the stated values of α we have $f'_\alpha(y) < 0$ on $(0,\overline{y})$. Suppose not. Suppose there exists a first $y_1 \in (0,\overline{y})$ such that $f'_\alpha(y_1) = 0$ with $f''_\alpha(y_1) > 0$. Multiply the ODE (2.1) by $f_\alpha(y)$ and integrate from 0 to y_1 to obtain

$$(c+1)S^2 + \frac{1}{2} - S\alpha = (2c+3)\int_0^{y_1} f_\alpha(t) f'_\alpha(t)^2\,dt - f_\alpha(y_1) f''_\alpha(y_1) > 0, \qquad (2.27)$$

which gives a contradiction since $\alpha \leqslant (c+1)S + 1/(2S)$ implies that the left hand side of (2.27) is less than or equal to 0. Thus $f'_\alpha(y) < 0$ on $(0, \overline{y})$ and the existence proof goes through as in Theorems 1 and 2. Also, the result of Corollary 1 also holds for the case $-1 < c \leqslant -3/2$. □

THEOREM 4. *Let* $-1 < c \leqslant 0$ *and* $S \geqslant 2/(c+1)$. *Let*

$$0 < \alpha_1 = \frac{(c+1)S - \sqrt{(c+1)^2S^2 - 4}}{2} \leqslant \alpha_2 = \frac{(c+1)S + \sqrt{(c+1)^2S^2 - 4}}{2}. \quad (2.28)$$

Then any solution of the IVP (2.1–2.4) *with* $f''_\alpha(0) = \alpha$ *where* $\alpha_1 \leqslant \alpha \leqslant \alpha_2$ *satisfies* $f'_\alpha(\infty) = 0$. *When* $S = 2/(c+1)$ *this interval collapses to a single point* $\alpha_1 = \alpha_2 = 1$ *corresponding to one solution to the BVP* (1.1, 1.3). *When* $S > 2/(c+1)$, *the values of* α *in this interval correspond to infinitely many solutions to the BVP* (1.1, 1.3).

Proof. Note that $f'_\alpha(0) = -1 < 0$ so that $f'_\alpha(y) < 0$ for small $y > 0$. We first show that for $f''_\alpha(0) = \alpha$ with $\alpha_1 \leqslant \alpha \leqslant \alpha_2$, $f'_\alpha(y) < 0$ cannot increase through zero. To show this we will assume $f'_\alpha(y)$ does increase through zero and derive a contradiction. If $f'_\alpha(y)$ were to increase through zero at some first $y_1 > 0$, then we note that as we have seen before we must have $f''_\alpha(y_1) > 0$.

Next consider the function $r(y) = f''_\alpha(y)/f'_\alpha(y)$. Note that the preceding paragraph implies that

$$\lim_{y \to y_1^-} r(y) = -\infty. \quad (2.29)$$

Further, $r(y)$ satisfies

$$r'(y) + r(y)^2 + (c+1)f_\alpha(y)r(y) - f'_\alpha(y) = 0, \quad (2.30)$$

with

$$r(0) = -\alpha, \quad (2.31)$$

and thus

$$r'(0) = -\alpha^2 + (c+1)S\alpha - 1. \quad (2.32)$$

Note that for $\alpha_1 \leqslant \alpha \leqslant \alpha_2$ we have $r(0) < 0$ and $r'(0) \geqslant 0$.

Next, differentiating (2.30) we obtain

$$r''(y) + (2r(y) + (c+1)f_\alpha(y))\, r'(y) + cf''_\alpha(y) = 0. \quad (2.33)$$

Since $f''_\alpha(0) = \alpha > 0$ and $f''_\alpha(y)$ cannot decrease through zero, we have that $f''_\alpha(y) > 0$ for as long as the solution to the IVP exists. (Again, at this point, we do not yet know if $f_\alpha(y)$ exists for all of $0 \leqslant y < \infty$.) Thus since $-1 < c \leqslant 0$ we have

$$r''(y) + (2r(y) + (c+1)f_\alpha(y))\, r'(y) = -cf''_\alpha(y) \geqslant 0, \quad (2.34)$$

or on integration

$$r'(y) = \left[r'(0) - \int_0^y cf''_\alpha(t) e^{\int_0^t 2r(s) + (c+1)f_\alpha(s)\, ds}\, dt \right] e^{-\int_0^t 2r(s) + (c+1)f_\alpha(s)\, ds} \geqslant 0. \quad (2.35)$$

Thus $r(y)$ is nondecreasing on its entire interval of existence, contradicting (2.29). This contradiction implies that $f'_\alpha(y)$ cannot increase through zero. Thus $f'_\alpha(y) < 0$ on its entire interval of existence.

We next show that the solution to the IVP $f_\alpha(y)$ does indeed exist for all $y > 0$. As noted earlier, $f''_\alpha(y) > 0$ on $(0,\overline{y})$. This along with the fact that $f'_\alpha(y)$ cannot increase through zero implies that $f'_\alpha(y)$ is bounded on its maximal interval of existence. Thus, on integration, $f_\alpha(y)$ is bounded as well. Finally, from (2.7) we then see that $f''_\alpha(y)$ is bounded, and so the solution of the IVP cannot cease to exist at a finite value of y, so that $\overline{y} = \infty$.

Since $f'_\alpha(y) < 0$ and $f''_\alpha(y) > 0$ for all $y > 0$, $f'_\alpha(\infty) = L \leqslant 0$ exists. If $f'_\alpha(y) \to L < 0$ then $f_\alpha(y) \to -\infty$ and thus from (2.1) and $-1 < c \leqslant 0$ we have

$$f'''_\alpha(y) = f'_\alpha(y)^2 - (c+1) f_\alpha(y) f''_\alpha(y) > 0 \tag{2.36}$$

for large y. This would imply, since $f''_\alpha(y) > 0$, that $f'_\alpha(y) > 0$ for y large contradicting $f'_\alpha(y) \to L < 0$. Therefore $f'_\alpha(\infty;\alpha) = 0$ for $\alpha_1 \leqslant \alpha \leqslant \alpha_2$ corresponding to infinitely many solutions to the BVP (1.1, 1.3). □

As we will discuss in Section 4, when $-1 < c < 0$ it appears from numerical integration that the actual continuum of solutions is larger than the interval $\alpha_1 \leqslant \alpha \leqslant \alpha_2$ listed in Theorem 4.

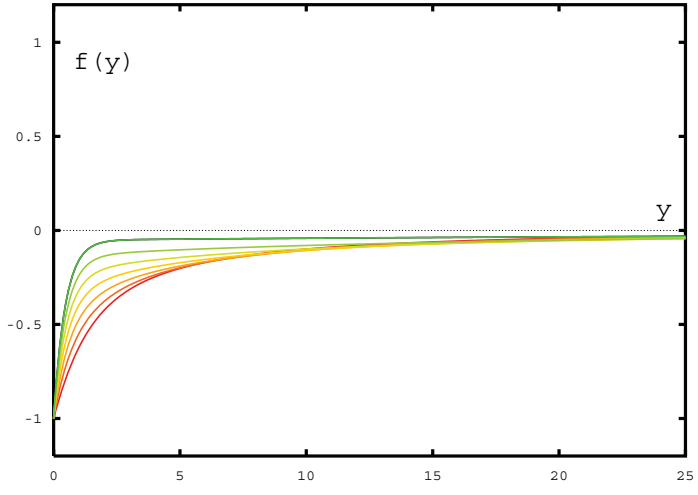


Figure 3: *Several solutions from the continuum of solutions guaranteed by Theorem 4. Parameter values:* $c = -0.5$, $S = 5.0$, *and* $f''_\alpha(0) = \alpha = 0.5,\ .75,\ 1.0,\ 1.25,\ 1.5,\ 1.75,\ 2.0$.

Figure 3 graphs numerically integrated solutions from the continuum given in Theorem 4 for parameter values $c = -0.5$ and $S = 5.0$. For these parameter values, any value of $f''_\alpha(0) = \alpha$ with $\alpha_1 = 0.5 \leqslant \alpha \leqslant 2.0 = \alpha_2$ gives a solution to the BVP. Solutions in the figure are given for $\alpha = 0.5, 0.75, 1.0, 1.25, 1.5, 1.75, 2.0$. As with the

previous Theorems, smaller values of α give slower convergence of $f'_\alpha(\infty)$ to 0. Also, the minimal value of α giving a solution appears to be smaller than the bound α_1 given in the Theorem, and the maximal value of α giving a solution appears to be larger than α_2. This is discussed in more detail in Section 4.

THEOREM 5. *Let*

$$c \geqslant 1 \quad \textit{and} \quad S > \frac{\sqrt{2(c+2)}}{\sqrt{3}(c+1)} > 0, \tag{2.37}$$

then in order to have $f'_\alpha(\infty) = 0$, *we must have*

$$0 < (c+1)S - \sqrt{(c+1)^2S^2 - 2(c+1)/3} = \underline{\alpha} < \alpha < \overline{\alpha} = (c+1)S. \tag{2.38}$$

Proof. We will assume that $f'_\alpha(\infty) = 0$ and derive a series of bounds on $f''_\alpha(0) = \alpha$, ultimately arriving at (2.38). First note that $f_\alpha(y) > 0$ for all $y > 0$. If not, since $f'_\alpha(y) < 0$ for all $y > 0$, there exists a y_0 such that $f_\alpha(y_0) = 0$ with $f_\alpha(y) < 0$ on (y_0, ∞). From the ODE (2.1) we then have $f'''_\alpha(y) = f'_\alpha(y)^2 - (c+1)f_\alpha(y)f''_\alpha(y) > 0$ for y large and thus $f'_\alpha(y)$ cannot approach zero from below giving a contradiction of $f'_\alpha(\infty) = 0$. Thus

$$f_\alpha(y) > 0 \quad \forall y > 0. \tag{2.39}$$

Next, we claim that $f'''_\alpha(0) < 0$. The argument for this will involve two cases, $c = 1$ and $c > 1$.

If $c = 1$, then differentiating (2.1) gives

$$f^{(4)}_\alpha(y) + 2f_\alpha(y)f'''_\alpha(y) = 0, \tag{2.40}$$

which implies that

$$f'''_\alpha(y) = f'''_\alpha(0) \exp\left(-2\int_0^y f_\alpha(t)\,dt\right). \tag{2.41}$$

If $f'''_\alpha(0) > 0$, then $f'''_\alpha(y) > 0$ for all $y > 0$ and we cannot have $f'_\alpha(\infty) = 0$. If $f'''_\alpha(0) = 0$, then $f_\alpha(y) = S - y + y^2/(4S)$, which again does not satisfy $f'_\alpha(\infty) = 0$. Thus we conclude that $f'''_\alpha(0) < 0$.

Next consider the case $c > 1$. If $f'''_\alpha(0) \geqslant 0$, then $f'''_\alpha(y)$ is positive for y just greater than zero. In order to satisfy $f'_\alpha(\infty) = 0$, there must exist a first y_3 with $f_\alpha(y_3) > 0$, $f'_\alpha(y_3) < 0$, $f''_\alpha(y_3) > 0$, $f'''_\alpha(y_3) = 0$ and $f^{(4)}_\alpha(y_3) \leqslant 0$. However, differentiating (2.1) and evaluating at y_3 we have

$$f^{(4)}_\alpha(y_3) = (1-c)f'_\alpha(y_3)f''_\alpha(y_3) > 0, \tag{2.42}$$

giving a contradiction. Thus for a solution we must have $f'''_\alpha(0) = 1 - (c+1)S\alpha < 0$ or

$$\alpha > \frac{1}{(c+1)S}. \tag{2.43}$$

Thus for any solution with $c \geqslant 1$ and $S > 0$ we have

$$f_\alpha(y) > 0,\ \ f'_\alpha(y) < 0,\ \ f''_\alpha(y) > 0,\ \text{ and }\ f'''_\alpha(y) < 0,\ \ \forall y > 0. \tag{2.44}$$

From these bounds and (2.1) we further conclude that

$$f_\alpha(\infty) = l \geqslant 0 \text{ exists and } f'_\alpha(\infty) = f''_\alpha(\infty) = f'''_\alpha(\infty) = 0. \tag{2.45}$$

To obtain the upper bound in (2.38), integrate the ODE (2.1) from 0 to ∞ to obtain

$$(c+2)\int_0^\infty f'_\alpha(t)^2\,dt = (c+1)S - \alpha > 0, \tag{2.46}$$

from which it follows that

$$\alpha < (c+1)S = \overline{\alpha}. \tag{2.47}$$

For the lower bound in (2.38), first differentiate (2.1) to obtain

$$f^{(4)}_\alpha(y) + (c+1)f_\alpha(y)f'''_\alpha(y) + (c-1)f'_\alpha(y)f''_\alpha(y) = 0. \tag{2.48}$$

Multiply (2.48) by $f'_\alpha(y)$ and integrate from 0 to ∞ and use (2.44) to obtain

$$(c+1)\int_0^\infty f_\alpha(t)f'_\alpha(t)f'''_\alpha(t)\,dt = -\frac{1}{2}\alpha^2 + (c+1)S\alpha - \frac{1}{3}(c+2) > 0, \tag{2.49}$$

or, after some rearrangement,

$$p(\alpha) = \alpha^2 - 2(c+1)S\alpha + \frac{2}{3}(c+2) < 0. \tag{2.50}$$

In order for the inequality (2.50) to have a nonempty solution set, the discriminant must be positive, which implies that

$$S > \frac{\sqrt{2(c+2)}}{\sqrt{3}(c+1)}. \tag{2.51}$$

When (2.51) holds we then have

$$\begin{aligned} \underline{\alpha} &= (c+1)S - \sqrt{(c+1)^2S^2 - 2(c+1)/3} \\ &< \alpha < (c+1)S + \sqrt{(c+1)^2S^2 - 2(c+1)/3} \end{aligned} \tag{2.52}$$

Combining (2.47) and (2.52) gives the desired result. Note also, that under the assumptions of the Theorem, the bound (2.43) is subsumed into (2.52). $\square$

3. Nonexistence results

In each of the following theorems, we assume a solution to the IVP (2.1–2.4) satisfying $f'_\alpha(\infty) = 0$ exists and then derive a contradiction. This will correspond to parameter values for which the BVP (1.1, 1.3) has no solution.

THEOREM 6. *If $c \geqslant -1$ and $S \leqslant 0$, then no solution to the IVP (2.1–2.4) can satisfy $f'_\alpha(\infty) = 0$. Consequently, the BVP (1.1, 1.3) has no solution.*

Proof. Using the bounds (2.5) and (2.6) (which apply to any solution when $S \leqslant 0$) and the ODE (2.1) we have

$$f'''_\alpha(y) = f'_\alpha(y)^2 - (c+1) f_\alpha(y) f''_\alpha(y) > 0, \quad \forall y > y_2, \tag{3.1}$$

from which we conclude that $f'_\alpha(y)$ cannot approach 0 from below. Thus the BVP has no solution. □

THEOREM 7. *If*

$$c \geqslant 1 \quad \text{and} \quad 0 < S \leqslant \frac{\sqrt{2(c+2)}}{\sqrt{3}(c+1)}, \tag{3.2}$$

then no solution to the IVP (2.1–2.4) can satisfy $f'_\alpha(\infty) = 0$. Thus, the BVP (1.1, 1.3) has no solution.

Proof. From the proof of Theorem 4, if $S \leqslant \sqrt{2(c+2)}/(\sqrt{3}(c+1))$, then $p(\alpha)$ defined in (2.50) satisfies $p(\alpha) \geqslant 0$ for all α and (2.49) cannot be satisfied. Thus the BVP has no solution. □

THEOREM 8. *If $1/2 \leqslant c < 1$ and $0 < S \leqslant 1/(c+1)$, then no solution to the IVP (2.1–2.4) can satisfy $f'_\alpha(\infty) = 0$. Consequently, the BVP (1.1, 1.3) has no solution.*

Proof. We first show that we cannot have $f'''_\alpha(0) \leqslant 0$. If $f'''_\alpha(0) = 1 - (c+1)S\alpha \leqslant 0$, then $f'''_\alpha(y) < 0$ for y just greater than 0. In fact, we must then have $f'''_\alpha(y) < 0$ for all $y > 0$, for if $f'''_\alpha(y)$ were to increase through 0 at some first y_3, then from (2.48) we have

$$f^{(4)}_\alpha(y_3) = (1-c) f'_\alpha(y_3) f''_\alpha(y_3) < 0, \tag{3.3}$$

contradicting $f^{(4)}_\alpha(y_3) \geqslant 0$ when $f'''_\alpha(y)$ increases through 0. Thus if $f'''_\alpha(0) \leqslant 0$, then $f'''_\alpha(y) < 0$ for all $y > 0$. Continuing with the assumption that $f'''_\alpha(0) \leqslant 0$ we obtain the bound

$$f'''_\alpha(0) = 1 - (c+1)S\alpha \leqslant 0 \ \text{ or } \ \alpha \geqslant \frac{1}{(c+1)S}. \tag{3.4}$$

However, integrating the ODE (2.1) from 0 to ∞ gives

$$(c+2)\int_0^\infty f'_\alpha(t)^2\, dt = (c+1)S - \alpha > 0 \ \text{ or } \ \alpha < (c+1)S. \tag{3.5}$$

Combining (3.4) and (3.5) gives

$$\frac{1}{(c+1)S} \leqslant \alpha < (c+1)S, \tag{3.6}$$

which gives a contradiction when $S \leqslant 1/(c+1)$.

Thus, if a solution exists, we must have

$$f'''_\alpha(0) = 1-(c+1)S\alpha > 0 \text{ or } \alpha < \frac{1}{(c+1)S}. \tag{3.7}$$

But in order to satisfy $f'_\alpha(\infty) = 0$ we must have $f'''_\alpha(y) < 0$ for y large with $f'''_\alpha(\infty) = 0$. Thus, $f^{(4)}_\alpha(y)$ must at some point y_5 have a positive maximum with $f_\alpha(y_5) > 0$, $f'_\alpha(y_5) < 0$, $f''_\alpha(y_5) > 0$, $f'''_\alpha(y_5) < 0$, $f^{(4)}_\alpha(y_5) > 0$, $f^{(5)}_\alpha(y_5) = 0$, $f^{(6)}_\alpha(y_5) \leqslant 0$. (As in the proof of Theorem 4, if $f_\alpha(0) = S > 0$, then $f > 0$ for all $y > 0$.) However, differentiating (2.1) three times and evaluating at such a point we have

$$f^{(6)}_\alpha(y_5) = 2(1-2c)f''_\alpha(y_5)f'''_\alpha(y_5) - (3c+1)f'_\alpha(y_5)f^{(4)}_\alpha(y_5) > 0, \tag{3.8}$$

giving a contradiction. Thus if $1/2 \leqslant c < 1$ and $0 < S \leqslant 1/(c+1)$, the BVP has no solution. □

THEOREM 9. *If $-1 < c < 1$ and $0 < S \leqslant 1/\sqrt{2(c+1)}$, then no solution to the IVP (2.1–2.4) can satisfy $f'_\alpha(\infty) = 0$. Thus, the BVP (1.1, 1.3) has no solution.*

Proof. Integrating the ODE (2.1) from 0 to ∞ gives

$$(c+2)\int_0^\infty f'_\alpha(t)^2\,dt = (c+1)S - \alpha > 0, \tag{3.9}$$

from which we conclude that $f''_\alpha(0) = \alpha < (c+1)S$. In fact, whenever $f'_\alpha(y)$ is in the range $-1 < f'_\alpha(y) < 0$, we must have $f''_\alpha(y) < (c+1)S$ as well. To see this, choose any $\tilde{y} > 0$ with $-1 < f'_\alpha(\tilde{y}) < 0$ and integrate (2.1) from $\tilde{y}$ to ∞ to obtain

$$(c+2)\int_{\tilde{y}}^\infty f'_\alpha(t)^2\,dt = -(c+1)f_\alpha(\tilde{y})f'_\alpha(\tilde{y}) - f''_\alpha(\tilde{y}). \tag{3.10}$$

Using the facts that $-1 < f'_\alpha(\tilde{y}) < 0$ and $0 < f_\alpha(\tilde{y}) < S$ we conclude

$$0 < (c+2)\int_{\tilde{y}}^\infty f'_\alpha(t)^2\,dt = -(c+1)f_\alpha(\tilde{y})f'_\alpha(\tilde{y}) - f''_\alpha(\tilde{y}) < (c+1)S - f''_\alpha(\tilde{y}), \tag{3.11}$$

which implies that $f''_\alpha(\tilde{y}) < (c+1)S$.

If $f''_\alpha(0) = \alpha \geqslant 0$, let $\hat{y} = 0$. If $f''_\alpha(0) = \alpha < 0$, let $\hat{y}$ be the point where $f'_\alpha(y)$ increases through -1. Thus $f'_\alpha(\hat{y}) = -1$ and $-1 < f'_\alpha(y) < 0$ for all $y > \hat{y}$. Since $f''_\alpha(y) < (c+1)S$ for all $y > \hat{y}$, we have

$$f'_\alpha(y) < -1 + (s+1)S(y-\hat{y}), \quad \forall y > \hat{y}. \tag{3.12}$$

Thus

$$f_\alpha(y) < S - (y-\hat{y}) + \frac{1}{2}(c+1)S(y-\hat{y})^2, \quad \forall y > \hat{y}. \tag{3.13}$$

Evaluating (3.13) at $y = \hat{y} + 1/[(c+1)S]$ we have

$$f_\alpha(\hat{y} + 1/[(c+1)S]) < S - \frac{1}{2(c+1)S}, \tag{3.14}$$

which implies that $f_\alpha(\hat{y} + 1/[(c+1)S]) \leqslant 0$ if $S \leqslant 1/\sqrt{2(c+1)}$, contradicting the fact that $f_\alpha(y) > 0$ for all $y > 0$. Thus no solution exists if $-1 < c < 1$ and $S \leqslant 1/\sqrt{2(c+1)}$. □

COROLLARY 3. *For $c = -1$ and any $S \in \mathbb{R}$, then no solution to the IVP (2.1–2.4) can satisfy $f'_\alpha(\infty) = 0$. Thus, the BVP (1.1, 1.3) has no solution.*

Proof. In this case $f'''_\alpha(y) = f'_\alpha(y)^2 \geqslant 0$ and $f'_\alpha(y)$ cannot approach 0 from below. □

THEOREM 10. *If $-3/2 \leqslant c < -1$ and $S \geqslant 0$, then no solution to the IVP (2.1–2.4) can satisfy $f'_\alpha(\infty) = 0$. Thus, the BVP (1.1, 1.3) has no solution.*

Proof. For large y we must have $f'_\alpha(y) < 0$, $f''_\alpha(y) > 0$ and $f'''_\alpha(y) < 0$. This last is true since $f'''_\alpha(y)$ cannot increase through 0, for if $f'''_\alpha(y) = 0$ and $f^{(4)}_\alpha(y) \geqslant 0$, (2.48) implies

$$f^{(4)}_\alpha(y) = (1-c) f'_\alpha(y) f''_\alpha(y) < 0, \tag{3.15}$$

giving a contradiction. Thus, for large y,

$$f_\alpha(y) = \frac{f'_\alpha(y)^2 - f'''_\alpha(y)}{(c+1)f''_\alpha(y)} < 0. \tag{3.16}$$

But $f_\alpha(0) = S \geqslant 0$, so there exists a $y_0 > 0$ such that $f_\alpha(y_0) = 0$ with $f_\alpha(y) < 0$ for all $y > y_0$. Multiplying the ODE (2.1) by $f_\alpha(y)$ and integrating from y_0 to ∞ gives

$$\frac{1}{2} f'_\alpha(y_0)^2 + (c+1) f_\alpha(\infty)^2 f'_\alpha(\infty) - (2c+3) \int_{y_0}^{\infty} f_\alpha(t) f'_\alpha(t)^2 \, dt = 0. \tag{3.17}$$

However, each term on the left side of (3.17) is non-negative, with $f'_\alpha(y_0)^2/2$ strictly positive, giving a contradiction. Thus no solution exists. □

The existence and nonexistence results of the previous two sections are depicted graphically in Figure 4.

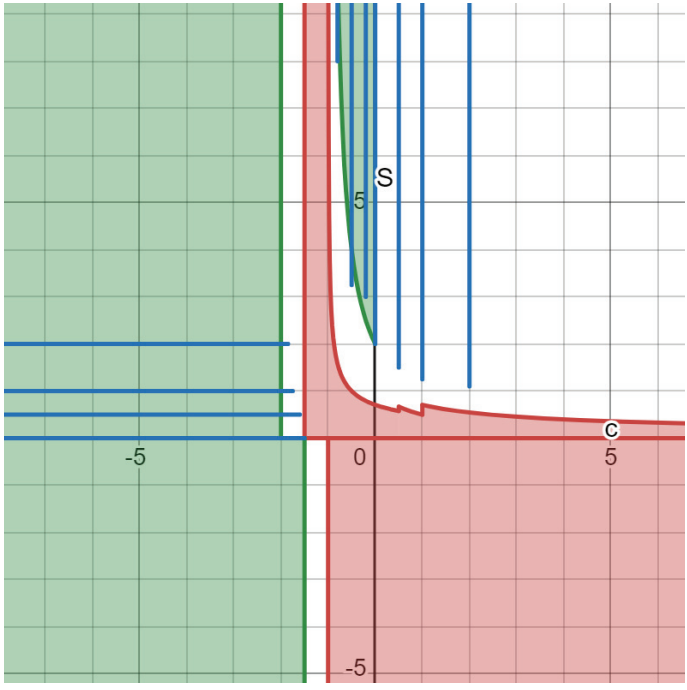


Figure 4: *Graph of parameter space with c on the horizontal axis and S on the vertical, illustrating regions of existence (green) and nonexistence (red) of solutions to the BVP. The blue vertical and horizontal lines show parameter values for which Merkin and Pop [6] find numerical solutions to the BVP.*

4. Discussion and open questions

The results here confirm and expand on many of the conjectures made by Merkin and Pop [6] for the BVP (1.1, 1.3). The present work also gives a more complete picture of the true nature of the solution set. Specifically, whereas Merkin and Pop [6] report a unique solution for values $c \leqslant -2$, we prove the existence of a continuum of solutions given by solutions of the IVP (2.1–2.4) where $f''_\alpha(0) = \alpha$ for any $\alpha \in (-\infty, (c+1)S]$. Also, Merkin and Pop [6] report dual solutions for $c > -1$ and $S > S_{crit}(c)$. Here for $-1 < c \leqslant 0$ we again prove the existence of a continuum of solutions given by $f''_\alpha(0) = \alpha$ for $\alpha \in [\alpha_1, \alpha_2]$ where α_1 and α_2 are defined in (2.28). The reason for these discrepancies seems fairly straightforward. The numerical procedure employed by Merkin and Pop is most likely identifying the extremal solutions on the continuum of solutions. For $c \leqslant -2$, their numerical results likely found the upper finite boundary on the interval $\alpha \in (-\infty, (c+1)S]$. Likewise, for $-1 < c \leqslant 0$, they find the finite endpoints of the bounded interval of existence $[\alpha_1, \alpha_2]$, accounting for their dual solutions found in this case.

However, we note for $-1 < c < 0$, $S \geqslant 2/(c+1)$ that while Theorem 4 proves existence of solutions for $\alpha \in [\alpha_1, \alpha_2]$, the true interval of existence appears to be larger. Through numerical integration we see that for these parameter values there is a larger interval $\alpha_{\min} < \alpha_1 \leqslant \alpha \leqslant \alpha_2 < \alpha_{\max}$ on which solutions exist. For example, when $-1 < c < 0$ and $S = 2/(c+1)$, we then have $\alpha_1 = \alpha_2 = 1$ and the interval $[\alpha_1, \alpha_2]$ collapses to a single point with $f''_\alpha(0) = 1$. However, numerical integration indicates that solutions exist for more than the single value $\alpha = 1$. The range of values of $f''_\alpha(0) = \alpha$ which give solutions to the BVP for several values of c with $S = 2/(c+1)$ is given in Table 1. The numerical solutions were calculated using the fourth order Runge-Kutta method with a step size of .001 on the interval $0 < y < y_{\max} = 1000$.

We also note that for the case $c = 0$, the results of Theorem 4 are optimal. For this case it is proven in [7] that no solution exists for $S < 2$, a unique solution exists for $S = 2$, and a continuum of solutions exists with $\alpha_1 \leqslant \alpha \leqslant \alpha_2$ for $S > 2$. Further, values of α outside this interval do not give solutions to the BVP.

Existence of a solution for the case $c > 0$ remains an open question. The major stumbling block in extending the proof of Theorem 4 to the case $c > 0$ is the change in sign of the term $cf''_\alpha(y)$ in equation (2.33). Numerically it does appear that there is an $S_{crit}(c)$ such that solutions do exist for $c > 0$ and $S > S_{crit}(c)$.

Given the results of this work, the natural conjecture would be that $S_{crit}(c) = 2/(c+1)$ for all $c > -1$. However, numerical integration of the problem suggests that $S_{crit}(c) < 2/(c+1)$ for $-1 < c < 0$ and $S_{crit}(c) > 2/(c+1)$ for $c > 0$. The exact value of S_{crit} remains and open question.

c	$\alpha_{\min}$	$\alpha_1 = \alpha_2$	$\alpha_{\max}$
-0.1	0.69	1	1.24
-0.3	0.28	1	1.41
-0.5	-0.30	1	1.52
-0.7	-1.78	1	1.61
-0.9	-5.20	1	1.69

Table 1: *Numerically calculated interval of existence of* $f''_\alpha(0) = \alpha$ *for various values of* c *when* $S = 2/(c+1)$.

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