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SHOOTING METHOD USING DIFFERENCE EQUATIONS

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Abstract. In this paper, we introduce an alternative fixed-point method to the classical shooting technique. Our approach solves a sequence of discrete initial value problems, which can be used to approximate the solution of a discrete boundary value problem, which in turn approximates the solution of a continuous boundary value problem, complete with error bounds. The key to the technique is the use of discrete initial value problems which are trivial to solve to approximate the trajectory of a solution of a continuous boundary value problem then using difference equations to approximate a solution. To demonstrate the method, we apply it to the right focal boundary value problem.

1. Introduction

The shooting method is a numerical technique for approximating solutions to boundary value problems by solving corresponding initial value problems. These initial value problems are typically approximated using numerical integration schemes, with iterative adjustments to the initial conditions. Foundational discussions of the method can be found in [5, 6, 8], while more recent advancements are presented in [2, 9]. Despite ongoing efforts to refine the classical approach, most improvements have focused on approximating the continuous initial value problem directly. In contrast, this paper proposes a different strategy: instead of applying the shooting method to the differential equation itself, we apply it to an associated difference equation. The trajectory of the resulting solution of the discrete initial value problem is then used to approximate the solution of the original continuous boundary value problem. This approach offers a key advantage over traditional techniques, as discrete initial-value problems are generally easier to solve.

For integers a and b with $a < b$, define the set

$$D_{[a,b]} = \{x \in \mathbb{Z} : a \leqslant x \leqslant b\} = \{a, a+1, \cdots, b-1, b\}.$$

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Let N be a natural number and define the Banach space

$$B := \{y | y : D_{[0,N+2]} \to \mathbb{R} \text{ and } y(0) = 0\}$$

which is endowed with the norm,

$$\|y\|_{\ell^\infty} = \max_{k \in D_{[0,N+2]}} |y(k)|.$$

The shooting method using difference equations begins with an application of the shooting method to a discrete boundary value problem, and we will illustrate the technique on right focal boundary value problems. See [7, 10] for an introductory treatment of the shooting method applied to discrete boundary value problems. The discrete right focal boundary value problem is given by

$$\Delta^2 y(t-1) + f(y(t)) = 0, \quad t \in D_{[1,N+1]}, \tag{1.1}$$

$$y(0) = \Delta y(N+1) = 0. \tag{1.2}$$

The corresponding discrete initial value problem to (1.1), (1.2) has initial conditions $y(0) = 0$ and $y(1) - y(0) = (\Delta y)(0) = y(1) =: y_1 \in \mathbb{R}$. With this in mind, let m be a non-negative real number and let $y_m \in B$ be the unique solution of the initial value problem given by

$$\Delta^2 y(t-1) + f(y(t)) = 0, \quad t \in D_{[1,N+1]}, \tag{1.3}$$

$$y(0) = 0 \quad \text{and} \quad y(1) = \frac{m}{N+2}, \tag{1.4}$$

which is recursively defined by

$$y_m(t+2) = 2y_m(t+1) - y_m(t) - f(y_m(t+1)), \ \ y_m(0) = 0, \ \text{ and } \ y_m(1) = \frac{m}{N+2}$$

for integers $t \in D_{[0,N]}$. One can show that

$$y_m(t) = \frac{tm}{N+2} - \sum_{s=1}^{t} (t-s) f(y_m(s)), \quad t \in D_{[0,N+2]}, \tag{1.5}$$

is the unique solution of the initial value problem (1.3), (1.4), where we employ the convention that

$$\sum_{s=i}^{j} g(s) = 0$$

for any function g when i and j are integers with $i > j$. In (3.5) and in Theorem 4 below, we will see why we have taken $y_m(1) = m/(N+2)$. In short, this is necessary for the approximation of a solution of the discrete boundary value problem to lead to an approximation of a solution to the right focal continuous boundary value problem.

Note that if m^* is a nonnegative real number with

$$y_{m^*}(N+2) = y_{m^*}(N+1),$$

then y_{m^*} is a solution of the boundary value problem (1.1), (1.2).

The first step in applying the shooting method to approximate the solution of a discrete boundary value problem involves solving a discrete initial value problem, which can be efficiently computed through iteration. The objective is to determine a value of m such that the solution to the initial value problem defined by equations (1.3) and (1.4) also satisfies the boundary conditions of the discrete boundary value problem (1.1) and (1.2). The solution takes the form

$$y_m(t) = \frac{tm}{N+2} - \sum_{s=1}^{t}(t-s)f(y_m(s)),$$

which naturally leads to a fixed-point formulation used to approximate the solution of the corresponding continuous boundary value problem. In this approach, the shooting method is first applied to a discrete boundary value problem via discrete initial value problems. The trajectory of the resulting discrete solution and the discrete solution is then used to approximate the continuous problem, with a bisection scheme employed to establish error bounds.

2. Preliminaries

For a non-negative real number m, define the function

$$h(m) = y_m(N+2) - y_m(N+1) = \frac{m}{N+2} - \sum_{s=1}^{N+1} f(y_m(s)).$$

In the following theorem, we verify that h is continuous assuming f is Lipschitz continuous.

THEOREM 1. *Let m be a non-negative real number. If f is Lipschitz continuous, then h is continuous at m.*

Proof. Let m be a non-negative real number and τ be the Lipschitz constant for f. For $\varepsilon > 0$, let

$$\delta = \min\left\{\frac{1}{2(2\tau+4)^{N+1}}, \frac{\varepsilon}{2(2\tau+4)^{N+1}}\right\}.$$

Suppose that n is a non-negative real number with

$$0 < |m-n| < \delta,$$

thus

$$|y_m(0) - y_n(0)| = |0-0| = 0 < |m-n|.$$

We will show that

$$|y_m(t) - y_n(t)| < (2\tau+4)^{t-1}|m-n|$$

for all positive integers t, with $t \leqslant N+2$, by finite induction.

Clearly the statement is true when $t = 1$, since

$$|y_m(1) - y_n(1)| = \left|\frac{m}{N+2} - \frac{n}{N+2}\right| < |m-n| = (2\tau+4)^0|m-n|.$$

Let $k \in D_{[1,N+1]}$ and suppose that

$$|y_m(t) - y_n(t)| < (2\tau+4)^{t-1}|m-n|$$

for all integers t, with $0 \leqslant t \leqslant k$. Hence

$$\begin{aligned}
&|y_m(k+1) - y_n(k+1)| \\
&= \left|2y_m(k) - y_m(k-1) - f(y_m(k)) - 2y_n(k) + y_n(k-1) + f(y_n(k))\right| \\
&\leqslant 2\left|y_m(k) - y_n(k)\right| + \left|y_m(k-1) - y_n(k-1)\right| + \left|f(y_m(k)) - f(y_n(k))\right| \\
&\leqslant 2\left|y_m(k) - y_n(k)\right| + \left|y_m(k-1) - y_n(k-1)\right| + \tau\left|y_m(k) - y_n(k)\right| \\
&\leqslant (\tau+2)(2\tau+4)^{k-1}|m-n| + (2\tau+4)^{k-2}|m-n| \\
&< (\tau+2)(2\tau+4)^{k-1}|m-n| + (\tau+2)(2\tau+4)^{k-1}|m-n| \\
&= (2\tau+4)^k|m-n|.
\end{aligned}$$

Thus, by mathematical induction, for every $t \in D_{[0,N+2]}$,

$$|y_m(t) - y_n(t)| < (2\tau+4)^{t-1}|m-n|,$$

and since $|m-n| < \delta$, we have

$$\|y_m - y_n\|_{\ell^\infty} < (2\tau+4)^{N+1}|m-n| < (2\tau+4)^{N+1}\delta \leqslant \min\left\{\frac{1}{2}, \frac{\varepsilon}{2}\right\},$$

hence

$$\|y_m - y_n\|_{\ell^\infty} < \frac{\varepsilon}{2}.$$

Therefore,

$$\begin{aligned}
|h(m) - h(n)| &= \left|y_m(N+2) - y_m(N+1) - y_n(N+2) + y_n(N+1)\right| \\
&\leqslant \left|y_m(N+2) - y_n(N+2)\right| + \left|y_m(N+1) - y_n(N+1)\right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus, h is continuous at m. □

Since we have established that h is continuous for all non-negative real values of m, it follows that h is continuous on the interval $[0,\infty)$. In Theorem 2, we demonstrate that if there exist positive real numbers c and d with $h(c)h(d) < 0$, then a slightly modified bisection method developed by Avery, Anderson, and Lyons [3] can be applied to approximate a solution to the boundary value problem given by equations (1.1) and (1.2). Since not all discrete boundary value problems admit solutions, this condition serves to verify the existence of a solution before attempting to compute one. Moreover, it provides a structured framework for selecting initial conditions in the associated initial value problems.

THEOREM 2. *Suppose that f is a non-negative, Lipschitz continuous function with constant τ. If there exist positive real numbers c and d with $h(c)h(d) < 0$, then there exists a θ in the interval with endpoints c and d such that*

$$h(\theta) = 0,$$

and thus y_θ is a solution of (1.1), (1.2). Moreover, there is a sequence $\{\theta_j\}_{j=0}^{\infty}$ in the interval with endpoints c and d such that

$$\theta_j \to \theta$$

with

$$|\theta - \theta_j| \leqslant \frac{d-c}{2^{j+1}}$$

and

$$\|y_\theta - y_{\theta_j}\|_{\ell^\infty} \leqslant \frac{(d-c)(\tau+2)^{N+1}}{2^{j-N}}.$$

Proof. For convenience we will assume $c < d$, the proof when $d < c$ is nearly identical. In Theorem 1, we verified that h is a continuous real-valued function. We have assumed that

$$h(c)h(d) < 0.$$

Therefore, by the intermediate value theorem, there exists a $\theta \in \big[c,d\big]$ such that $h(\theta) = 0$, and thus y_θ is a solution of (1.1), (1.2).

Now, let

$$c_0 = c, \;\; d_0 = d \;\text{ and }\; \theta_0 = \frac{c_0 + d_0}{2}.$$

Recursively define the sequences $\{c_n\}_{n=0}^{\infty}$, $\{d_n\}_{n=0}^{\infty}$, and $\{\theta_n\}_{n=0}^{\infty}$ by

$$c_{n+1} = \theta_n, \;\; d_{n+1} = d_n \;\text{ and }\; \theta_{n+1} = \frac{c_{n+1} + d_{n+1}}{2}$$

if $h(\theta_n)h(d_n) \leqslant 0$, and

$$c_{n+1} = c_n, \;\; d_{n+1} = \theta_n \;\text{ and }\; \theta_{n+1} = \frac{c_{n+1} + d_{n+1}}{2}$$

if $h(c_n)h(\theta_n) \leqslant 0$ and $h(\theta_n)h(d_n) \neq 0$. Observe that for each whole number n that

$$h(c_n)h(d_n) \leqslant 0.$$

Thus, by the intermediate value theorem, there is a $\theta \in [c_n, d_n]$ such that $h(\theta) = 0$. By induction,

$$d_j - c_j = \frac{d_{j-1} - c_{j-1}}{2} = \frac{d_0 - c_0}{2^j} = \frac{d-c}{2^j},$$

and since θ_j is the midpoint of the interval $[c_j, d_j]$ and $\theta \in [c_j, d_j]$, we have that

$$\theta_j - c_j = \frac{d-c}{2^{j+1}} \;\text{ and }\; d_j - \theta_j = \frac{d-c}{2^{j+1}}.$$

So, we either have that

$$\theta \in [c_j, \theta_j] \text{ or } \theta \in [\theta_j, d_j],$$

and hence

$$|\theta - \theta_j| \leqslant \frac{d-c}{2^{j+1}}.$$

Also, in the proof of Theorem 1, we showed that

$$\|y_\theta - y_{\theta_j}\|_{\ell^\infty} \leqslant (2\tau+4)^{N+1}|\theta - \theta_j|.$$

Hence,

$$\|y_\theta - y_{\theta_j}\|_{\ell^\infty} \leqslant (2\tau+4)^{N+1}\left(\frac{d-c}{2^{j+1}}\right) = \frac{(d-c)(2\tau+4)^{N+1}}{2^{j+1}} = \frac{(d-c)(\tau+2)^{N+1}}{2^{j-N}}.$$

Thus,

$$\lim_{j\to\infty} y_{\theta_j} = y_\theta. \quad \square$$

3. Second-order right focal boundary value problem

While solving the discrete second-order right focal boundary value problem (1.1), (1.2) using a shooting method is interesting, the value of this approach increases when it is used to approximate a solution of a continuous second-order right focal boundary value problem. We will consider the continuous second-order right focal boundary value problem given by

$$x''(t) + r(x(t)) = 0, \quad t \in (0,1), \tag{3.1}$$

$$x(0) = x'(1) = 0 \tag{3.2}$$

where

$$r(x) = (N+2)^2 f(x).$$

Note that, if we assume that $r : \mathbb{R} \to [0,\infty)$ (or f) is continuously differentiable, then we will have that r is Lipschitz continuous on any closed set. The standard fixed point method to solving (3.1), (3.2) is to convert the boundary value problem to a fixed point problem using the associated Green's function. Every solution of (3.1), (3.2) is a fixed point of

$$Hx(t) = \int_0^1 G(t,s)r(x(s))\,ds, \tag{3.3}$$

with

$$G(t,s) = \min\{t,s\}$$

being the corresponding Green's function of (3.1), (3.2). See Duffy [4] for a thorough treatment of Green's functions and Agarwal [1] for a discussion of right focal boundary value problems. The operator

$$H : C[0,1] \to C^2[0,1]$$

is completely continuous where the norm $\|\cdot\|_\infty$ is the usual supremum norm on the Banach space $E = C[0,1]$. For a non-negative real number m, establish the initial conditions

$$x(0) = 0 \text{ and } x'(0) = m. \tag{3.4}$$

It was shown in [3] that solutions of the initial value problem (3.1), (3.4) are fixed points of the operator A defined by

$$Ax(t) = mt - \int_0^t (t-s)r(x(s))\,ds \tag{3.5}$$

on the Banach space E. Also, if x^* is a fixed point of A and

$$m = \int_0^1 r(x^*(s))ds,$$

then x^* is a solution of (3.1), (3.2). In [3], a bisection method was introduced that involved partitioning the interval $[0,1]$ into subintervals, allowing the Banach contraction principle to be applied on each one. By piecing together the resulting solutions of these initial value problems, a global solution to the boundary value problem was obtained. Although effective, the approach in [3] is complex and computationally intensive. In contrast, the shooting method presented here offers a simpler bisection scheme due to the ease in solving, not approximating, discrete boundary value problems. The error bounds are easily calculated as the solutions of the initial value problems are fixed points of the operators (1.5) and (3.5).

Let $i \in D_{[0,N+1]}$ and $t \in \left[i/(N+2),(i+1)/(N+2)\right]$ and suppose $y_m \in B$ is the unique solution of the initial value problem (1.3), (1.4). Define $y_m^* \in E = C[0,1]$ by

$$y_m^*(t) = y_m(i) + \big((N+2)t - i\big)\big(y_m(i+1) - y_m(i)\big),$$

and then define $w_m \in E = C[0,1]$ by

$$w_m(t) = Ay_m^*(t) = mt - \int_0^t (t-s)r(y_m^*(s))\,ds. \tag{3.6}$$

In what follows in this section, we show how to find m so that Hw_m is as close as one wants to get to a solution of the boundary value problem (3.1), (3.2). In the following theorems, we provide the bounds which are needed to verify that Hw_m approximates a solution of the continuous boundary value problem (3.1), (3.2). We note that $Hw_m \in E$ and satisfies the boundary conditions (3.2).

THEOREM 3. *Let m be a non-negative real number, $y_m \in B$ be the unique solution of the initial value problem (1.3), (1.4) with $f(x) = r(x)/(N+2)^2$ where r is a Lipschitz continuous function with Lipschitz constant ψ, C be a positive real number with $\max\{\|y_m\|_\infty\} \leqslant C$, and M be a positive real number with $r(x) \leqslant M$ for all $x \in [0,C]$. Then, for $t \in D_{[0,N+1]}$,*

$$\max_{t \in D_{[0,N+1]}} \left|\Delta y_m(t) - \Delta w_m\Big(\frac{t}{N+2}\Big)\right| \leqslant \frac{0.5M + 2C\psi}{(N+2)^2} \equiv M_1$$

and

$$\left\|y_m^* - w_m\right\|_\infty \leqslant \frac{(2.5M+2m+2C\psi)(N+2)+M}{(N+2)^2} \equiv M_2.$$

Proof. Let $f(x) = r(x)/(N+2)^2$. Thus, for w_m defined in (3.6) and an integer $t \in D_{[0,N+2]}$ we have that

$$w_m\Big(\frac{t}{N+2}\Big) = \frac{mt}{N+2} - \int_0^{\frac{t}{N+2}} \Big(\frac{t}{N+2} - s\Big) r(y_m^*(s))\, ds.$$

Thus, it follows for $t \in D_{[0,N+1]}$ that we have

$$\begin{aligned}\Delta w_m(t) =& \frac{m(t+1)}{N+2} - \int_0^{\frac{t+1}{N+2}} \Big(\frac{t+1}{N+2} - s\Big) r(y_m^*(s))\, ds \\ & - \frac{mt}{N+2} + \int_0^{\frac{t}{N+2}} \Big(\frac{t}{N+2} - s\Big) r(y_m^*(s))\, ds \\ =& \frac{m}{N+2} - \int_{\frac{t}{N+2}}^{\frac{t+1}{N+2}} \Big(\frac{t+1}{N+2} - s\Big) r(y_m^*(s))\, ds - \int_0^{\frac{t}{N+2}} \frac{r(y_m^*(s))}{N+2}\, ds,\end{aligned}$$

and hence

$$\left|\Delta w_m(t)\right| \leqslant \frac{m}{N+2} + \Big(\frac{1}{N+2}\Big)\Big(\frac{1}{N+2}\Big)M + \frac{M}{N+2} = \frac{(M+m)(N+2)+M}{(N+2)^2}.$$

Since $y_m \in B$ is the unique solution of the initial value problem (1.3), (1.4), for an integer $t \in D_{[0,N+2]}$,

$$y_m(t) = \frac{mt}{N+2} - \sum_{s=1}^{t} (t-s) \frac{r(y_m(s))}{(N+2)^2}.$$

Moreover, for $t \in \big[i/(N+2), (i+1)/(N+2)\big]$ with $i \in D_{[0,N+1]}$ we have defined

$$y_m^*(t) = y_m(i) + \big((N+2)t - i\big)\big(y_m(i+1) - y_m(i)\big),$$

and hence for any $t \in D_{[0,N+2]}$,

$$y_m(t) = y_m^*(t).$$

Thus, for $t \in D_{[0,N+1]}$, we have

$$\begin{aligned}\Delta y_m^*(t) =& \frac{m(t+1)}{N+2} - \sum_{s=1}^{t+1} (t+1-s) \frac{r(y_m^*(s))}{(N+2)^2} - \frac{mt}{N+2} + \sum_{s=1}^{t} (t-s) \frac{r(y_m^*(s))}{(N+2)^2} \\ =& \frac{m}{N+2} - \sum_{s=1}^{t} \frac{r(y_m^*(s))}{(N+2)^2},\end{aligned}$$

and hence

$$\left|\Delta y_m^*(t)\right| \leqslant \frac{m}{N+2} + \frac{M}{N+2} = \frac{M+m}{N+2}.$$

For $s \in [0,1]$, define

$$y_{m,c}(s) = y_m\big(\lceil (N+2)s \rceil\big).$$

Since y_m is concave down, there is a $t_m \in D_{[0,N+2]}$ with $y_m(i)$ increasing for $i \leqslant t_m$ and decreasing for $i \geqslant t_m$. Hence, for $s \in D_{[1,N+2]}$,

$$\max_{\tau \in \left[\frac{s-1}{N+2}, \frac{s}{N+2}\right]} \left|y_m^*(\tau) - y_{m,c}(\tau)\right| = y_m\Big(\frac{s}{N+2}\Big) - y_m\Big(\frac{s-1}{N+2}\Big) \text{ when } s \leqslant t_m$$

and

$$\max_{\tau \in \left[\frac{s-1}{N+2}, \frac{s}{N+2}\right]} \left|y_m^*(\tau) - y_{m,c}(\tau)\right| = y_m\Big(\frac{s-1}{N+2}\Big) - y_m\Big(\frac{s}{N+2}\Big) \text{ when } s \geqslant t_m.$$

Thus, for $t \in D_{[0,N+1]}$, we now estimate $\left|\Delta y_m(t) - \Delta w_m\big(t/(N+2)\big)\right|$ using the concavity of y_m,

$$\begin{aligned}
&\left|\Delta y_m(t) - \Delta w_m\Big(\frac{t}{N+2}\Big)\right| \\
=&\Big|\frac{m}{N+2} - \sum_{s=1}^{t} \frac{r(y_m(s))}{(N+2)^2} - \frac{m}{N+2} \\
&\quad + \int_{\frac{t}{N+2}}^{\frac{t+1}{N+2}} \Big(\frac{t+1}{N+2} - \tau\Big) r(y_m^*(\tau))\, d\tau + \int_0^{\frac{t}{N+2}} \frac{r(y_m^*(\tau))}{N+2}\, d\tau \Big| \\
=&\Big| - \sum_{s=1}^{t} \Big(\frac{1}{N+2}\Big) \int_{\frac{s-1}{N+2}}^{\frac{s}{N+2}} r(y_m(s))\, d\tau \\
&\quad + \int_{\frac{t}{N+2}}^{\frac{t+1}{N+2}} \Big(\frac{t+1}{N+2} - \tau\Big) r(y_m^*(\tau))\, d\tau + \int_0^{\frac{t}{N+2}} \frac{r(y_m^*(\tau))}{N+2}\, d\tau \Big| \\
=&\Big| - \sum_{s=1}^{t} \Big(\frac{1}{N+2}\Big) \int_{\frac{s-1}{N+2}}^{\frac{s}{N+2}} r(y_{m,c}(\tau))\, d\tau \\
&\quad + \int_{\frac{t}{N+2}}^{\frac{t+1}{N+2}} \Big(\frac{t+1}{N+2} - \tau\Big) r(y_m^*(\tau))\, d\tau + \sum_{s=1}^{t} \Big(\frac{1}{N+2}\Big) \int_{\frac{s-1}{N+2}}^{\frac{s}{N+2}} r(y_m^*(\tau))\, d\tau \Big| \\
=&\Big| \int_{\frac{t}{N+2}}^{\frac{t+1}{N+2}} \Big(\frac{t+1}{N+2} - \tau\Big) r(y_m^*(\tau))\, d\tau \\
&\quad + \sum_{s=1}^{t} \Big(\frac{1}{N+2}\Big) \int_{\frac{s-1}{N+2}}^{\frac{s}{N+2}} r(y_m^*(\tau)) - r(y_{m,c}(\tau))\, d\tau \Big|.
\end{aligned}$$

Applying the M bound on r and the Lipschitz condition, we obtain

$$\begin{aligned}
&\left|\Delta y_m(t) - \Delta w_m\Big(\frac{t}{N+2}\Big)\right| \\
&\leqslant \int_{\frac{t}{N+2}}^{\frac{t+1}{N+2}} \Big(\frac{t+1}{N+2} - \tau\Big) M\, d\tau + \sum_{s=1}^{t} \Big(\frac{1}{N+2}\Big) \int_{\frac{s-1}{N+2}}^{\frac{s}{N+2}} \Big|r(y_m^*(\tau)) - r(y_{m,c}(\tau))\Big|\, d\tau \\
&\leqslant \frac{M}{2(N+2)^2} + \sum_{s=1}^{t} \Big(\frac{1}{N+2}\Big) \int_{\frac{s-1}{N+2}}^{\frac{s}{N+2}} \psi \Big|y_m^*(\tau) - y_{m,c}(\tau)\Big|\, d\tau \\
&= \frac{M}{2(N+2)^2} + \sum_{s=1}^{t_m} \Big(\frac{1}{N+2}\Big) \int_{\frac{s-1}{N+2}}^{\frac{s}{N+2}} \psi \Big|y_m^*(\tau) - y_{m,c}(\tau)\Big|\, d\tau \\
&\quad + \sum_{s=t_m+1}^{N+1} \Big(\frac{1}{N+2}\Big) \int_{\frac{s-1}{N+2}}^{\frac{s}{N+2}} \psi \Big|y_m^*(\tau) - y_{m,c}(\tau)\Big|\, d\tau \\
&\leqslant \frac{M}{2(N+2)^2} + \sum_{s=1}^{t_m} \Big(\frac{\psi}{N+2}\Big) \int_{\frac{s-1}{N+2}}^{\frac{s}{N+2}} \Big(y_m(s) - y_m(s-1)\Big)\, d\tau \\
&\quad + \sum_{s=t_m+1}^{N+1} \Big(\frac{\psi}{N+2}\Big) \int_{\frac{s-1}{N+2}}^{\frac{s}{N+2}} \Big(y_m(s-1) - y_m(s)\Big)\, d\tau \\
&= \frac{M}{2(N+2)^2} + \sum_{s=1}^{t_m} \Big(\frac{\psi}{N+2}\Big)\Big(\frac{y_m(s) - y_m(s-1)}{N+2}\Big) \\
&\quad + \sum_{s=t_m+1}^{N+1} \Big(\frac{\psi}{N+2}\Big)\Big(\frac{y_m(s-1) - y_m(s)}{N+2}\Big) \\
&\leqslant \frac{M}{2(N+2)^2} + \Big(\frac{\psi}{N+2}\Big)\Big(\frac{y_m(t_m) - y_m(0)}{N+2}\Big) + \Big(\frac{\psi}{N+2}\Big)\Big(\frac{y_m(t_m) - y_m(N+1)}{N+2}\Big) \\
&\leqslant \frac{M}{2(N+2)^2} + \frac{C\psi}{(N+2)^2} + \frac{C\psi}{(N+2)^2} \\
&\leqslant \frac{0.5M + 2C\psi}{(N+2)^2}.
\end{aligned}$$

For $i \in D_{[0,N+1]}$, we have that

$$y_m(i) - w_m(i) = \sum_{s=0}^{i-1} \Big(\Delta y_m(s) - \Delta w_m(s)\Big)$$

and

$$\begin{aligned}
|y_m(i) - w_m(i)| &= \Big|\sum_{s=0}^{i-1} \Big(\Delta y_m(s) - \Delta w_m(s)\Big)\Big| \\
&\leqslant \sum_{s=0}^{i-1} \Big|\Delta y_m(s) - \Delta w_m(s)\Big| \\
&\leqslant \sum_{s=0}^{i-1} \frac{0.5M + 2C\psi}{(N+2)^2} < \frac{0.5M + 2C\psi}{N+2}.
\end{aligned}$$

It remains to bound $\|y_m^* - w_m\|_\infty$. Thus, for $t \in [i/(N+2),(i+1)/(N+2)]$,

$$\begin{aligned}
|y_m^*(t) - w_m(t)| &\leqslant |y_m^*(t) - y_m^*(i)| + |y_m^*(i) - w_m(i)| + |w_m(i) - w_m(t)| \\
&\leqslant |y_m^*(i+1) - y_m^*(i)| + |y_m^*(i) - w_m(i)| + |w_m(i) - w_m(i+1)| \\
&\leqslant |\Delta y_m^*(i)| + \frac{0.5M + 2C\psi}{N+2} + |\Delta w_m(i)| \\
&\leqslant \frac{M+m}{N+2} + \frac{0.5M + 2C\psi}{N+2} + \frac{(M+m)(N+2)+M}{(N+2)^2} \\
&= \frac{(2.5M + 2m + 2C\psi)(N+2)+M}{(N+2)^2},
\end{aligned}$$

and hence

$$\|y_m^* - w_m\|_\infty \leqslant \frac{(2.5M + 2m + 2C\psi)(N+2)+M}{(N+2)^2}. \quad \square$$

THEOREM 4. *Let m be a non-negative real number, and let $f(x) = r(x)/(N+2)^2$ where r is a Lipschitz continuous function with Lipschitz constant ψ. Suppose $y_m \in B$ is the unique solution of the initial value problem (1.3), (1.4), and $z_m \in E$ is the unique solution of the initial value problem (3.1), (3.4). Assume there exists a positive real number C such that $\max\{\|y_m\|_\infty, \|z_m\|_\infty\} \leqslant C$, and a positive real number M such that $r(x) \leqslant M$ for all $x \in [0,C]$. Then, we have the estimate*

$$\|z_m - w_m\|_\infty \leqslant \frac{Me^{1+\psi}}{2(N+2)^2} + \frac{((2.5M + 2m + 2C\psi)(N+2)+M)(N+1)e^{1+\psi}}{N(N+2)^2} \equiv M_3.$$

Proof. Let $i \in D_{[1,N+2]}$, we will prove by finite induction that for all $t \in [0,i]$ that

$$\max_{t\in[0,i]} \left| z_m\Big(\frac{t}{N+2}\Big) - w_m\Big(\frac{t}{N+2}\Big) \right| \leqslant \delta_i$$

where

$$\delta_i = \Big(\delta_1 + \frac{d\psi(i-1)}{N}\Big) p^{i-1}$$

for

$$\delta_1 = \frac{M}{2(N+2)^2}, \quad d = \frac{(2.5M + 2m + 2C\psi)(N+2)+M}{(N+2)^2}, \quad \text{and} \quad p = \frac{N+2+\psi}{N+1}.$$

Note that

$$w_m\Big(\frac{t}{N+2}\Big) = \frac{mt}{N+2} - \int_0^{\frac{t}{N+2}} \Big(\frac{t}{N+2} - s\Big)\, r(y_m^*(s))\, ds$$

and

$$z_m\Big(\frac{t}{N+2}\Big) = \frac{mt}{N+2} - \int_0^{\frac{t}{N+2}} \Big(\frac{t}{N+2} - s\Big)\, r(z_m(s))\, ds.$$

For $i=1$ and $t\in[0,1]$, by Taylor's Theorem, for some $t_0\in(0,t)$,

$$z_m\Big(\frac{t}{N+2}\Big)-w_m\Big(\frac{t}{N+2}\Big)=\left(\frac{r\big(z_m\big(\frac{t_0}{N+2}\big)\big)-r\big(y_m^*\big(\frac{t_0}{N+2}\big)\big)}{(N+2)^2}\right)\left(\frac{(t-0)^2}{2}\right)$$

since

$$z_m(0)=w_m(0)$$

and

$$z_m'\Big(\frac{0}{N+2}\Big)=w_m'\Big(\frac{0}{N+2}\Big).$$

Thus,

$$\left|z_m\Big(\frac{t}{N+2}\Big)-w_m\Big(\frac{t}{N+2}\Big)\right|\leqslant\frac{M}{2(N+2)^2}=\delta_1.$$

Recall from Theorem 3,

$$\big\|y_m^*-w_m\big\|_\infty\leqslant d.$$

For $i\in D_{[1,N+1]}$ and

$$\delta_i=\Big(\delta_1+\frac{d\psi(i-1)}{N}\Big)p^{i-1},$$

suppose that

$$\max_{t\in[0,i]}\left|z_m\Big(\frac{t}{N+2}\Big)-w_m\Big(\frac{t}{N+2}\Big)\right|\leqslant\delta_i.$$

Thus, for $t\in[i,i+1]$, by Taylor's Theorem, for some $t_i\in(i,t)$, we have that

$$\begin{aligned}
&\left|z_m\Big(\frac{t}{N+2}\Big)-w_m\Big(\frac{t}{N+2}\Big)\right|\\
=&\left|z_m\Big(\frac{i}{N+2}\Big)-w_m\Big(\frac{i}{N+2}\Big)\right.\\
&+\Big(\frac{1}{N+2}\Big)\left(\int_0^{\frac{i}{N+2}}r(z_m(s))-r(y_m^*(s))\,ds\right)(t-i)\\
&\left.+\left(\frac{r\big(z_m\big(\frac{t_i}{N+2}\big)\big)-r\big(y_m^*\big(\frac{t_i}{N+2}\big)\big)}{(N+2)^2}\right)\left(\frac{(t-i)^2}{2}\right)\right|\\
\leqslant&\delta_i+\Big(\frac{i}{(N+2)^2}\Big)\psi\max_{w\in\left[0,\frac{i}{N+2}\right]}\left|z_m(w)-y_m^*(w)\right|+\frac{\psi\max_{q\in\left[\frac{i}{N+2},\frac{i+1}{N+2}\right]}\left|z_m(q)-y_m^*(q)\right|}{(N+2)^2}\\
\leqslant&\delta_i+\Big(\frac{i}{(N+2)^2}\Big)\psi\left(\max_{q\in\left[0,\frac{i}{N+2}\right]}\left|z_m(q)-w_m(q)\right|+\max_{q\in\left[0,\frac{i}{N+2}\right]}\left|w_m(q)-y_m^*(q)\right|\right)\\
&+\frac{\psi\big(\max_{q\in\left[\frac{i}{N+2},\frac{i+1}{N+2}\right]}\left|z_m(q)-w_m(q)\right|+\max_{q\in\left[\frac{i}{N+2},\frac{i+1}{N+2}\right]}\left|w_m(q)-y_m^*(q)\right|\big)}{(N+2)^2},
\end{aligned}$$

and hence

$$\begin{aligned}&\Big(1-\frac{\psi}{(N+2)^2}\Big)\max_{t\in[0,i+1]}\Big|z_m\Big(\frac{t}{N+2}\Big)-w_m\Big(\frac{t}{N+2}\Big)\Big|\\ \leqslant&\Big(1+\frac{\psi}{N+2}\Big)\delta_i+\Big(\frac{1}{N+2}+\frac{1}{(N+2)^2}\Big)\psi d.\end{aligned}$$

Since

$$\frac{(N+2)^2}{(N+2)^2-\psi}\leqslant\frac{(N+2)^2}{(N+2)^2-(N+2)}=\frac{N+2}{N+1},$$

we have

$$\begin{aligned}&\max_{t\in[0,i+1]}\Big|z_m\Big(\frac{t}{N+2}\Big)-w_m\Big(\frac{t}{N+2}\Big)\Big|\\ \leqslant&\Big(\frac{N+2}{N+1}\Big)\Big(\frac{N+2+\psi}{N+2}\Big)\delta_i+\Big(\frac{N+2}{N+1}\Big)\Big(\frac{N+3}{(N+2)^2}\Big)\psi d\\ \leqslant&\Big(\frac{N+2+\psi}{N+1}\Big)\Big(\delta_1+\frac{d\psi(i-1)}{N}\Big)p^{i-1}+\Big(\frac{N+2}{N+1}\Big)\Big(\frac{N+3}{(N+2)^2}\Big)\psi d\\ \leqslant&\delta_1 p^i+\frac{d\psi(i-1)p^i}{N}+\frac{\psi d}{N}\\ \leqslant&\delta_1 p^i+\frac{d\psi(i-1)p^i}{N}+\frac{\psi d p^i}{N}=\delta_{i+1}.\end{aligned}$$

Thus, by the principle of mathematical induction, we have verified that for all $i\in D_{[1,N+1]}$,

$$\max_{t\in[0,i+1]}\Big|z_m\Big(\frac{t}{N+2}\Big)-w_m\Big(\frac{t}{N+2}\Big)\Big|\leqslant\delta_{i+1}=\Big(\delta_1+\frac{d\psi i}{N}\Big)p^i.$$

Therefore,

$$\|z_m-w_m\|_\infty\leqslant\Big(\delta_1+\frac{d\psi(N+1)}{N}\Big)p^{N+1},$$

and since

$$p^{N+1}=\Big(1+\frac{1+\psi}{N+1}\Big)^{N+1}\leqslant e^{1+\psi},$$

we have that

$$\|z_m-w_m\|_\infty\leqslant\frac{Me^{1+\psi}}{2(N+2)^2}+\frac{((2.5M+2m+2C\psi)(N+2)+M)(N+1)e^{1+\psi}}{N(N+2)^2}.\quad\square$$

In the following theorem, we provide a bound on

$$\Big|\frac{(z_m)'(1)}{N+2}-\Delta y_m\Big(N+1\Big)\Big|.$$

This will be used to provide conditions when

$$h(m) = y_m(N+2) - y_m(N+1) = \frac{m}{N+2} - \sum_{s=1}^{N+1} f(y_m(s))$$

and

$$g(m) = m - \int_0^1 r(z_m(s))\, ds$$

are of the same sign, which is necessary to verify the existence of a solution from the bisection scheme for both the discrete boundary value problem (1.1), (1.2) and the continuous boundary value problem (3.1), (3.2).

THEOREM 5. *Let m be a non-negative real number, $y_m \in B$ be the unique solution of the initial value problem* (1.3), (1.4), *$z_m \in E$ be the unique solution of the initial value problem* (3.1), (3.4) *with $f(x) = r(x)/(N+2)^2$ where r is a Lipschitz continuous function with Lipschitz constant ψ, C be a positive real number with $\max\{\|y_m\|_\infty, \|z_m\|_\infty\} \leqslant C$, and M be a positive real number with $r(x) \leqslant M$ for all $x \in [0, C]$. Then,*

$$\left| \frac{(z_m)'(1)}{N+2} - \Delta y_m(N+1) \right| \leqslant \frac{\psi(M_3 + M_2)}{N+2} + \frac{M}{2(N+2)^2} + M_1 \equiv M_4.$$

Proof. From Theorem 3,

$$\left| \Delta w_m\left(\frac{N+1}{N+2}\right) - \Delta y_m(N+1) \right| \leqslant M_1.$$

From Theorem 3 and Theorem 4,

$$\begin{aligned} \left| z_m'(1) - w_m'(1) \right| &= \left| m - \int_0^1 r(z_m(s))ds - m + \int_0^1 r(y_m^*(s))ds \right| \\ &= \left| \int_0^1 r(z_m(s)) - r(y_m^*(s))ds \right| \\ &\leqslant \psi \|z_m - y_m^*\|_\infty \\ &\leqslant \psi\left(\|z_m - w_m\|_\infty + \|w_m - y_m^*\|_\infty \right) \\ &\leqslant \psi(M_3 + M_2). \end{aligned}$$

Thus,

$$\left| \frac{z_m'(1)}{N+2} - \frac{w_m'(1)}{N+2} \right| \leqslant \frac{\psi(M_3 + M_2)}{N+2}.$$

We also have that

$$\begin{aligned} &\left| \frac{(w_m)'(1)}{N+2} - \Delta w_m\left(\frac{N+1}{N+2}\right) \right| \\ &= \left| \frac{m}{N+2} - \left(\frac{1}{N+2}\right) \int_0^1 r(y_m^*(s))ds - \frac{m}{N+2} \right. \\ &\quad \left. + \int_{\frac{N+1}{N+2}}^{\frac{N+2}{N+2}} (1-s) r(y_m^*(s))\, ds + \int_0^{\frac{N+1}{N+2}} \frac{r(y_m^*(s))}{N+2}\, ds \right| \end{aligned}$$

$$
\begin{aligned}
&= \Big| \Big(\frac{-1}{N+2} \Big) \int_{\frac{N+1}{N+2}}^{1} r(y_m^*(s)) ds + \int_{\frac{N+1}{N+2}}^{1} (1-s) r(y_m^*(s))\, ds \Big| \\
&= \Big| \int_{\frac{N+1}{N+2}}^{1} \Big(\frac{N+1}{N+2} - s \Big) r(y_m^*(s))\, ds \Big| \\
&\leqslant M \Big| \int_{\frac{N+1}{N+2}}^{1} \Big(\frac{N+1}{N+2} - s \Big)\, ds \Big| = \frac{M}{2(N+2)^2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Big| \frac{(z_m)'(1)}{N+2} - \Delta y_m\Big(N+1\Big) \Big| &\leqslant \Big| \frac{(z_m)'(1)}{N+2} - \frac{(w_m)'(1)}{N+2} \Big| \\
&\quad + \Big| \frac{(w_m)'(1)}{N+2} - \Delta w_m\Big(\frac{N+1}{N+2} \Big) \Big| \\
&\quad + \Big| \Delta w_m\Big(\frac{N+1}{N+2} \Big) - \Delta y_m\big(N+1\big) \Big| \\
&\leqslant \frac{\psi\big(M_3 + M_2\big)}{N+2} + \frac{M}{2(N+2)^2} + M_1. \quad \square
\end{aligned}
$$

In the following theorem, we provide a bound on $\big|z_m - z_n\big|$ which will be the final inequality needed to approximate a solution of the continuous boundary value problem (3.1), (3.2). The proof of this result follows the technique presented in Theorem 4 and its proof.

THEOREM 6. *Let m and n be non-negative real numbers, and let $f(x) = r(x)/(N+2)^2$ where r is a Lipschitz continuous function with Lipschitz constant ψ. Suppose $\kappa \in \mathbb{N}$ satisfies $\psi < 2\kappa^2$, and let $z_m, z_n \in E$ be the unique solutions to the initial value problem (3.1), (3.4) corresponding to m and n, respectively. Then the following inequality holds:*

$$
\|z_m - z_n\|_\infty \leqslant \Big(\frac{2\kappa}{2\kappa^2 - \psi} \Big)^{\kappa} (\kappa + \psi)^{\kappa - 1} |m - n| \equiv M_5 |m - n|.
$$

Moreover,

$$
g(m) = m - \int_0^1 r(z_m(s))\, ds
$$

is continuous.

Proof. Let $\kappa \in \mathbb{N}$ with $\psi < 2\kappa^2$, and let $z_m \in E$ be the unique solution of the initial value problem (3.1), (3.4). Then, for $t \in [0,1]$, we have that

$$
z_m(t) = mt - \int_0^t (t-s) r(z_m(s))\, ds.
$$

Thus, for an integer $t \in D_{[0,\kappa]}$, we have that

$$
z_m\Big(\frac{t}{\kappa} \Big) = \frac{mt}{\kappa} - \int_0^{\frac{t}{\kappa}} \Big(\frac{t}{\kappa} - s \Big) r(z_m(s))\, ds,
$$

and similarly, for z_n, we have that

$$z_n\Big(\frac{t}{\kappa}\Big) = \frac{nt}{\kappa} - \int_0^{\frac{t}{\kappa}} \Big(\frac{t}{\kappa} - s\Big) r(z_n(s))\, ds.$$

For $i \in D_{[1,\kappa]}$, let

$$\|z_m - z_n\|_i = \max_{t\in[0,i]} \Big|z_m\Big(\frac{t}{\kappa}\Big) - z_n\Big(\frac{t}{\kappa}\Big)\Big|,$$

and let $\tau_i \in [0,i]$ with

$$\|z_m - z_n\|_i = \Big|z_m\Big(\frac{\tau_i}{\kappa}\Big) - z_n\Big(\frac{\tau_i}{\kappa}\Big)\Big|.$$

For $i \in D_{[1,\kappa]}$, define

$$\varepsilon_i = \frac{2^i(\kappa)^i(\kappa+\psi)^{i-1}|m-n|}{(2\kappa^2-\psi)^i}.$$

By the principle of finite induction, we will show that for all $i \in D_{[1,\kappa]}$,

$$\|z_m - z_n\|_i \leqslant \varepsilon_i.$$

For $i = 1$, by Taylor's Theorem, and some $t_0 \in (0,\tau_1)$, we have that

$$\begin{aligned}
\|z_m - z_n\|_1 &= \Big|z_m\Big(\frac{\tau_1}{\kappa}\Big) - z_n\Big(\frac{\tau_1}{\kappa}\Big)\Big| \\
&= \left|\Big(\frac{m}{\kappa} - \frac{n}{\kappa}\Big)(\tau_1 - 0) + \left(\frac{z_m''\big(\frac{t_0}{\kappa}\big) - z_n''\big(\frac{t_0}{\kappa}\big)}{\kappa^2}\right)\left(\frac{(\tau_1-0)^2}{2}\right)\right| \\
&\leqslant \frac{|m-n|}{\kappa} + \left(\frac{\big|r\big(z_m\big(\frac{t_0}{\kappa}\big)\big) - r\big(z_n\big(\frac{t_0}{\kappa}\big)\big)\big|}{\kappa^2}\right)\Big(\frac{1}{2}\Big) \\
&\leqslant \frac{|m-n|}{\kappa} + \left(\frac{\psi\big|z_m\big(\frac{\tau_1}{\kappa}\big) - z_n\big(\frac{\tau_1}{\kappa}\big)\big|}{2\kappa^2}\right),
\end{aligned}$$

and since

$$z_m(0) = z_n(0) = 0,$$

we have that

$$z_m'\Big(\frac{0}{\kappa}\Big) = \frac{m}{\kappa} \text{ and } z_n'\Big(\frac{0}{\kappa}\Big) = \frac{n}{\kappa}.$$

Therefore,

$$\|z_m - z_n\|_1 \leqslant \frac{|m-n|}{\kappa} + \frac{\psi\|z_m - z_n\|_1}{2(\kappa)^2},$$

and thus

$$\|z_m - z_n\|_1\Big(1 - \frac{\psi}{2\kappa^2}\Big) \leqslant \frac{|m-n|}{\kappa}$$

so that

$$\|z_m - z_n\|_1 \leqslant \frac{2\kappa|m-n|}{2\kappa^2-\psi} = \varepsilon_1.$$

Let $i \in D_{[1,\kappa-1]}$ and suppose that for all $j \in D_{[1,i]}$,

$$\|z_m - z_n\|_j \leqslant \varepsilon_j.$$

Either $\tau_{i+1} \in [0,i]$ or $\tau_{i+1} \in (i,i+1]$, we proceed by cases.

Case 1: $\tau_{i+1} \in [0,i]$.

Thus,

$$\|z_m - z_n\|_{i+1} = \|z_m - z_n\|_i$$

and since

$$\varepsilon_{i+1} = \frac{2\kappa(\kappa+\psi)}{(2\kappa^2-\psi)}\varepsilon_i > \varepsilon_i,$$

we have that

$$\|z_m - z_n\|_{i+1} = \|z_m - z_n\|_i \leqslant \varepsilon_i < \varepsilon_{i+1}.$$

Case 2: $\tau_{i+1} \in (i,i+1]$.

Thus, by Taylor's Theorem for some $t_i \in (i,\tau_{i+1})$, doing an expansion around $t = i$,

$$\begin{aligned}
\|z_m - z_n\|_{i+1} &= \left| z_m\left(\frac{\tau_{i+1}}{\kappa}\right) - z_n\left(\frac{\tau_{i+1}}{\kappa}\right)\right| \\
&= \left| z_m\left(\frac{i}{\kappa}\right) - z_n\left(\frac{i}{\kappa}\right) + \left(\frac{1}{\kappa}\right)\left(\int_0^{\frac{i}{\kappa}} r(z_m(s)) - r(z_n(s))\,ds\right)(\tau_{i+1}-i)\right. \\
&\quad \left. + \left(\frac{r\left(z_m\left(\frac{t_i}{\kappa}\right)\right) - r\left(z_n\left(\frac{t_i}{\kappa}\right)\right)}{\kappa^2}\right)\left(\frac{(\tau_{i+1}-i)^2}{2}\right)\right| \\
&\leqslant \left| z_m\left(\frac{\tau_i}{\kappa}\right) - z_n\left(\frac{\tau_i}{\kappa}\right)\right| + \left(\frac{1}{\kappa}\right)\int_0^{\frac{i}{\kappa}} \psi\left|z_m(\tau_i) - z_n(\tau_i)\right| ds \\
&\quad + \left(\frac{\psi(z_m(\tau_{i+1}) - z_n(\tau_{i+1}))}{\kappa^2}\right)\left(\frac{1^2}{2}\right) \\
&= \varepsilon_i + \frac{\psi\varepsilon_i i}{\kappa^2} + \frac{\psi\|z_m - z_n\|_{i+1}}{2\kappa^2} \\
&= \varepsilon_i + \frac{\psi\varepsilon_i}{\kappa} + \frac{\psi\|z_m - z_n\|_{i+1}}{2\kappa^2},
\end{aligned}$$

and hence

$$\|z_m - z_n\|_{i+1} \leqslant \frac{(\kappa+\psi)\varepsilon_i}{\kappa} + \frac{\psi\|z_m - z_n\|_{i+1}}{2\kappa^2}$$

so

$$\|z_m - z_n\|_{i+1} \leqslant \frac{2\kappa(\kappa+\psi)\varepsilon_i}{2\kappa^2-\psi} = \varepsilon_{i+1}.$$

Consequently, by the principle of mathematical induction, we have that for all $i \in D_{[1,\kappa]}$,

$$\|z_m - z_n\|_i \leqslant \frac{2^i\kappa^i(\kappa+\psi)^{i-1}|m-n|}{(2\kappa^2-\psi)^i}.$$

Therefore,

$$\|z_m - z_n\|_\infty = \|z_m - z_n\|_\kappa \leqslant \frac{2^\kappa \kappa^\kappa (\kappa+\psi)^{\kappa-1}|m-n|}{(2\kappa^2-\psi)^\kappa}.$$

Moreover, let $m > 0$ and $\varepsilon > 0$. For any $n > 0$ with

$$|m-n| < \frac{(2\kappa^2-\psi)^\kappa \varepsilon}{\left((2\kappa^2-\psi)^\kappa + \psi 2^\kappa \kappa^\kappa (\kappa+\psi)^{\kappa-1}\right)},$$

we have that

$$\begin{aligned}
|g(m)-g(n)| &= \left|m - \int_0^1 r(z_m(s))\,ds - n + \int_0^1 r(z_n(s))\,ds\right| \\
&\leqslant |m-n| + \left|\psi \int_0^1 z_m(s) - z_n(s)\,ds\right| \\
&\leqslant |m-n| + \psi \|z_m - z_n\|_\infty \\
&\leqslant \frac{\left((2\kappa^2-\psi)^\kappa + \psi 2^\kappa \kappa^\kappa (\kappa+\psi)^{\kappa-1}\right)|m-n|}{(2\kappa^2-\psi)^\kappa} \\
&\leqslant \varepsilon.
\end{aligned}$$

Thus, g is continuous. □

THEOREM 7. *Assume the hypotheses of Theorems 3–6 hold and let* $f(x) = r(x)/(N+2)^2$ *where* r *is a Lipschitz continuous function with Lipschitz constant* ψ. *Also let* c *and* d *be positive real numbers and suppose* $h(c)h(d) < 0$ *and for all positive integers* j *let* θ_j *be the midpoint of the interval with endpoints* c_j *and* d_j *generated by the bisection method in Theorem 2. If* ν *is a positive integer with*

$$g(c_{\nu-1})h(c_{\nu-1}) > 0 \text{ and } g(d_{\nu-1})h(d_{\nu-1}) > 0,$$

then there exists a λ *in the interval with endpoints* $c_{\nu-1}$ *and* $d_{\nu-1}$ *such that*

$$g(\lambda) = 0 \text{ and thus } z_\lambda \text{ is a solution of (3.1), (3.2).}$$

Moreover,

$$\|z_\lambda - Hw_{\theta_\nu}\|_\infty \leqslant \left(\frac{\psi}{2}\right)\left(\frac{(d-c)M_5}{2^{\nu+1}} + M_3\right),$$

where H *is defined in (3.3) and* w_{θ_ν} *is defined in (3.6).*

Proof. We will assume that $c < d$, the proof when $d < c$ is nearly identical. In Theorem 1, we verified that h is a continuous real-valued function, and in Theorem 6, we verified that g is a continuous real-valued function. Since we have assumed that f is a non-negative, Lipschitz continuous function (inherited from r) and that $h(c)h(d) < 0$ from the work in Theorem 2, there exists a sequence of intervals $\{[c_j, d_j]\}_{j=0}^\infty$ whose midpoints $\{\theta_j\}_{j=0}^\infty$ converge to θ with $\theta \in [c_j, d_j]$ for all positive integers j,

$$|\theta - \theta_j| \leqslant \frac{d-c}{2^{j+1}},$$

and $h(\theta) = 0$ thus y_θ is a solution of (1.1), (1.2).

From Theorem 2,

$$h(c_{\nu-1}) < 0 \text{ and } h(d_{\nu-1}) > 0,$$

and we have assumed that

$$g(c_{\nu-1})h(c_{\nu-1}) > 0 \text{ and } g(d_{\nu-1})h(d_{\nu-1}) > 0.$$

Therefore,

$$g(c_{\nu-1}) < 0 \text{ and } g(d_{\nu-1}) > 0.$$

Thus, since g is continuous, by the intermediate value theorem, there is a $\lambda \in [c_{\nu-1}, d_{\nu-1}]$ such that $g(\lambda) = 0$, and hence z_λ is a solution of (3.1), (3.2). Moreover, either

$$\lambda \in [c_{\nu-1}, \theta_\nu] \text{ or } \lambda \in [\theta_\nu, d_{\nu-1}],$$

and thus

$$|\lambda - \theta_\nu| \leqslant \frac{d-c}{2^{\nu+1}}.$$

Hence,

$$\begin{aligned}
\|z_\lambda - Hw_{\theta_\nu}\|_\infty &= \|Hz_\lambda - Hw_{\theta_\nu}\|_\infty \\
&= \sup_{t\in[0,1]} \left| \int_0^1 G(t,s)\big(r(z_\lambda(s)) - r(w_{\theta_\nu}(s))\big)\, ds \right| \\
&\leqslant \sup_{t\in[0,1]} \psi \int_0^1 G(t,s)|z_\lambda(s) - w_{\theta_\nu}(s)|\, ds \\
&\leqslant \sup_{t\in[0,1]} \psi\|z_\lambda - w_{\theta_\nu}\|_\infty \int_0^1 G(t,s)\, ds \\
&\leqslant \frac{\psi\|z_\lambda - w_{\theta_\nu}\|_\infty}{2} \\
&\leqslant \frac{\psi\big(\|z_\lambda - z_{\theta_\nu}\|_\infty + \|z_{\theta_\nu} - w_{\theta_\nu}\|_\infty\big)}{2} \\
&\leqslant \left(\frac{\psi}{2}\right)\Big(M_5|\lambda - \theta_\nu| + M_3\Big) \\
&\leqslant \left(\frac{\psi}{2}\right)\left(\frac{(d-c)M_5}{2^{\nu+1}} + M_3\right). \quad \square
\end{aligned}$$

4. Example – approximating a solution to a second-order right focal boundary value problem

For $t \in (0,1)$, consider the boundary value problem

$$\begin{aligned}
x''(t) + \frac{4}{1 + |x(t)|e^{-|x(t)|}} &= 0, \\
x(0) = 0, \quad x'(1) &= 0.
\end{aligned}$$

Here, we note that

$$r(x) = \frac{4}{1+|x|e^{-|x|}}$$

is Lipschitz continuous with constant $\psi = 4$.

For $t \in D_{[1,N+1]}$, the associated discrete initial value problem is

$$\Delta^2 y(t-1) + f(y(t)) = 0,$$
$$y(0) = 0, \quad y(1) = \frac{\theta_i}{N+2},$$

which can be recursively expressed as

$$y_{\theta_i}(t+2) = 2y_{\theta_i}(t+1) - y_{\theta_i}(t) - f(y_{\theta_i}(t+1)), \quad y_{\theta_i}(0) = 0, \quad y_{\theta_i}(1) = \frac{\theta_i}{N+2},$$

for integers $t \in D_{[0,N]}$, where $f(x) = r(x)/(N+2)^2$.

Noting that $h(0) < 0$ and $h(4) > 0$, we apply Theorem 2 with initial interval $c_0 = 0$ and $d_0 = 4$. Using Mathematica, we evaluate $h(\theta_i)$ to identify the intervals where it changes sign. Table 1 summarizes these intervals and the corresponding signs of $h(\theta_i)$ for $N = 10^7$. Note this means each evaluation of $h(m)$ requires ten million iterations of the recurrence relation.

Let

$$M = 5, \quad \kappa = 4, \quad C = 2, \quad c = 0, \quad d = 4, \quad m = \frac{1595257}{524288}.$$

Then,

$$M_1 = \frac{0.5M + 2C\psi}{(N+2)^2} < 3.4 \cdot 10^{-13},$$
$$M_2 = \frac{(2.5M + 2m + 2C\psi)(N+2) + M}{(N+2)^2} < 5.0 \cdot 10^{-6},$$
$$M_3 = \frac{Me^{1+\psi}}{2(N+2)^2} + \frac{((2.5M + 2m + 2C\psi)(N+2) + M)(N+1)e^{1+\psi}}{N(N+2)^2} < 5.2 \cdot 10^{-4},$$
$$M_5 = \left(\frac{2\kappa}{2\kappa^2 - \psi}\right)^{\kappa} (\kappa + \psi)^{\kappa - 1} < 3.5,$$

and set

$$\delta = M_4 = \frac{\psi(M_3 + M_2)}{N+2} + \frac{M}{2(N+2)^2} + M_1 < 2.1 \cdot 10^{-10}.$$

Hence, by Theorem 5,

$$\left| \frac{g(m)}{N+2} - h(m) \right| \leqslant \delta, \quad \text{so that} \quad h(m) - \delta < \frac{g(m)}{N+2} < h(m) + \delta.$$

Taking into account the intervals $[c_{19}, d_{19}]$,

$$0 < h(d_{19}) - \delta \leqslant \frac{g(d_{19})}{N+2} \quad \text{and} \quad \frac{g(c_{19})}{N+2} \leqslant h(c_{19}) + \delta < 0,$$

i	c_i	d_i	θ_i	$h(\theta_i)$
0	0	4	2	negative
1	2	4	3	negative
2	$\frac{6}{2}$	$\frac{8}{2}$	$\frac{7}{2}$	positive
3	$\frac{12}{4}$	$\frac{14}{4}$	$\frac{13}{4}$	positive
4	$\frac{24}{8}$	$\frac{26}{8}$	$\frac{25}{8}$	positive
5	$\frac{48}{16}$	$\frac{50}{16}$	$\frac{49}{16}$	positive
6	$\frac{96}{32}$	$\frac{98}{32}$	$\frac{97}{32}$	negative
7	$\frac{194}{64}$	$\frac{196}{64}$	$\frac{195}{64}$	positive
8	$\frac{388}{128}$	$\frac{390}{128}$	$\frac{389}{128}$	negative
9	$\frac{778}{256}$	$\frac{780}{256}$	$\frac{779}{256}$	positive
10	$\frac{1556}{512}$	$\frac{1558}{512}$	$\frac{1557}{512}$	negative
11	$\frac{3114}{1024}$	$\frac{3116}{1024}$	$\frac{3115}{1024}$	negative
12	$\frac{6230}{2048}$	$\frac{6232}{2048}$	$\frac{6231}{2048}$	negative
13	$\frac{12462}{4096}$	$\frac{12464}{4096}$	$\frac{12463}{4096}$	positive
14	$\frac{24924}{8192}$	$\frac{24926}{8192}$	$\frac{24925}{8192}$	negative
15	$\frac{49850}{16384}$	$\frac{49852}{16384}$	$\frac{49851}{16384}$	negative
16	$\frac{99702}{32768}$	$\frac{99704}{32768}$	$\frac{99703}{32768}$	negative
17	$\frac{199406}{65536}$	$\frac{199408}{65536}$	$\frac{199407}{65536}$	negative
18	$\frac{398814}{131072}$	$\frac{398816}{131072}$	$\frac{398815}{131072}$	positive
19	$\frac{797628}{262144}$	$\frac{797630}{262144}$	$\frac{797629}{262144}$	positive

Table 1: *Values of c_i, d_i, θ_i, and the sign of $h(\theta_i)$.*

we conclude that $g(d_{19}) > 0$ and $g(c_{19}) < 0$. Thus, $\nu = 20$. Therefore, we define

$$c_{20} = \frac{1595256}{524288}, \quad d_{20} = \frac{1595258}{524288}.$$

By Theorem 7, there exists

$$\lambda \in [c_{20}, d_{20}]$$

such that z_λ is a solution of (3.1), (3.2). Let

$$\theta_{20} = \frac{c_{20} + d_{20}}{2} = \frac{1595257}{524288}.$$

Then, Theorem 7 guarantees

$$\|z_\lambda - Hw_{\theta_{20}}\|_\infty \leqslant \frac{\psi}{2}\left(\frac{(d-c)M_5}{2^{21}} + M_3\right) < 0.0011.$$

That is, $Hw_{\theta_{20}}$ lies in the set E, satisfies the boundary conditions (3.2), and approximates a solution z_λ of the boundary value problem (3.1), (3.2) to approximately one-thousandth.

5. Conclusions and next steps

Solving discrete initial value problems can be done recursively. Consequently, the shooting method for discrete boundary value problems relies on this recursive structure to establish the existence of solutions and compute approximate solutions. The method essentially reduces to finding an interval $[c,d]$ such that $h(c)h(d) < 0$, since solutions correspond to the zeros of the real-valued function $h(x)$. The appeal of the shooting method using difference equations lies in the relative ease of evaluating h – and, in principle, graphing it – to identify intervals where it changes sign. In this context, the initial guess for the trajectory is far more than a mere guess; it provides a systematic starting point.

However, further work is needed to refine this approach. This includes integrating computational tools, extending applications to a broader class of boundary value problems, identifying common themes and strategies, and simplifying error bounds for practical implementation. Most importantly, it is valuable to compare this method with other computational techniques for approximating solutions to boundary value problems. In particular, a computational comparison with the classical shooting method, using Euler or Runge-Kutta schemes to solve the underlying initial value problems, can provide insight into efficiency and accuracy.

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