Differential
Equations
& Applications
Volume 18, Number 2 (2026), 119–138
doi:10.7153/dea-2026-18-08

EXISTENCE AND NONEXISTENCE OF SIGN–CHANGING SOLUTIONS FOR LINEARLY PERTURBED SUPERLINEAR EQUATIONS ON EXTERIOR DOMAINS

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(*Communicated by C. O. Alves*)

Abstract. In this paper, we study the radial solutions of $\Delta u + K(|x|)f(u) + (N-2)^2 u/|x|^{2+(N-2)\delta} = 0$, $0 < \delta < 2$ in the exterior of the ball of radius $R > 0$ in $\mathbb{R}^N$ where f grows superlinearly at infinity and is singular at zero with $f(u) \sim -1/|u|^{q-1}u$ and $0 < q < 1$ for small u. We assume $K(|x|) \sim |x|^{-\alpha}$ for large $|x|$ and establish the existence of an infinite number of sign-changing solutions when $N + q(N-2) < \alpha < 2(N-1)$. We also prove nonexistence for $0 < \alpha \leqslant 2$.

1. Introduction

In this paper, we are interested in radial solutions of

$$\Delta u + K(|x|)f(u) + \frac{(N-2)^2 u}{|x|^{2+(N-2)\delta}} = 0, \quad 0 < \delta < 2 \quad \text{on} \quad \mathbb{R}^N \backslash B_R, \tag{1}$$

$$u = 0 \quad \text{on} \quad \partial B_R, \quad u \to 0 \quad \text{as} \quad |x| \to \infty \tag{2}$$

when $N > 2$ and where B_R is the ball of radius $R > 0$ centered at the origin. Assuming $u(x) = u(|x|) = u(r)$, then this becomes

$$u'' + \frac{N-1}{r}u' + K(r)f(u) + \frac{(N-2)^2 u}{r^{2+(N-2)\delta}} = 0 \quad \text{for} \quad R < r < \infty, \tag{3}$$

$$u(R) = 0, \quad \lim_{r\to\infty} u(r) = 0. \tag{4}$$

Numerous papers have proved the existence of *positive* solutions of these equations with various nonlinearities $f(u)$ and for various functions $K(|x|) \sim |x|^{-\alpha}$ with $\alpha > 0$. See, for example, [1], [5], [6], [11]–[13].

Here, we prove the existence of multiple solutions including *sign-changing* solutions for this equation. We have also proved the existence of *sign-changing* solutions in other recent papers: [2]–[4], [8]–[10].

Mathematics subject classification (2020): 34B40, 35B05.
Keywords and phrases: Exterior domains, semilinear, superlinear, radial, sign-changing.
This research is supported by Simons grant MP-TSM-00002319.
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We assume $f:\mathbb{R}\backslash\{0\}\to\mathbb{R}$ is odd, locally Lipschitz, and

$$f(u)=|u|^{p-1}u+g(u) \quad \text{with} \quad p>1 \quad \text{for large} \quad |u| \quad \text{and} \quad \lim_{u\to\infty}\frac{|g(u)|}{|u|^p}=0. \tag{H1}$$

We also assume there exists a locally Lipschitz $g_1:\mathbb{R}\to\mathbb{R}$ such that

$$f(u)=-\frac{1}{|u|^{q-1}u}+g_1(u) \quad \text{with} \quad 0<q<1 \quad \text{for small} \quad |u| \quad \text{and} \quad g_1(0)=0. \tag{H2}$$

In addition, we assume

$$f \quad \text{has a unique positive zero,} \quad \beta, \quad \text{with} \quad f<0 \quad \text{on} \quad (0,\beta) \quad \text{and} \quad f>0 \quad \text{on} \quad (\beta,\infty). \tag{H3}$$

Let $F(u)=\int_0^u f(t)\,dt$. Since f is odd, F is even. Also, since $0<q<1$ by $(H2)$, it follows that F is continuous with $F(0)=0$. Also,

$$F \quad \text{has a unique positive zero,} \quad \gamma, \quad \text{where} \quad 0<\beta<\gamma \quad \text{with} \quad F<0 \quad \text{on} \quad (0,\gamma) \quad \text{and} \quad F>0 \quad \text{on} \quad (\gamma,\infty). \tag{5}$$

In addition, it follows from $(H3)$ that there exists $F_0>0$ and $f_0>0$ such that

$$F(u)\geqslant -F_0 \quad \text{and} \quad |u|^{q-1}uf(u)\geqslant -f_0 \quad \text{for all} \quad u\in\mathbb{R}\backslash\{0\}. \tag{6}$$

We also assume $K>0$ and K' are continuous on $[R,\infty)$. In addition, we assume

$$\frac{rK'}{K}>-2(N-1), \quad \text{there exist positive} \quad K_0 \quad \text{and} \quad K_1 \quad \text{such that} \quad \frac{K_0}{r^\alpha}\leqslant K\leqslant\frac{K_1}{r^{\alpha_1}} \quad \text{with} \quad N+q(N-2)<\alpha_1\leqslant\alpha<2(N-1) \quad \text{on} \quad [R,\infty). \tag{H4}$$

REMARK 1. The reason for our interest in this problem with the extra term $(N-2)^2u/r^{2+(N-2)\delta}$ is that we will see in Section 2 that after a change of variables we will ultimately be examining the following boundary problem

$$v''+h(t)f(v)+\frac{v}{t^{2-\delta}}=0 \tag{7}$$

$$v(0)=0=v(R^{2-N}). \tag{8}$$

Now if we examine the linear part of this problem, we have

$$v''+\frac{v}{t^{2-\delta}}=0. \tag{9}$$

When $\delta>0$ it is well-known that all solutions of (9) are continuous and have only a *finite number of zeros* on $[0,R^{2-N}]$. In fact, the general solution of (9) is

$$y=c_1t^{\frac{1}{2}}J_{\frac{1}{\delta}}\left(\frac{2}{\delta}t^{\frac{\delta}{2}}\right)+c_2t^{\frac{1}{2}}Y_{\frac{1}{\delta}}\left(\frac{2}{\delta}t^{\frac{\delta}{2}}\right) \tag{10}$$

where $J_{1/\delta}$ and $Y_{1/\delta}$ are the Bessel functions of order $1/\delta$ of the first and second kind, and c_1, c_2 are constants.

However, when $\delta = 0$ the solution of

$$v'' + \frac{v}{t^2} = 0 \tag{11}$$

is

$$v_a = c_1 t^{\frac{1}{2}} \cos\left(\frac{\sqrt{3}}{2}\ln t\right) + c_2 t^{\frac{1}{2}} \sin\left(\frac{\sqrt{3}}{2}\ln t\right).$$

Thus we see that the solutions are continuous and have *infinitely many zeros* on $[0, R^{2-N}]$.

So then ultimately the challenge for us is what happens to solutions of (7)–(8) in the case when $0 < \delta < 2$? Are there solutions of (7)–(8) that have infinitely many zeros on $(0, R^{2-N})$ as there are for (11) or are there infinitely many solutions of (7)–(8) where each solution has only a finite number of zeros on $(0, R^{2-N})$? We were able to show it is this second situation which occurs.

In this paper, we prove the following:

THEOREM 1. (Main) *Let $N > 2$ and assume $(H1)$–$(H4)$. Then there exists $R_0 > 0$ and a non-negative integer n_0 so that if $R \geqslant R_0$ then there are infinitely many $C^1[R,\infty)$ solutions, u_n, of (3)–(4) such that u_n has exactly n zeros on (R,∞) for each non-negative integer $n \geqslant n_0$.*

REMARK 2. The addition of the linear term adds a number of difficulties to the proofs of several of the lemmas in Section 2. A number of integrals that arise are only defined when $0 < \delta < 2$. In addition, the constant $1/\delta$ appears as an upper bound in a number of estimates.

A similar problem but without the additional linear term was investigated in earlier papers. In [3] we proved Theorem 1 in the case $\alpha > 2(N-1)$. In [10] we proved Theorem 1 in the case $N + q(N-2) < \alpha < 2(N-1)$ and we proved nonexistence for $0 < \alpha < N + q(N-2)$.

REMARK 3. Solutions of (3)–(4) have continuous second derivatives except at points where $u(r_0) = 0$ due to the fact that $\lim_{u\to 0} |f(u)| = \infty$. Solutions, however, do turn out to be $C^1[R,\infty)$. Therefore, by a $C^1[R,\infty)$ *solution of* (3)–(4) we mean $u \in C^1[R,\infty)$ such that $1/|u|^q$ is integrable on $[R,\infty)$ and

$$r^{N-1}u' + \int_R^r t^{N-1}Kf(u) + \int_R^r \frac{u(t)}{t^{3-N+(N-2)\delta}}\,dt = R^{N-1}u'(R) \quad \text{for} \quad r \geqslant R$$
$$u(R) = 0, \quad \lim_{r\to\infty} u(r) = 0.$$

In this paper, we also consider the case where

$$-\frac{rK'}{K} < \alpha \quad \text{with} \quad 0 < \alpha \leqslant 2 \quad \text{on} \quad [R,\infty). \tag{$H5$}$$

We also prove:

THEOREM 2. *Assume* $(H1)$–$(H3)$ *and* $(H5)$. *If* $R>0$ *and* $N>2$, *then there are no* $C^1[R,\infty)$ *solutions of* (3)–(4) *when* $0<\alpha\leqslant 2$.

2. Preliminaries for Theorem 1

Let u solve (3)–(4) and suppose $(H1)$–$(H4)$ are satisfied. Now let $u(r)=v(r^{2-N})$. Then we see (3)–(4) is equivalent to

$$v''(t)+h(t)f(v(t))+\frac{v(t)}{t^{2-\delta}}=0 \quad \text{for} \quad 0<t<R^{2-N}, \tag{12}$$

$$v(0)=0, \quad v(R^{2-N})=0 \tag{13}$$

where

$$h(t)=\frac{t^{\frac{2(N-1)}{2-N}}K(t^{\frac{1}{2-N}})}{(N-2)^2}>0, \quad 0<\delta<2. \tag{14}$$

It follows from $(H4)$ that $h'<0$.

We now define $\tilde{\alpha}=(2(N-1)-\alpha)/(N-2)$ and $\tilde{\alpha}_1=(2(N-1)-\alpha_1)/(N-2)$. Note that the assumption $N+q(N-2)<\alpha_1\leqslant\alpha<2(N-1)$ from $(H4)$ implies

$$0<\tilde{\alpha}+q\leqslant\tilde{\alpha}_1+q<1. \tag{15}$$

It also follows from $(H4)$ and (14) that there exist $h_0>0$ and $h_1>0$ such that

$$h_0t^{-\tilde{\alpha}}\leqslant h\leqslant h_1t^{-\tilde{\alpha}_1} \quad \text{for} \quad 0<t<R^{2-N}, \tag{16}$$

and from this and (15) it follows that $h(t)/t^q$ and $h(t)$ are integrable on $(0,R^{2-N}]$.

We now consider the following initial value problem

$$v_a''(t)+h(t)f(v_a(t))+\frac{v_a(t)}{t^{2-\delta}}=0 \quad \text{for} \quad t>0, \tag{17}$$

$$v_a(0)=0, \quad v_a'(0)=a>0. \tag{18}$$

REMARK 4. We do not consider the case when $\delta=0$ with the initial conditions in (18) because if there is such a solution then we see that $tv_a''\sim -a$ for small t and so then dividing by t and integrating gives that $v_a'\sim -a\ln(t)$ which blows up as t goes to 0 and so we cannot have $v_a(0)=0$ and $v_a'(0)=-a$ when $\delta=0$.

LEMMA 2.1. *Let* $N>2$, $a>0$, *and assume* $(H1)$–$(H4)$. *Then there exists an* $\varepsilon(a)>0$ *such that there is a unique solution of* (17)–(18) *on* $[0,\varepsilon(a)]$. *In addition,* $\varepsilon(a)$ *can be chosen uniformly in* a *in compact subsets of* $(0,\infty)$.

Proof. We first assume v_a solves (17)–(18), then integrate (17) on $(0,t)$ and use (18) along with $(H2)$ to obtain

$$v_a'=a-\int_0^t h(x)f(v_a(x))\,dx-\int_0^t x^{\delta-2}v_a(x)\,dx. \tag{19}$$

Now, integrate (19) on $(0,t)$ to obtain

$$v_a = at - \int_0^t \int_0^s h(x) f(v_a(x))\, dx\, ds - \int_0^t \int_0^s x^{\delta-2} v_a(x)\, dx\, ds. \tag{20}$$

Next, let $w(t) = v_a(t)/t$. Then, we obtain $w(0) = a > 0$ and

$$w(t) = a - \frac{1}{t}\int_0^t \int_0^s h(x) f(xw(x))\, dx\, ds - \frac{1}{t}\int_0^t \int_0^s x^{\delta-1} w(x)\, dx\, ds \quad \text{for} \quad t > 0. \tag{21}$$

Now using the contraction mapping principle, we prove the existence of a solution of (17)–(18) on $[0,\varepsilon]$ for some $\varepsilon > 0$. Let

$$\mathscr{B} := \left\{ w \in C[0,\varepsilon] \mid w(0) = a \text{ and } |w(t) - a| \leqslant \frac{a}{2} \text{ on } [0,\varepsilon] \right\},$$

where $C[0,\varepsilon]$ is the set of all continuous functions on $[0,\varepsilon]$, $\varepsilon > 0$, and

$$||w|| = \sup_{x\in[0,\varepsilon]} |w(x)|.$$

Then $(\mathscr{B}, ||.||)$ is a Banach space [7]. Now define a map T on $\mathscr{B}$ by:

$$Tw(t) = \begin{cases} a & \text{for} \quad t = 0 \\ a - \frac{1}{t}\int_0^t \int_0^s h(x) f(xw(x))\, dx\, ds - \frac{1}{t}\int_0^t \int_0^s x^{\delta-1} w(x)\, dx\, ds & \text{for} \quad 0 < t \leqslant \varepsilon. \end{cases}$$

Next, note that for any $w \in \mathscr{B}$, we have

$$0 < \frac{a}{2} \leqslant w(x) \leqslant \frac{3a}{2} \quad \text{on} \quad [0,\varepsilon]. \tag{22}$$

Then $|-1/(x^q w^q(x))| \leqslant 2^q x^{-q}/a^q$ on $(0,\varepsilon]$. Since g_1 is locally Lipschitz by $(H2)$ and $g_1(0) = 0$, therefore there exists $L_1 > 0$ such that for some $\varepsilon_1 > 0$

$$|g_1(xw(x))| \leqslant \frac{3aL_1}{2} x \quad \text{on} \quad [0,\varepsilon_1]. \tag{23}$$

Let $\varepsilon_2 := \min\{\varepsilon, \varepsilon_1\}$. Then it follows from this, (16), and (23)

$$|h(x) f(xw(x))| \leqslant \frac{2^q h_1}{a^q} x^{-(\tilde{\alpha}+q)} + \frac{3ah_1L_1}{2} x^{1-\tilde{\alpha}} \quad \text{on} \quad (0,\varepsilon_2].$$

Integrating on $(0,t) \subset (0,\varepsilon_2)$, and using (15), we obtain

$$\int_0^t |h(x) f(xw(x))|\, dx \leqslant C_1 t^{1-\tilde{\alpha}-q} + C_2 t^{2-\tilde{\alpha}} \to 0 \quad \text{as} \quad t \to 0^+, \tag{24}$$

where $C_1 := 2^q h_1/(a^q(1-\tilde{\alpha}-q))$ and $C_2 := 3ah_1L_1/(2(1-\tilde{\alpha}))$.

Integrating (24) again, we obtain

$$\frac{1}{t}\int_0^t \int_0^s |h(x) f(xw(x))|\, dx\, ds \to 0 \quad \text{as} \quad t \to 0^+. \tag{25}$$

Since $|x^{\delta-1}w(x)| \leqslant 3ax^{\delta-1}/2$ on $(0,\varepsilon_2)$ $(\because w \in \mathscr{B})$, integrating both sides of this inequality on $(0,t) \subset (0,\varepsilon_2)$ gives

$$\int_0^t |x^{\delta-1}w(x)|\,dx \leqslant \frac{3a}{2\delta}t^{\delta}.$$

Integrating again on $(0,t) \subset (0,\varepsilon_2)$, we obtain

$$\frac{1}{t}\int_0^t\int_0^s \left|x^{\delta-1}w(x)\right|\,dx\,ds \leqslant \frac{3a}{2\delta(\delta+1)}t^{\delta} \to 0 \quad \text{as} \quad t \to 0^+. \tag{26}$$

We now show that $Tw \in \mathscr{B}$ for each $w \in \mathscr{B}$ if $\varepsilon_2 > 0$ is sufficiently small. For any $w \in \mathscr{B}$, $Tw(0) = a$ and it is clear from the definition and (25)–(26) that $Tw(t)$ is continuous for any $t \in (0,\varepsilon_2]$. It is also clear from (25) that

$$|Tw(t)-a| \leqslant \frac{1}{t}\int_0^t\int_0^s |h(x)f(xw(x))|\,dx\,ds + \frac{1}{t}\int_0^t\int_0^s |x^{\delta-1}w(x)|\,dx\,ds \to 0 \quad \text{as } t \to 0^+.$$

Thus $|Tw(t)-a| \leqslant a/2$ for sufficiently small $t > 0$. Therefore $T : \mathscr{B} \longrightarrow \mathscr{B}$ if ε_2 is sufficiently small.

We next prove that T is a contraction mapping if ε is sufficiently small. Let $w_1, w_2 \in \mathscr{B}$. Then

$$|Tw_1 - Tw_2| \leqslant \frac{1}{t}\int_0^t\int_0^s |h(x)[f(xw_1)-f(xw_2)]|\,dx\,ds + \frac{1}{t}\int_0^t\int_0^s |x^{\delta-1}(w_1-w_2)|\,dx\,ds. \tag{27}$$

Note that by $(H2)$

$$|f(xw_1)-f(xw_2)| \leqslant \frac{1}{x^q}\left|\frac{1}{{w_1}^q} - \frac{1}{{w_2}^q}\right| + L_1 x|w_1-w_2|,$$

where L_1 is the Lipschitz constant for g_1. Then using the mean value theorem for any $x \in (0,\varepsilon)$ there is a c_x with $a/2 \leqslant c_x \leqslant 3a/2$ such that

$$\frac{1}{x^q}\left|\frac{1}{{w_1}^q} - \frac{1}{{w_2}^q}\right| = \frac{q}{x^q c_x^{q+1}}|w_1-w_2| \leqslant \frac{2^{q+1}q}{a^{q+1}}|w_1-w_2|x^{-q}.$$

It follows from this, (15), (16), and integrating on $(0,t)$ that

$$\int_0^t |h(x)(f(xw_1)-f(xw_2))|\,dx \leqslant \frac{2^{q+1}qh_2}{(1-\tilde{\alpha}-q)a^{q+1}}|w_1-w_2|t^{1-(\tilde{\alpha}+q)} + \frac{L_1h_2}{2-\tilde{\alpha}}|w_1-w_2|t^{2-\tilde{\alpha}}.$$

Integrating again on $(0,t)$, we obtain

$$\int_0^t\int_0^s |h(x)(f(xw_1)-f(xw_2))|\,dx\,ds \leqslant C_3|w_1-w_2|t^{2-(\tilde{\alpha}+q)} + C_4|w_1-w_2|t^{3-\tilde{\alpha}},$$

where $C_3 = 2^{q+1}qh_1/((1-\tilde{\alpha}-q)(2-\tilde{\alpha}-q)a^{q+1})$ and $C_4 = L_1h_1/((2-\tilde{\alpha})(3-\tilde{\alpha}))$. Thus

$$\frac{1}{t}\int_0^t\int_0^s |h(x)(f(xw_1)-f(xw_2))|\,dx\,ds \leqslant (C_3\varepsilon^{1-(\tilde{\alpha}+q)}+C_4\varepsilon^{2-\tilde{\alpha}})||w_1-w_2|| \quad \text{on} \quad (0,\varepsilon). \tag{28}$$

Also, since $|x^{\delta-1}(w_1-w_2)| \leqslant x^{\delta-1}||w_1-w_2||$, then integrating twice on $(0,t)\subset(0,\varepsilon)$, we obtain

$$\frac{1}{t}\int_0^t\int_0^s |x^{\delta-1}(w_1-w_2)|\,dx\,ds \leqslant \frac{\varepsilon^\delta}{\delta(\delta+1)}||w_1-w_2||. \tag{29}$$

Using (28) and (29) in (27), we see that

$$|Tw_1-Tw_2| \leqslant C_\varepsilon||w_1-w_2||,$$

where $C_\varepsilon := C_3\varepsilon^{1-(\tilde{\alpha}+q)}+C_4\varepsilon^{2-\tilde{\alpha}}+\varepsilon^\delta/(\delta(\delta+1))$. Note that $C_\varepsilon \to 0$ as $\varepsilon \to 0^+$. Therefore, $0<C_\varepsilon<1$ for sufficiently small $\varepsilon>0$, and hence T is a contraction on $\mathscr{B}$. Then by the contraction mapping principle there exists a unique $w\in\mathscr{B}$ such that $Tw=w$ on $[0,\varepsilon]$ for some $\varepsilon>0$, and hence $v_a(t)=tw(t)$ is a solution of (17)–(18) on $[0,\varepsilon]$.

Moreover, from (22), we see that $at/2 \leqslant v_a \leqslant 3at/2$. So v_a is bounded on $[0,\varepsilon]$. Also, from (25) and (26) it follows that v_a' is bounded on $[0,\varepsilon]$. This completes Lemma 2.1. □

LEMMA 2.2. *Let $N>2$ and assume $(H1)$–$(H4)$. Suppose v_a solves (17)–(18) with $a>0$ on some maximal half-open interval $[0,d)$ with $\varepsilon \leqslant d \leqslant R^{2-N}$. Then there exists a constant $C>0$ where C is independent of d such that $|v_a(t)|\leqslant C$ and $|v_a'(t)|\leqslant C$ on $[0,d]\subset[0,R^{2-N}]$. (In addition, we will see in the proof that C can be chosen independently of a for compact subsets of $(0,\infty)$). In particular, it follows from this that v_a and v_a' are continuous, can be extended to all of $[0,R^{2-N}]$, and that v_a varies continuously with $a>0$.*

Proof. Note it follows from (17)–(18) that

$$\left(\frac{1}{2}v_a'^2(t)+\frac{v_a^2(t)}{2t^{2-\delta}}+h(t)F(v_a(t))\right)' \leqslant h'(t)F(v_a(t)).$$

Integrating both sides on $[\varepsilon,t]$ where $[0,\varepsilon]$ is the interval of existence obtained in Lemma 2.1, we obtain

$$\frac{1}{2}v_a'^2(t)+h(t)F(v_a(t)) \leqslant \frac{1}{2}v_a'^2(\varepsilon)+\frac{v_a^2(\varepsilon)}{2\varepsilon^{2-\delta}}+h(\varepsilon)F(v_a(\varepsilon))+\int_\varepsilon^t h'(s)F(v_a(s))\,ds.$$

It follows from (6) that $F\geqslant -F_0$ for some F_0. Since $h>0$, therefore $hF\geqslant -hF_0$, $h'F\leqslant -F_0h'$ $(\because h'<0$ from the comment after (14)), and hence

$$v_a'^2(t) \leqslant v_a'^2(\varepsilon)+\frac{v_a^2(\varepsilon)}{\varepsilon^{2-\delta}}+2h(\varepsilon)(F(v_a(\varepsilon))+F_0) \quad \text{on} \quad [\varepsilon,d].$$

Taking the square root on both sides, we obtain $|v_a'(t)| \leqslant C_5$ on $[\varepsilon, d]$, where

$$C_5 := \sqrt{v_a'^2(\varepsilon) + \frac{v_a^2(\varepsilon)}{\varepsilon^{2-\delta}} + 2h(\varepsilon)(F(v_a(\varepsilon)) + F_0)}.$$

Also, since $v_a(t) = v_a(\varepsilon) + \int_\varepsilon^t v_a'(s)\,ds$, then $|v_a(t)| \leqslant C_6$ on $[\varepsilon, d]$ where $C_6 := C_5(R^{2-N} - \varepsilon) + |v_a(\varepsilon)|$.

Now combining this with the bounds obtained for v_a and v_a' on $[0, \varepsilon]$ in Lemma 2.1, we see that v_a and v_a' are uniformly bounded on $[0, d]$ by a constant that only depends on ε and R. It follows from this that v_a and v_a' can be extended to be defined on $[0, R^{2-N}]$, and v_a, v_a' are uniformly bounded on $[0, R^{2-N}]$. It also follows from this that v_a varies continuously with $a > 0$. This completes the proof. □

LEMMA 2.3. *Let $N > 2$, and assume $(H1)$–$(H4)$ or $(H1)$–$(H3)$, and $(H5)$. Let v_a solve (17)–(18). If $v_a(z_a) = 0$ and M_a is a local maximum (or local minimum) with $0 \leqslant z_a < M_a$ and $v_a' \geqslant 0$ (or $v_a' \leqslant 0$) on (z_a, M_a), then there is a $C_7 > 0$ such that $|v_a(M_a)| \geqslant C_7$ where C_7 is independent of a. In addition, there exists $R_0 > 0$ such that $|v_a(M_a)| > \gamma$ if $R > R_0$.*

Proof. Let us suppose $v_a > 0$ and $v_a' \geqslant 0$ on (z_a, M_a). If $v_a(M_a) > \gamma$ then we are done and we can just use $C_7 = \gamma$. So let us suppose $0 < v_a(M_a) \leqslant \gamma$. Then rewrite (17) as

$$((t - z_a)v_a' - v_a)' + (t - z_a)hg_1(v_a) + (t - z_a)\frac{v_a}{t^{2-\delta}} = \frac{(t - z_a)h}{v_a^q}.$$

Integrate on (z_a, t) to obtain

$$(t - z_a)v_a' - v_a + \int_{z_a}^t (s - z_a)hg_1(v_a)\,ds + \int_{z_a}^t (s - z_a)\frac{v_a}{s^{2-\delta}}\,ds = \int_{z_a}^t \frac{(s - z_a)h}{v_a^q}\,ds.$$

Now let $t = M_a$ to get

$$\int_{z_a}^{M_a} (s - z_a)hg_1(v_a)\,ds + \int_{z_a}^{M_a} (s - z_a)\frac{v_a}{s^{2-\delta}}\,ds = \int_{z_a}^{M_a} \frac{(s - z_a)h}{v_a^q}\,ds + v_a(M_a). \tag{30}$$

Since v_a is increasing on (z_a, M_a), then we see

$$\int_{z_a}^{M_a} (s - z_a)\frac{v_a}{s^{2-\delta}}\,ds \leqslant \int_{z_a}^{M_a} \frac{v_a}{s^{1-\delta}}\,ds \leqslant \frac{v_a(M_a)(M_a^\delta - z_a^\delta)}{\delta}$$

and

$$\int_{z_a}^{M_a} \frac{(s - z_a)h}{v_a^q}\,ds \geqslant \frac{1}{v_a^q(M_a)} \int_{z_a}^{M_a} (s - z_a)h\,ds. \tag{31}$$

Since $v_a(M_a) \leqslant \gamma$, then estimating the leftmost term of (30) we obtain

$$\int_{z_a}^{M_a} (s - z_a)hg_1(v_a)\,ds \leqslant \left(\max_{[0,\gamma]} |g_1|\right) \int_{z_a}^{M_a} (s - z_a)h\,ds. \tag{32}$$

Combining (31)–(32) and inserting into (30) gives

$$\Big(\max_{[0,\gamma]}|g_1|\Big)\int_{z_a}^{M_a}(s-z_a)h\,ds \geqslant \frac{1}{v_a^q(M_a)}\int_{z_a}^{M_a}(s-z_a)h\,ds + v_a(M_a)\Big(1-\frac{\big(M_a^\delta - z_a^\delta\big)}{\delta}\Big). \tag{33}$$

Case 1. $\big(1-(M_a^\delta - z_a^\delta)/\delta\big) \geqslant 0$
In this case, it follows from (33) that

$$v_a(M_a) \geqslant 1/\Big(\max_{[0,\gamma]}|g_1|\Big)^{\frac{1}{q}}. \tag{34}$$

Case 2. $\big(1-(M_a^\delta - z_a^\delta)/\delta\big) < 0$
In this case (33) becomes

$$\Big(\max_{[0,\gamma]}|g_1|\Big)\int_{z_a}^{M_a}(s-z_a)h\,ds + v_a(M_a)\Big(\frac{\big(M_a^\delta - z_a^\delta\big)}{\delta}-1\Big) \geqslant \frac{1}{v_a^q(M_a)}\int_{z_a}^{M_a}(s-z_a)h\,ds. \tag{35}$$

Thus,

$$\Big(\max_{[0,\gamma]}|g_1|\Big)\int_{z_a}^{M_a}(s-z_a)h\,ds + \frac{R^{(2-N)\delta}\gamma}{\delta} \geqslant \frac{1}{v_a^q(M_a)}\int_{z_a}^{M_a}(s-z_a)h\,ds. \tag{36}$$

Hence

$$v_a^q(M_a) \geqslant 1/\Big(\max_{[0,\gamma]}|g_1| + \frac{R^{(2-N)\delta}\gamma}{\delta\int_{z_a}^{M_a}(s-z_a)h\,ds}\Big). \tag{37}$$

Finally, we will find a lower bound for $\int_{z_a}^{M_a}(s-z_a)h\,ds$. Whether we assume $(H4)$ or $(H5)$ it follows that $2(N-1)K + rK' \geqslant 0$ which implies that h is decreasing, and hence we see

$$\int_{z_a}^{M_a}(s-z_a)h\,ds \geqslant h(R^{2-N})\int_{z_a}^{M_a}(s-z_a)\,ds = \frac{h(R^{2-N})(M_a-z_a)^2}{2}.$$

Since we are in case 2, then we see $M_a \geqslant (z_a^\delta+\delta)^{1/\delta}$, and hence

$$\int_{z_a}^{M_a}(s-z_a)h\,ds \geqslant \frac{h(R^{2-N})(M_a-z_a)^2}{2} \geqslant \frac{h(R^{2-N})}{2}\Big(\big(z_a^\delta+\delta\big)^{\frac{1}{\delta}} - z_a\Big)^2.$$

Now note that if we let $b(x) = (x^\delta+\delta)^{1/\delta} - x$ where $x \in [0, R^{2-N}]$ and $0 < \delta < 2$, then we see $b(x)$ is decreasing for $1 < \delta < 2$, is constant for $\delta = 1$, and is increasing for $0 < \delta < 1$. So in all cases we see $b(x) \geqslant b_0 = \min\{\delta^{1/\delta}, (R^{(2-N)\delta}+\delta)^{1/\delta} - R^{2-N})\} > 0$. Thus we see

$$\int_{z_a}^{M_a}(s-z_a)h\,ds \geqslant h(R^{2-N})\int_{z_a}^{M_a}(s-z_a)\,ds = \frac{h(R^{2-N})(M_a-z_a)^2}{2} \geqslant \frac{b_0^2 h(R^{2-N})}{2}.$$

Then from (37), we obtain

$$v_a(M_a) \geqslant 1/\Big(\max_{[0,\gamma]}|g_1| + \frac{2R^{(2-N)\delta}\gamma}{\delta b_0^2 h(R^{2-N})}\Big)^{\frac{1}{q}}. \tag{38}$$

Finally, combining (34), (38) and the possibility that $v_a(M_a) \geqslant \gamma$, we let

$$C_7 = \min\left\{\gamma, \frac{1}{\left(\max\limits_{[0,\gamma]}|g_1| + \frac{2R^{(2-N)\delta}\gamma}{\delta b_0^2 h(R^{2-N})}\right)^{\frac{1}{q}}}, \frac{1}{\left(\max\limits_{[0,\gamma]}|g_1|\right)^{\frac{1}{q}}}\right\}.$$

Hence $|v_a(M_a)| \geqslant C_7 > 0$. This completes the first part of the lemma.

For the second part of the lemma, we again rewrite (17) as

$$((t-z_a)v_a' - v_a)' + (t-z_a)hf(v_a) + (t-z_a)\frac{v_a}{t^{2-\delta}} = 0.$$

Integrating on (z_a, M_a) we obtain

$$v_a(M_a) = \int_{z_a}^{M_a}(t-z_a)h(t)f(v_a(t))\,dt + \int_{z_a}^{M_a}(t-z_a)\frac{v_a(t)}{t^{2-\delta}}\,dt. \tag{39}$$

Note that

$$\int_{z_a}^{M_a}(t-z_a)\frac{v_a(t)}{t^{2-\delta}}\,dt \leqslant \frac{v_a(M_a)}{\delta}\left(M_a^\delta - z_a^\delta\right) \leqslant \frac{v_a(M_a)}{\delta}R^{(2-N)\delta}.$$

Since $R^{(2-N)\delta}/\delta \to 0$ as $R \to \infty$, therefore there exists $R_1 > 0$ such that

$$\int_{z_a}^{M_a}(t-z_a)\frac{v_a(t)}{t^{2-\delta}}\,dt \leqslant \frac{1}{2}v_a(M_a) \quad \text{for} \quad R \geqslant R_1. \tag{40}$$

Also, using (16) and $(H1)$ this then gives $f(v_a) \leqslant C_* v_a^p$ for some $C_* > 0$, and so we obtain

$$\int_{z_a}^{M_a}(t-z_a)h(t)f(v_a(t))\,dt \leqslant C_* h_1 \int_{z_a}^{M_a} t^{1-\tilde{\alpha}_1}v_a^p(t)\,dt \leqslant \frac{C_* h_1 v_a^p(M_a)}{2-\tilde{\alpha}_1}R^{(2-N)(2-\tilde{\alpha}_1)}. \tag{41}$$

Now using (40) and (41) in (39), we obtain for $R > R_1$

$$v_a^{p-1}(M_a) \geqslant \frac{2-\tilde{\alpha}_1}{2h_1}R^{(N-2)(2-\tilde{\alpha}_1)} \to \infty \quad \text{as} \quad R \to \infty.$$

Hence there exists $R_0 > R_1$ such that $v_a(M_a) > \gamma$ if $R > R_0$ for all $a > 0$. This completes the lemma. □

LEMMA 2.4. *Let $N > 2$, $a > 0$, and assume $(H1)$–$(H4)$. Suppose v_a solves (17)–(18). Then v_a has at most a finite number of zeros on $[0, R^{2-N}]$.*

Proof. Suppose by way of contradiction that v_a has an infinite number of zeros, z_k, and without loss of generality, let us assume $z_k < z_{k+1}$. Then v_a would have an infinite number of extrema, M_k, with $z_k < M_k < z_{k+1}$. Since z_k and M_k are in the compact set $[0, R^{2-N}]$, it follows that z_k would have a limit point z, and also we would have $M_k \to z$. Now, by the mean value theorem and Lemma 2.3, $C_7 \leqslant |v_a(M_k)| = |v_a(M_k) - v_a(z_k)| = |v_a'(c_k)||M_k - z_k|$ for $z_k < c_k < M_k$. However, $|M_k - z_k| \to 0$. Thus $|v_a'(c_k)| \to \infty$, contradicting that the $|v_a'(c_k)|$ are bounded by Lemma 2.2. Thus, v_a has at most a finite number of zeros. □

LEMMA 2.5. *Let $N > 2$, $a > 0$, and assume $(H1)$–$(H4)$. Suppose v_a solves (17)–(18). Then*

$$\max_{[0,\, R^{2-N}]} |v_a| \to \infty \quad as \quad a \to \infty.$$

Proof. Suppose by way of contradiction that there is a $C_8 > 0$ such that $|v_a(t)| \leqslant C_8$ for all $t \in [0,\, R^{2-N}]$ and for all $a > 0$. Since g_1 is continuous, there is a $C_9 > 0$ such that $|g_1(v_a(t))| \leqslant C_9$ for all $t \in [0,\, R^{2-N}]$.

Now, consider the following initial value problem

$$y''(t) + \frac{y(t)}{t^{2-\delta}} = 0 \quad \text{for} \quad t > 0, \tag{42}$$

$$y(0) = 0, \quad y'(0) = 1. \tag{43}$$

Note that this is equivalent to Bessel's equation of order $1/\delta$ and in fact $y(t) = At^{1/2} J_{1/\delta}(2t^{\delta/2}/\delta)$ with $A > 0$. Now multiply (17) by $y(t)$, (42) by $v_a(t)$ and subtract to obtain

$$(y(t)v_a'(t) - v_a(t)y'(t))' + h(t)y(t)f(v_a(t)) = 0. \tag{44}$$

Let $J_{\frac{1}{\delta},1}$ be the first positive zero of the Bessel function $J_{\frac{1}{\delta}}$. Using $(H2)$ we see $h(t)y(t)f(v_a) \leqslant h(t)y(t)g_1(v_a)$ and integrating (44) on $(0,t) \subset (0,t_1)$ where $t_1 > 0$ is sufficiently small and $t_1 < J_{1/\delta,1}$, we obtain

$$y(t)v_a'(t) - v_a(t)y'(t) + \int_0^t h(s)y(s)g_1(v_a(s))\, ds \geqslant 0.$$

Then rewriting the above inequality, we get

$$\left(\frac{v_a(t)}{y(t)}\right)' + \frac{1}{y^2(t)} \int_0^t h(s)y(s)g_1(v_a(s))\, ds \geqslant 0 \quad \forall\, t \in (0,t_1).$$

Integrating again on $(0,t) \subset (0,t_1)$, we see

$$v_a(t) + y(t) \int_0^t \frac{1}{y^2(t)} \int_0^s h(x)y(x)g_1(v_a(x))\, dx\, ds \geqslant ay(t). \tag{45}$$

Now, since $y(t) \geqslant 0$ on $[0,t_1]$, then from (42) we see $y'' \leqslant 0$ on $[0,t_1]$. Therefore, integrating this twice on $(0,t)$ where $0 < t < t_1$, we obtain

$$y(t) \leqslant t \quad \forall\, t \in [0,t_1]. \tag{46}$$

In addition, since $y(0) = 0$ and $y'(0) = 1$, there exists a t_0 with $0 < t_0 < t_1$ such that $y(t) \geqslant t/2$ for all $t \in [0,t_0]$. Also, since $y(t) > 0$ for all $t \in [t_0,t_1)$, then there exists a $C_{y_1} > 0$ such that $y(t) \geqslant C_{y_1}$ on $[t_0,t_1]$, and hence $y(t) \geqslant C_{y_2} t$ on $[t_0,t_1]$ where $C_{y_2} := C_{y_1}/t_1$. Now set $C_y := \min\{1/2, C_{y_2}\}$. Then $y(t) \geqslant C_y t$ for all $t \in [0,t_1]$, and using (46) we see

$$C_y t \leqslant y(t) \leqslant t \quad \forall\, t \in [0,t_1]. \tag{47}$$

Using (16) and (47) in (45), we obtain

$$|ay(t)| \leqslant \left| v_a(t) + y(t) \int_0^t \frac{1}{y^2(t)} \int_0^s h(x)y(x)g_1(v_a(x))\, dx\, ds \right|$$
$$\leqslant C_8 + \frac{C_9 h_1 t_1^{2-\tilde{\alpha}}}{C_y^2(1-\tilde{\alpha})(2-\tilde{\alpha})} \quad \text{on} \quad [0,t_1]. \tag{48}$$

Now notice that $|ay(t)| \to \infty$ as $a \to \infty$ for any fixed t with $0 < t < t_1$. However, the right-hand side of (48) is bounded for all a and the left-hand side is not. This is a contradiction. Thus

$$\max_{[0,\, t_1]} |v_a| \to \infty \quad \text{as} \quad a \to \infty. \tag{49}$$

Since $\max\limits_{t\in[0,\, R^{2-N}]} |v_a(t)| \geqslant \max\limits_{t\in[0,\, t_1]} |v_a(t)|$, therefore we see $\max\limits_{[0,\, R^{2-N}]} |v_a| \to \infty$ as $a \to \infty$. □

LEMMA 2.6. *Let $N > 2$ and assume $(H1)$–$(H4)$. Suppose v_a solves (17)–(18). Then v_a has a first local maximum, M_a, if $a > 0$ is sufficiently large. In addition, $M_a \to 0$ and $v_a(M_a) \to \infty$ as $a \to \infty$.*

Proof. Suppose not. Therefore, assume that v_a is non-decreasing on $[0, R^{2-N}]$.

Claim. For sufficiently large $a > 0$, there exists a $t_{a,\beta} \in (0, R^{2-N})$ such that $v_a(t_{a,\beta}) = \beta$ and $t_{a,\beta} \to 0$ as $a \to \infty$, where $t_{a,\beta}$ is the smallest value of t for which $v_a(t_{a,\beta}) = \beta$.

Proof of Claim. Rearranging terms and integrating (44) twice on $(0,t) \subset (0, J_{1/\delta,1}) \cap (0, R^{2-N})$, we obtain

$$v_a(t) + y(t) \int_0^t \frac{1}{y^2(t)} \int_0^s h(x)y(x)f(v_a(x))\, dx\, ds = ay(t) \quad \forall\, t \in (0, J_{\frac{1}{\delta},1}) \cap (0, R^{2-N}). \tag{50}$$

Now, if $v_a < \beta$ for all a, then (50) implies

$$v_a(t) \geqslant ay(t) = aAt^{\frac{1}{2}} J_{\frac{1}{\delta}}\left(\frac{2}{\delta} t^{\frac{\delta}{2}}\right) \quad \forall\, t \in (0, J_{\frac{1}{\delta},1}) \cap (0, R^{2-N}). \tag{51}$$

Now let $0 < t_0 < J_{1/\delta,1}$ and $t_0 < R^{2-N}$, then $ay(t_0) = aAt_0^{1/2} J_{1/\delta}\left(2t_0^{\delta/2}/\delta\right) \to \infty$ as $a \to \infty$, so we see that v_a exceeds β, which contradicts our assumption. Therefore, there exists a smallest positive $t_{a,\beta}$ such that $v_a(t_{a,\beta}) = \beta$ and $0 < t_{a,\beta} < t_0$. In addition, evaluating (51) at $t_{a,\beta}$, we see

$$\frac{\beta}{a} \geqslant At_{a,\beta}^{\frac{1}{2}} J_{\frac{1}{\delta}}\left(\frac{2}{\delta} t_{a,\beta}^{\frac{\delta}{2}}\right).$$

It follows from this that $t_{a,\beta} \to 0$ as $a \to \infty$. This proves the claim.

Continuing now with our assumption that v_a is non-decreasing on $[0, R^{2-N}]$, then we see that given $\varepsilon > 0$, $\min\limits_{[\varepsilon,\, R^{2-N}]} |v_a| = v_a(\varepsilon) = \max\limits_{t\in[0,\, \varepsilon]} v_a(t) \to \infty$ as $a \to \infty$ by (49).

Thus $v_a \to \infty$ uniformly on $[\varepsilon,\ R^{2-N}]$. Then for any $t \in [\varepsilon,\ R^{2-N}]$, we obtain

$$\frac{h(t)f(v_a(t))}{v_a(t)} + \frac{1}{t^{2-\delta}} \geqslant \frac{h(t)f(v_a(t))}{v_a(t)} \geqslant h(t)v_a^{p-1}(t)\Big(1 - \Big|\frac{g(v_a(t))}{v_a^p(t)}\Big|\Big)$$
$$\geqslant h(R^{2-N})v_a^{p-1}(t)\Big(1 - \Big|\frac{g(v_a(t))}{v_a^p(t)}\Big|\Big). \tag{52}$$

Since $v_a \to \infty$ uniformly as $a \to \infty$ on $[\varepsilon,\ R^{2-N}]$, then $\big|g(v_a(t))/v_a^p(t)\big| \to 0$ uniformly as $a \to \infty$ by $(H1)$, and since h is bounded from below on $[\varepsilon,\ R^{2-N}]$, then the right-hand side of (52) goes to ∞ as $a \to \infty$, *i.e.*

$$\frac{h(t)f(v_a(t))}{v_a(t)} + \frac{1}{t^{2-\delta}} \to \infty \quad \text{uniformly on} \quad [\varepsilon,\ R^{2-N}] \quad \text{as} \quad a \to \infty. \tag{53}$$

Now let $T_a := \min\limits_{[\varepsilon,\ R^{2-N}]} \big[h(t)f(v_a(t))/v_a(t) + 1/t^{2-\delta}\big]$. Then $T_a \to \infty$ as $a \to \infty$ by (53). Next, let $\tilde{y}_a$ be the solution of

$$\tilde{y}_a'' + T_a\tilde{y}_a = 0 \tag{54}$$

with the same initial conditions as v_a at $t = \varepsilon$. Then $\tilde{y}_a$ is a linear combination of $\sin(\sqrt{T_a}t)$ and $\cos(\sqrt{T_a}t)$, and it is well known that $\tilde{y}_a$ has a zero on $\big[\varepsilon, \varepsilon + \pi/\sqrt{T_a}\big]$. Then for a sufficiently large $\big[\varepsilon, \varepsilon + \pi/\sqrt{T_a}\big] \subset [\varepsilon,\ R^{2-N}]$ and so $\tilde{y}_a$ has a zero, $z_{\tilde{y}_a}$, on $[\varepsilon,\ R^{2-N}]$. Hence $\tilde{y}_a$ has a local maximum, M_{y_a}, on this interval with $\tilde{y}_a$ increasing on $[\varepsilon,\ M_{\tilde{y}_a}]$.

Next, rewrite (18) as

$$v_a'' + \Big(\frac{h(t)f(v_a(t))}{v_a(t)} + \frac{1}{t^{2-\delta}}\Big)v_a = 0. \tag{55}$$

Multiplying (55) by $\tilde{y}_a$, (54) by v_a, and subtracting gives

$$(v_a'\tilde{y}_a - v_a\tilde{y}_a')' + \Big(\Big(\frac{h(t)f(v_a(t))}{v_a(t)} + \frac{1}{t^{2-\delta}}\Big) - T_a\Big)v_a\tilde{y}_a = 0.$$

Integrating on $(\varepsilon, M_{\tilde{y}_a})$ and recalling $\tilde{y}_a(M_{\tilde{y}_a}) > 0$, $\tilde{y}_a'(M_{\tilde{y}_a}) = 0$ and $v_a'(M_{\tilde{y}_a}) > 0$ gives

$$\tilde{y}_a(M_{\tilde{y}_a})v_a'(M_{\tilde{y}_a}) + \int_{\varepsilon_m}^{M_{\tilde{y}_a}} \Big(\Big(\frac{h(t)f(v_a(t))}{v_a(t)} + \frac{1}{t^{2-\delta}}\Big) - T_a\Big)v_a\tilde{y}_a\,ds = 0.$$

However, this is impossible because the first term on the left-hand side is positive, since $\tilde{y}_a(M_{\tilde{y}_a}) > 0$, $v_a'(M_{\tilde{y}_a}) > 0$, and by the line after (53) the second term on the left-hand side is also positive for a sufficiently large. Thus we obtain a contradiction and so v_a must have a first local maximum, M_a, on $[0, R^{2-N}]$ if a is sufficiently large. It follows then from Lemma 2.5 that $v_a(M_a) \to \infty$ as $a \to \infty$.

Next, we show $M_a \to 0$ as $a \to \infty$.

Suppose M_a does not go to 0 as $a \to \infty$. Then, there is a $C_M > 0$ such that $M_a > C_M$ for large a. Again, since $t_{a,\beta} \to 0$ as $a \to \infty$ (by the claim at the beginning

of this lemma), it follows that $t_{a,\beta} < M_a/2 < M_a$ for large a and from (17) we see that $h(t)f(v_a(t)) + v(t)/t^{2-\delta} > 0$, so $v_a'' < 0$ on $(t_{a,\beta}, M_a)$. Therefore, using the concavity of v_a, we obtain

$$v_a\left(\lambda_a t_{a,\beta} + (1-\lambda_a)M_a\right) \geqslant \lambda_a \beta + (1-\lambda_a)v_a(M_a) \quad \text{for} \quad 0 < \lambda_a < 1. \tag{56}$$

Now choose λ_a so that $\lambda_a t_{a,\beta} + (1-\lambda_a)M_a = M_a/2$, that is $\lambda_a = M_a/(2(M_a - t_{a,\beta})) = 1/(2\left(1 - t_{a,\beta}/M_a\right)$. Then we see $t_{a,\beta}/M_a \leqslant t_{a,\beta}/C_M \to 0$ and so $\lambda_a \to 1/2$ as $a \to \infty$. Then by (56),

$$v_a\Big(\frac{M_a}{2}\Big) \geqslant \lambda_a\beta + (1-\lambda_a)v_a(M_a) = \frac{\beta}{2(1-\frac{t_{a,\beta}}{M_a})} + \Big(1 - \frac{1}{2(1-\frac{t_{a,\beta}}{M_a})}\Big)v_a(M_a) \to \infty$$
$$\text{as} \quad a \to \infty. \tag{57}$$

Next, we integrate (17) on (t, M_a) with $M_a/2 < t < M_a$. Recalling that v_a is increasing on $(M_a/2, M_a)$, and $v_a \to \infty$ on $(M_a/2, M_a)$ from (57), it follows from (*H1*) that $f(v_a) \geqslant C_f v_a^p$ with $C_f > 0$ on $(M_a/2, M_a)$, and we obtain

$$v_a' = \int_t^{M_a} h f(v_a)\, ds \geqslant \int_t^{M_a} C_f h v_a^p\, ds \geqslant C_f v_a^p \int_t^{M_a} h\, ds \quad \text{on} \quad \Big(\frac{M_a}{2}, M_a\Big).$$

Thus,

$$\frac{v_a'}{v_a^p} \geqslant C_f \int_t^{M_a} h\, ds \quad \text{on} \quad \Big(\frac{M_a}{2}, M_a\Big).$$

Integrating on (t, M_a) and recalling $p > 1$, we obtain

$$v_a^{1-p}(t) \geqslant v_a^{1-p}(t) - v_a^{1-p}(M_a) \geqslant (p-1)C_f \int_t^{M_a}\int_s^{M_a} h\, dx\, ds \quad \text{on} \quad \Big(\frac{M_a}{2}, M_a\Big). \tag{58}$$

Now, evaluate (58) at $t = M_a/2$. Since $v_a(M_a/2) \to \infty$ as $a \to \infty$ from (57) , then

$$0 \leftarrow v_a^{1-p}\Big(\frac{M_a}{2}\Big) \geqslant (p-1)C_f \int_{\frac{M_a}{2}}^{M_a}\int_t^{M_a} h\, ds\, dt \quad \text{as} \quad a \to \infty.$$

However, as $a \to \infty$ the right-hand side goes to a positive constant (since $M_a \geqslant C_M > 0$), and the left-hand side goes to 0, so we obtain a contradiction. Thus, $M_a \to 0$ as $a \to \infty$. This completes the proof. □

LEMMA 2.7. *Let $N > 2$ and assume $(H1)$–$(H4)$. Suppose v_a solves (17)–(18). Then v_a has a first zero, z_a, with $0 < z_a < R^{2-N}$ if $a > 0$ is sufficiently large. In addition, $z_a \to 0$ as $a \to \infty$.*

Proof. From Lemma 2.6, we know that v_a has a first local maximum, M_a with $0 < M_a < R^{2-N}$ for sufficiently large $a > 0$. Now suppose by way of contradiction

that $v_a > 0$ on (M_a, R^{2-N}) for all a sufficiently large. Then $\left(v_a'^2/(2h) + F(v_a)\right)' = -v_a'^2 h'/(2h^2) - v_a v_a'/(ht^{2-\delta}) \geqslant 0$ on (M_a, R^{2-N}) when $v_a' \leqslant 0$. It follows then that

$$\frac{1}{2}\frac{v_a'^2}{h} + F(v_a) \geqslant F(v_a(M_a)) \text{ on } \left(M_a, R^{2-N}\right). \tag{59}$$

Claim. For sufficiently large $a > 0$, if $v_a > 0$ on (M_a, R^{2-N}) then $v_a' < 0$ on (M_a, R^{2-N}).

Proof of Claim. Suppose there is a first critical point t_a with $M_a < t_a < R^{2-N}$, $v_a'(t_a) = 0$ and $v_a(t_a) > 0$. Then from (59), we have $F(v_a(t_a)) \geqslant F(v_a(M_a))$. Since $v_a(M_a) \to \infty$ as $a \to \infty$ from Lemma 2.6, therefore $v_a(M_a) > \gamma$ for sufficiently large $a > 0$. So $F(v_a(M_a)) > 0$, and hence $F(v_a(t_a)) > 0$ for sufficiently large $a > 0$ and thus $v_a(t_a) > \gamma$.

On the other hand, since $v_a' < 0$ on (M_a, t_a), if $v_a''(t_a) < 0$ then v_a has a local maximum at t_a which contradicts that $v_a' < 0$ on (M_a, t_a). Thus $v_a''(t_a) \geqslant 0$. Then from (17) we have $f(v_a(t_a)) \leqslant 0$. It follows then that $v_a(t_a) < \beta$, which contradicts that $v_a(t_a) > \gamma > \beta$. Hence the claim is true.

Next rewriting and integrating (59) on (M_a, R^{2-N}) gives

$$\int_{M_a}^{R^{2-N}} \sqrt{h}\, ds \leqslant \int_{v_a(R^{2-N})}^{v_a(M_a)} \frac{dt}{\sqrt{2}\sqrt{F(v_a(M_a)) - F(t)}} \leqslant \int_0^{v_a(M_a)} \frac{dt}{\sqrt{2}\sqrt{F(v_a(M_a)) - F(t)}}. \tag{60}$$

Now, rewriting the right-hand side, letting $t = v_a(M_a)s$ and changing variables on the right-most integral in (60), we obtain

$$\int_0^{v_a(M_a)} \frac{dt}{\sqrt{2}\sqrt{F(v_a(M_a)) - F(t)}} = \frac{v_a(M_a)}{\sqrt{2}\sqrt{F(v_a(M_a))}} \int_0^1 \frac{ds}{\sqrt{1 - \frac{F(v_a(s))}{F(v_a(M_a))}}}.$$

We know from earlier that $v_a(M_a) \to \infty$ as $a \to \infty$ and so it follows from the superlinearity of f (from $(H1)$) that $v_a(M_a)/(\sqrt{2}\sqrt{F(v_a(M_a))}) \to 0$, and

$$\int_0^1 \frac{1}{\sqrt{1 - F(v_a(s))/F(v_a(M_a))}} ds \to \int_0^1 (1 - s^{p+1})^{-1/2} ds < \infty \quad \text{as} \quad a \to \infty.$$

Thus, it follows that the right-hand side of (60) goes to zero as $a \to \infty$ and therefore, we see

$$\int_{M_a}^{R^{2-N}} \sqrt{h}\, ds \to 0 \quad \text{as} \quad a \to \infty. \tag{61}$$

However, we know $M_a \to 0$ from Lemma 2.6, and so the left-hand side of (60) goes to $\int_0^{R^{2-N}} \sqrt{h}\, ds > 0$ as $a \to \infty$, which contradicts (61). Thus, v_a must have a first zero, z_a, with $M_a < z_a < R^{2-N}$ and $v_a' < 0$ on $(M_a, z_a]$.

Now, rewriting (59) and integrating on (M_a, z_a) gives

$$\int_{M_a}^{z_a} \sqrt{h}\, ds \leqslant \int_0^{v_a(M_a)} \frac{dt}{\sqrt{2}\sqrt{F(v_a(M_a)) - F(t)}}. \tag{62}$$

Again, since f is superlinear, the right-hand side of (62) goes to zero as $a \to \infty$, therefore $z_a - M_a \to 0$ as $a \to \infty$. Since $M_a \to 0$ as $a \to 0$, by Lemma 2.6 it follows that $z_a = (z_a - M_a) + M_a \to 0$ as $a \to \infty$. □

LEMMA 2.8. *Let $N > 2$ and assume $(H1)$–$(H4)$. Suppose v_a solves (17)–(18). Let $0 < z_a < R^{2-N}$. If $v_a(z_a) = 0$, then $v_a'(z_a) \neq 0$ if $R > R_0$. In addition, $|v_a'(z_a)| \to \infty$ as $a \to \infty$.*

Proof. Suppose $v_a(z_a) = 0$ for some $0 < z_a < R^{2-N}$. Then there exists a local extremum on $(0, z_a)$, so suppose without loss of generality there is a local maximum, M_a, with $M_a < z_a$ such that $v_a'(M_a) = 0$, and $v_a'(t) < 0$ for all $t \in (M_a, z_a]$. From Lemma 2.3, we know that $|v_a(M_a)| > \gamma$ for $R > R_0$, so $F(v_a(M_a)) > 0$. Since $\left(v_a'^2/(2h) + F(v_a)\right)' \geqslant 0$ on $[M_a, z_a]$, then $v_a'^2/(2h) + F(v_a)$ is non-decreasing on $[M_a, z_a]$, and therefore

$$0 < F(v_a(M_a)) \leqslant \frac{1}{2}\frac{v_a'^2(z_a)}{h(z_a)}. \tag{63}$$

Hence from the above inequality, we see

$$|v_a'(z_a)| > 0 \quad \text{for } R > R_0.$$

So, from Lemma 2.7 we know that $v_a(M_a) \to \infty$ and so from (63), we see

$$v_a'^2(z_a) \geqslant 2h(z_a)F(v_a(M_a)) \geqslant 2h(R_0^{2-N})F(v_a(M_a)) \to \infty \quad \text{as} \quad a \to \infty.$$

Therefore, $|v_a'(z_a)| \to \infty$ as $a \to \infty$. □

We may now repeat this argument on (z_a, R^{2-N}) and show that v_a has as many zeros as desired as $a \to \infty$ and at each zero, z_a, we have $|v_a'(z_a)| \to \infty$ as $a \to \infty$.

3. Proof of Theorem 1

First, assume $R > R_0$ where R_0 is obtained in Lemma 2.3. Let $a > 0$. From Lemma 2.4, we know that v_a has at most a finite number of zeros on $(0, R^{2-N})$. Now, let n be a non-negative integer, and let

$$S_{n,R} = \{a > 0 \mid v_a \text{ has exactly } n \text{ zeros on } (0, R^{2-N})\}.$$

From Lemma 2.4, we know some of the $S_{n,R}$ are non-empty so let n_0 be the smallest value of n such that $S_{n_0,R} \neq \emptyset$. It follows from Lemma 2.7 that $S_{n_0,R}$ is bounded from above. Let

$$a_{n_0} := \sup S_{n_0,R}.$$

Now we prove that $v_{a_{n_0}}$ has n_0 zeros on $(0, R^{2-N})$. For $a < a_{n_0}$, v_a has exactly n_0 zeros on $(0, R^{2-N})$ and at each zero, z, we have $v_a'(z) \neq 0$ by Lemma 2.8. Therefore, by continuity with respect to a we see that $v_{a_{n_0}}$ has at least n_0 zeros on $(0, R^{2-N})$. If $v_{a_{n_0}}$ has an $(n_0+1)^{st}$ zero on $(0, R^{2-N})$, then so would v_a for a slightly smaller than

a_{n_0} (by continuity with respect to a) contradicting the definition of a_{n_0}. Thus $v_{a_{n_0}}$ has exactly n_0 zeros on $(0,R^{2-N})$.

Next suppose $v_{a_{n_0}}(R^{2-N})\neq 0$. Without loss of generality suppose $v_{a_{n_0}}(R^{2-N})>0$. Now for $a>a_{n_0}$, v_a has an $(n_0+1)^{st}$ zero on $(0,R^{2-N})$ by definition of a_{n_0} and so again by continuity with respect to a it would follow that $v_{a_{n_0}}$ has an $(n_0+1)^{st}$ zero, a contradiction. Thus $v_{a_{n_0}}(R^{2-N})\leqslant 0$. Similarly, we can show that $v_{a_{n_0}}(R^{2-N})\geqslant 0$. Therefore $v_{a_{n_0}}(R^{2-N})=0$ and by Lemma 2.8, $v'_{a_{n_0}}(R^{2-N})\neq 0$.

Now since $v'_{a_{n_0}}(R^{2-N})\neq 0$, it follows that if a is slightly larger that a_{n_0} then v_a has exactly (n_0+1) zeros on $(0,R^{2-N})$. Thus, $S_{n_0+1,R}$ is non-empty. Let

$$a_{n_0+1}:=\sup S_{n_0+1,R}.$$

In a similar way, we can show $v_{a_{n_0+1}}$ has exactly (n_0+1) zeros on $(0,R^{2-N})$ and $v_{a_{n_0+1}}(R^{2-N})=0$, $v'_{a_{n_0+1}}(R^{2-N})\neq 0$.

Continuing in this way, we see that for every $n\geqslant n_0$ there exists a solution of (17)–(18) with exactly n zeros on $(0,R^{2-N})$ and $v_{a_n}(R^{2-N})=0$. This completes the proof. □

4. Proof of Theorem 2

Assume $(H1)$–$(H3)$ and $(H5)$ where $0<\alpha\leqslant 2$. Suppose there is a $C^1[R,\infty)$ solution u of (3)–(4). In particular, we assume $1/|u|^q$ is integrable. Set

$$E(r):=\frac{\frac{1}{2}u'^2(r)+\frac{u^2(r)}{2r^{2+(N-2)\delta}}}{K(r)}+F(u(r)).$$

Claim 1. Let u be a $C^1[R,\infty)$ solution of (3)–(4). Then $E'\leqslant 0$ and $E(r)\to 0$ as $r\to\infty$.

Proof of Claim 1. A straightforward calculation shows

$$E'(r)=-\Big[2(N-1)+\frac{rK'(r)}{K(r)}\Big]\frac{u'^2(r)}{2rK(r)}-\Big[2+(N-2)\delta+\frac{rK'(r)}{K(r)}\Big]\frac{u^2(r)}{2r^{3+(N-2)\delta}}. \quad (64)$$

By $(H5)$ we have $\alpha+rK'/K>0$ so since $\delta>0$ and $0<\alpha\leqslant 2$ it follows that $2+(N-2)\delta+rK'/K\geqslant\alpha+rK'/K>0$ and $2(N-1)+rK'/K\geqslant\alpha+rK'/K>0$, therefore $E'(r)\leqslant 0$ on $[R,\infty)$, so E is non-increasing on $[R,\infty)$ when $0<\alpha\leqslant 2$. Also $E\geqslant -F_0$ from (6), so E is bounded from below. Therefore, there exists L such that

$$\lim_{r\to\infty}E(r)=L.$$

In addition, since $u\to 0$ as $r\to\infty$ implies $F(u)\to 0$ and also $(u(r)^2/r^{2+(N-2)\delta})/K(r)\leqslant u(r)^2/r^{(2-\alpha)+(N-2)\delta}\to 0$ as $r\to\infty$ ($\because\ 0<\alpha\leqslant 2$ and $(H5)$), and since $u'^2/(2K)\geqslant 0$, therefore $u'^2/(2K)\to L$ as $r\to\infty$ and hence $L\geqslant 0$.

Now, we show that $L=0$. Suppose not. Suppose $L>0$, so then $u'^2/(2K)\to L>0$ as $r\to\infty$ and thus there exists $r_L>0$ such that

$$\frac{u'^2}{2K}>\frac{1}{2}L>0\quad\forall\, r\geqslant r_L. \tag{65}$$

So $u'(r)\neq 0$ for all $r\geqslant r_L$. Without loss of generality, assume $u'(r)<0$ for all $r\geqslant r_L$. Now taking the square root in (65) and using $(H5)$, we see

$$-u'(r)>\sqrt{\frac{L}{2}}C_{10}r^{-\frac{\alpha}{2}}\quad\forall\, r>r_L,\quad\text{and for some}\quad C_{10}>0.$$

Integrating on (r_L,r), we obtain

$$0\leftarrow u(r)\leqslant\frac{\sqrt{\frac{L}{2}}C_{10}r_L^{1-\frac{\alpha}{2}}}{1-\frac{\alpha}{2}}+u(r_L)-\frac{\sqrt{\frac{L}{2}}C_{10}}{1-\frac{\alpha}{2}}r^{1-\frac{\alpha}{2}}\to-\infty\quad\text{as}\quad r\to\infty,$$

which is a contradiction. This proves the claim.

Claim 2. Let u be a $C^1[R,\infty)$ solution of (3)–(4). If $z\in[R,\infty)$ and $u(z)=0$, then $u'(z)\neq 0$.

Proof of Claim 2. Suppose by way of contradiction that $u(z)=u'(z)=0$. Then $E(z)=0$. From Claim 1, we know that E is non-increasing and $E(r)\to 0$ as $r\to\infty$, therefore $E(r)=0$ for all $r\geqslant z$. Hence $E'(r)=0$ for all $r\geqslant z$, and this with (64) gives

$$0\geqslant-\Big[2(N-1)+\frac{rK'(r)}{K(r)}\Big]\frac{u'^2(r)}{2rK(r)}=\Big[2+(N-2)\delta+\frac{rK'(r)}{K(r)}\Big]\frac{u^2(r)}{2r^{3+(N-2)\delta}}\geqslant 0,$$

which forces

$$u(r)=u'(r)=0\quad\forall\, r\geqslant z. \tag{66}$$

But then $1/|u|^q$ is not integrable on $[R,\infty)$ and so u is not a solution. Thus the claim is true.

Claim 3. Let u be a $C^1[R,\infty)$ solution of (3)–(4). Then u can have at most a finite number of zeros.

Proof of Claim 3. Suppose not. Let u have infinitely many zeros, say z_k, with $z_k<z_{k+1}\ \forall\, k\in\mathbb{N}$. Then u must have infinitely many extrema, say m_k, such that

$$z_k<m_k<z_{k+1},\ \forall\, k\in\mathbb{N}.$$

Case 1. $z_k\to z^*$ for some $R<z^*<\infty$

Then $u(z^*)=0$ and $u'(z^*)=0$ $(\because\ m_k\to z^*$ and $u(m_k)=0\,\forall\, k)$, which contradicts Claim 2.

Case 2. $z_k\to\infty$

In this case, $m_k\to\infty$. Then by Lemma 2.3, we obtain

$$u(m_k)\geqslant C_7>0,$$

but this contradicts that $u(r)\to 0$ as $r\to\infty$. This proves the claim.

We continue now with the proof of Theorem 2. We know that u can have at most finitely many zeros. So, suppose u has n zeros z_i $(i = 1,2,\cdots,n)$ such that $z_i < z_{i+1}$, $i = 1,2,3,\cdots,n-1$. Since $u'(z_n) \neq 0$ by Claim 2, therefore without loss of generality, we may assume that $u(r) \geqslant 0$ for all $r \geqslant z_n$. Then u must be monotone for large r. Otherwise, u would have infinitely many extrema, m_k with $m_k \to \infty$ and by Lemma 2.3 we obtain

$$u(m_k) \geqslant C_7 > 0 \quad \text{for positive local maxima } `m_k.$$

Since $m_k \to \infty$, then this contradicts that $u(r) \to 0$ as $r \to \infty$. Thus u is monotone for large r.

Now, since $u(r) \to 0$ as $r \to \infty$ and u is monotone for large r, therefore without loss of generality, we may assume that there is an $r_0 > R$ such

$$0 \leqslant u(r) \leqslant \frac{\beta}{2} \quad \text{and} \quad u'(r) \leqslant 0 \quad \forall\, r > r_0. \tag{67}$$

Next, we multiply (3) by $r^{1+\delta/2}u^q$, integrate by parts on (r_0, r) and obtain

$$r^{1+\frac{\delta}{2}}u^q u' \geqslant C_{11} - \int_{r_0}^{r} t^{1+\frac{\delta}{2}}K(t)u^q(t)f(u(t))\,dt, \tag{68}$$

where $C_{11} := r_0^{1+\delta/2}u^q(r_0)u'(r_0) + u^{q+1}(r_0)r_0^{\delta/2-(N-2)\delta}/(\delta/2-(N-2)\delta)$. For large r we have $u \to 0$, so it follows from $(H2)$ and (67) that $-u^q f(u) > 1/2$ for large r. Thus, from (68), we obtain

$$u^q u' \geqslant \frac{C_{11}}{r^{1+\frac{\delta}{2}}} + \frac{1}{2r^{1+\frac{\delta}{2}}}\int_{r_0}^{r} t^{1+\frac{\delta}{2}}K(t)\,dt. \tag{69}$$

Integrating again on (r_0, r) gives

$$u^{q+1} \geqslant u^{q+1}(r_0) + \frac{2C_{11}(q+1)}{\delta}\left(r^{-\frac{\delta}{2}} + r_0^{-\frac{\delta}{2}}\right) + \frac{q+1}{2}\int_{r_0}^{r}\frac{1}{t^{1+\frac{\delta}{2}}}\int_{r_0}^{t} s^{1+\frac{\delta}{2}}K(s)\,ds\,dt. \tag{70}$$

Now, recalling $(H5)$ we see that $r_0^{\alpha}K(r_0) \leqslant r^{\alpha}K(r)$ for $r \geqslant r_0$. Then we estimate the integral in (70) and we obtain

$$\begin{aligned}\int_{r_0}^{r}\frac{1}{t^{1+\frac{\delta}{2}}}\int_{r_0}^{t} s^{1+\frac{\delta}{2}}K(s)\,ds\,dt &\geqslant r_0^{\alpha}K(r_0)\int_{r_0}^{r}\frac{1}{t^{1+\frac{\delta}{2}}}\int_{r_0}^{t} s^{1+\frac{\delta}{2}-\alpha}\,ds\,dt \\ &= \frac{r_0^{\alpha}K(r_0)}{2+\frac{\delta}{2}-\alpha}\int_{r_0}^{r}\frac{1}{t^{1+\frac{\delta}{2}}}\left(t^{2+\frac{\delta}{2}-\alpha} - r_0^{2+\frac{\delta}{2}-\alpha}\right)dt. \end{aligned}\tag{71}$$

If $0 < \alpha < 2$, then it follows from (71) that

$$\begin{aligned}\int_{r_0}^{r}\frac{1}{t^{1+\frac{\delta}{2}}}\int_{r_0}^{t} s^{1+\frac{\delta}{2}}K(s)\,ds\,dt &\geqslant \frac{r_0^{\alpha}K(r_0)}{2+\frac{\delta}{2}-\alpha}\left(\frac{r^{2-\alpha}-r_0^{2-\alpha}}{2-\alpha} + \frac{2r_0^{2+\frac{\delta}{2}-\alpha}}{\delta}\left(r^{-\frac{\delta}{2}} - r_0^{-\frac{\delta}{2}}\right)\right) \\ &\to \infty \quad \text{as} \quad r \to \infty. \end{aligned}\tag{72}$$

If $\alpha = 2$, then it follows from (71) that

$$\int_{r_0}^{r} \frac{1}{t^{1+\frac{\delta}{2}}} \int_{r_0}^{t} s^{1+\frac{\delta}{2}} K(s)\, ds\, dt \geqslant \frac{r_0^{\alpha} K(r_0)}{2+\frac{\delta}{2}-\alpha} \left(\ln(r) - \ln(r_0) - \frac{2r_0^{\frac{\delta}{2}}}{\delta} \left(r^{-\frac{\delta}{2}} - r_0^{-\frac{\delta}{2}} \right) \right) \to \infty$$

$$\text{as} \quad r \to \infty. \tag{73}$$

Now, notice that the first two terms on the right-hand side of (70) are bounded, so it follows from (72)–(73) that the right-hand side of (70) goes to infinity as $r \to \infty$, but by assumption the left-hand side of (70) goes to zero as $r \to \infty$. Thus, we obtain a contradiction, so the theorem is proved. □

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(Received October 31, 2025)

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