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IMPROVED OSCILLATION CRITERIA OF GENERAL CAUCHY—EULER TYPE FOURTH–ORDER DELAY DIFFERENTIAL EQUATIONS

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Abstract. This paper addresses the fourth-order strongly noncanonical delay differential equation of the form

$$\left(p_3(t)\left(p_2(t)\left(p_1(t)x'(t)\right)'\right)'\right)'+q(t)x(\tau(t))=0$$

with

$$\int_{t_0}^{\infty}\frac{1}{p_j(t)}dt<\infty,\quad j=1,2,3.$$

Employing a robust analytical framework, we develop improved oscillation criteria inspired by traditional approaches like those of Hille and Nehari. By transforming noncanonical forms into the corresponding canonical equivalents, we establish conditions that govern the oscillatory nature of equations of this type more effectively. The improvement of the new result over the known ones is demonstrated via an example.

1. Introduction

In this paper, we are concerned with the oscillatory behavior of the fourth-order strongly noncanonical delay differential equation

$$\left(p_3(t)\left(p_2(t)\left(p_1(t)x'(t)\right)'\right)'\right)'+q(t)x(\tau(t))=0,\quad t\in I:=[t_0,\infty),\tag{1.1}$$

where $p_j\in C^{(4-j)}(I,\mathbb{R})$, $j=1,2,3$, $q,\tau\in C(I,\mathbb{R})$, t_0 is a fixed real number such that $p_j(t)>0$, $q(t)>0$, and the delay function τ satisfies $\tau(t)\leqslant t$ and $\tau(t)\to\infty$ as $t\to\infty$. For the sake of simplicity, we define the operator:

$$T_0x=x,\quad T_1x=p_1x',\quad T_2x=p_2\left(p_1x'\right)',\quad T_3x=p_3\left(p_2\left(p_1x'\right)'\right)',\quad T_4x=(T_3x)'$$

on I and so we can rewrite (1.1) in the form

$$T_4x(t)+q(t)x(\tau(t))=0,\quad t\in I.\tag{1.2}$$

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Here after we assume that equation (1.1) (or (1.2)) is in strongly noncanonical form, i.e.,

$$\int_{t_0}^{\infty} \frac{1}{p_j(t)}dt < \infty, \quad j = 1,2,3. \tag{1.3}$$

We would also like to point out that equation (1.1) is said to be in canonical form, if

$$\int_{t_0}^{\infty} \frac{1}{p_j(t)}dt = \infty, \quad j = 1,2,3.$$

Under a solution of (1.1), we mean a nontrivial function $x \in C([t_x,\infty),\mathbb{R})$, $t_x \in I$, which has the property $T_j x \in C^1([t_x,\infty),\mathbb{R})$ for $j = 0,1,2,3$ and satisfies (1.1) on $[t_x,\infty)$. Our attention is restricted to those solutions x of (1.1) which exist on I and satisfy the condition

$$\sup\{|x(t)| : T \leqslant t < \infty\} > 0 \quad \text{for any } T \geqslant t_x.$$

DEFINITION 1. A solution of (1.1) is said to be *oscillatory* if it has an unbounded set of zeros on $[t_x,\infty)$; otherwise it is called *nonoscillatory*. Equation (1.1) itself is termed oscillatory if all its solutions are oscillatory.

Since many indications shown the importance of the fourth-order differential equations in the applications as well as in the number of mathematical problems involved (see [5, 9]), the subject of the qualitative theory of such equations has gained momentum in the past four decades. In particular, the oscillation theory of fourth-order functional differential equations has attracted great attention which is clear from the extensive research done in the area, see [1, 2, 3, 4, 6, 8, 10, 12] together with many references therein.

Basically, there are two methods for the investigation of oscillatory behavior of fourth-order differential equations with noncanonical form. One method is to find the equivalent canonical form to the noncanonical equation considered and then obtain the oscillation criteria of the original equation via its canonical form, see [1, 12]. The second approach is to eliminate all possible classes of nonoscillatory solutions, see [2, 4, 10] and the references therein.

In this paper, we use the first method, that is, we reduce the noncanonical equation (1.1) into canonical form without assuming any extra conditions. This approach reduces the number of classes of nonoscillatory (say positive) solutions to two instead of the usual eight classes for noncanonical equations, and hence to obtain oscillation criteria it is enough to find conditions for the elimination two sets of nonoscillatory solutions. For this reason, this approach significantly simplifies finding criteria for oscillation of solutions. Therefore, the results obtained in this paper are new and improve the known results and this fact is illustrated via an example.

2. Oscillation results

It is clear that condition (1.3) enables us to define the following notation:

$$\lambda_i(t) = \int_t^{\infty} \frac{1}{p_i(s)}ds, \quad \lambda_{ij}(t) = \int_t^{\infty} \frac{1}{p_i(s)}\lambda_j(s)ds, \quad \lambda_{ijk}(t) = \int_t^{\infty} \frac{1}{p_i(s)}\lambda_{jk}(s)ds,$$

where $i,j,k \in \{1,2,3\}$ are mutually different,

$$\mu(t) := \lambda_{12}(t)\lambda_{23}(t) - \lambda_2(t)\lambda_{123}(t),$$

$$r_1(t) := \frac{p_1(t)\lambda_{123}^2(t)}{\mu(t)}, \quad r_2(t) := \frac{p_2(t)\mu^2(t)}{\lambda_{321}(t)\lambda_{123}(t)}, \quad r_3(t) := \frac{p_3(t)\lambda_{321}^2(t)}{\mu(t)},$$

and

$$p(t) := q(t)\lambda_{321}(t)\lambda_{123}(\tau(t)).$$

We are now ready to state the following theorem, a proof of which can be found in [1].

THEOREM 1. (see [1, Theorem 1, Corollary 1]) *The strongly noncanonical differential equation* (1.1) *can be written in canonical form as*

$$D_4y(t) + p(t)y(\tau(t)) = 0, \quad t \in I, \tag{2.1}$$

where

$$y(t) = x(t)/\lambda_{123}(t), \quad D_0y = y, \quad D_1y = r_1y', \quad D_2y = r_2\Big(r_1y'\Big)',$$
$$D_3y = r_3\Big(r_2\Big(r_1y'\Big)'\Big)', \quad D_4y = \Big(D_3y\Big)' = \Big(r_3\Big(r_2\Big(r_1y'\Big)'\Big)'\Big)',$$

and

$$\int_{t_0}^{\infty} \frac{1}{r_j(t)}dt = \infty, \quad j = 1,2,3. \tag{2.2}$$

From Theorem 1, we see that the oscillation of strongly noncanonical equation (1.1) reduced to that of the canonical equation (2.1), and hence we have the following immediate results.

COROLLARY 1. *A function x is a solution of the strongly noncanonical equation* (1.1) *if and only if $y(t) = x(t)/\lambda_{123}(t)$ is a solution of the canonical equation* (2.1).

COROLLARY 2. *The strongly noncanonical equation* (1.1) *is nonoscillatory (oscillatory) if and only if the canonical equation* (2.1) *is nonoscillatory (oscillatory).*

In view of Corollary 2, we are ready to investigate the oscillation property of (1.1) using the canonical equation (2.1). Since (1.1) (or (2.1)) is linear, it is enough to consider only possible eventually positive solutions to (1.1) (or (2.1)) instead of nonoscillatory solutions. On the other hand, since (2.1) is in canonical form, there are only two classes of positive solutions, as stated in the following lemma. This lemma follows from a generalization of the well-known Kiguradze lemma [7] applied to (2.1) under condition (2.2).

LEMMA 1. *Assume that $y(t)$ is an eventually positive solution of (2.1). Then, either*

$$y(t) \in \mathcal{N}_1 \Longleftrightarrow D_1 y(t) > 0, \quad D_2 y(t) < 0, \quad D_3 y(t) > 0, \quad D_4 y(t) \leqslant 0,$$

or

$$y(t) \in \mathcal{N}_3 \Longleftrightarrow D_1 y(t) > 0, \quad D_2 y(t) > 0, \quad D_3 y(t) > 0, \quad D_4 y(t) \leqslant 0,$$

eventually. Accordingly, all eventually positive solutions to (2.1) can be expressed $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3$.

For notational purpose, we set:

$$\pi_1(t) := \int_{t_*}^{t} \frac{1}{r_3(s)} ds, \quad \pi_2(t) := \int_{t_*}^{t} \frac{\pi_1(s)}{r_2(s)} ds, \quad \pi_3(t) := \int_{t_*}^{t} \frac{\pi_2(s)}{r_1(s)} ds,$$

$$\alpha_1(t) := \int_{t_*}^{t} \frac{1}{r_1(s)} ds, \quad \alpha_2(t) := \frac{1}{r_2(t)} \int_{t}^{\infty} \frac{1}{r_3(u)} \int_{u}^{\infty} p(s) \frac{\alpha_1(\tau(s))}{\alpha_1(s)} ds du,$$

and

$$\alpha_3(t) := p(t) \frac{\pi_3(\tau(t))}{\pi_1(t)},$$

where $t_* \geqslant t_0$ is sufficiently large.

LEMMA 2. *Let $y(t)$ be a positive solution of (2.1). Then*

(*i*) *if $y(t) \in \mathcal{N}_1$, then $y(t)/\alpha_1(t)$ is decreasing;*

(*ii*) *if $y(t) \in \mathcal{N}_3$, then $y(t)/\pi_3(t)$ is decreasing and $y(t) \geqslant \pi_3(t) D_2 y(t)/\pi_1(t)$.*

Proof. The proof is similar to [12, Lemma 2.2] and hence the details are omitted. □

Next, we state a particular case of [8, Theorem 2].

LEMMA 3. *Assume that $\widehat{p}(t) > 0$ and $\widehat{q}(t) > 0$ on I. If the differential inequality*

$$\left(\widehat{p}(t) u'(t)\right)' + \widehat{q}(t) u(t) \leqslant 0$$

has an eventually positive solution, then the differential equation

$$\left(\widehat{p}(t) u'(t)\right)' + \widehat{q}(t) u(t) = 0 \tag{2.3}$$

also has an eventually positive solution.

It is well known (see [11]) that if

$$\int_{t_0}^{\infty} \frac{1}{\widehat{p}(t)} dt = \infty \quad \text{and} \quad \liminf_{t \to \infty} \left(\int_{t_0}^{t} \frac{1}{\widehat{p}(s)} ds \right) \int_{t}^{\infty} \widehat{q}(s) ds > \frac{1}{4}, \tag{2.4}$$

then equation (2.3) is oscillatory.

In what follows, we compare the oscillatory behavior of (1.1) with a second-order linear differential equation of type (2.3).

THEOREM 2. *If the second-order linear canonical differential equations*

$$\big(r_1(t)\omega'(t)\big)' + \alpha_2(t)\omega(t) = 0 \tag{2.5}$$

and

$$\big(r_3(t)z'(t)\big)' + \alpha_3(t)z(t) = 0 \tag{2.6}$$

are oscillatory, then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be an eventually positive solution of (1.1), say $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \geqslant t_1$, for some $t_1 \in I$. Thus, by Corollary 1, $y(t) = x(t)/\lambda_{123}(t)$ is a positive solution of (2.1), and so in view of Lemma 1, we deduce that either $y(t) \in \mathscr{N}_1$ or $y(t) \in \mathscr{N}_3$.

We first assume that $y(t) \in \mathscr{N}_1$. In view of Lemma 2(i), we see that

$$y(\tau(t)) \geqslant \frac{\alpha_1(\tau(t))}{\alpha_1(t)} y(t).$$

Using this estimate in (2.1) yields

$$D_4 y(t) + p(t)\frac{\alpha_1(\tau(t))}{\alpha_1(t)} y(t) \leqslant 0.$$

Integrating the latter inequality from t to ∞, we obtain

$$D_3 y(t) \geqslant \int_t^\infty p(s)\frac{\alpha_1(\tau(s))}{\alpha_1(s)} y(s)ds \geqslant y(t)\int_t^\infty p(s)\frac{\alpha_1(\tau(s))}{\alpha_1(s)}ds.$$

A second integration gives

$$D_2 y(t) + \Big(\int_t^\infty \frac{1}{r_3(u)}\int_u^\infty p(s)\frac{\alpha_1(\tau(s))}{\alpha_1(s)}dsdu\Big) y(t) \leqslant 0,$$

or

$$\big(r_1(t)y'(t)\big)' + \alpha_2(t)y(t) \leqslant 0. \tag{2.7}$$

Letting $\omega(t) = y(t)$ in (2.7) and then using Lemma 3, we see that $\omega(t)$ is a positive solution of equation (2.5), which contradicts the fact that (2.5) oscillates. The contradiction obtained shows that $\mathscr{N}_1 = \emptyset$.

Next, we assume that $y(t) \in \mathscr{N}_3$. From Lemma 2(ii), we have

$$y(\tau(t)) \geqslant \frac{\pi_3(\tau(t))}{\pi_3(t)} y(t) \quad \text{and} \quad y(t) \geqslant \frac{\pi_3(t)}{\pi_1(t)} D_2 y(t).$$

Using two estimates above in (2.1), we obtain

$$D_4 y(t) + p(t)\frac{\pi_3(\tau(t))}{\pi_1(t)} D_2 y(t) \leqslant 0. \tag{2.8}$$

Letting $z(t) = D_2 y(t)$ in (2.8), we see that $z(t)$ is a positive solution of the differential inequality

$$\left(r_3(t)z'(t)\right)' + \alpha_3(t)z(t) \leqslant 0,$$

and so by Lemma 3, the function $z(t)$ is also a positive solution of equation (2.6), which contradicts the fact that (2.6) oscillates. The contradiction obtained shows that $\mathcal{N}_3 = \emptyset$. As a result, we conclude that $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3 = \emptyset$. The proof is complete. □

Based on (2.4) and Theorem 2, we can easily obtain the following Hille and Nehari [11] type oscillation criterion for (1.1).

THEOREM 3. *If*

$$\liminf_{t\to\infty} \alpha_1(t) \int_t^\infty \alpha_2(s)ds > \frac{1}{4} \tag{2.9}$$

and

$$\liminf_{t\to\infty} \pi_1(t) \int_t^\infty \alpha_3(s)ds > \frac{1}{4}, \tag{2.10}$$

then equation (1.1) *is oscillatory.*

In the following, we present an example to show the importance and novelty of the main result over the known ones.

EXAMPLE 1. Consider the strongly noncanonical fourth-order equation

$$\left(t^2\left(t^2\left(t^2x'(t)\right)'\right)'\right)' + at^2x(\theta t) = 0, \quad t \geqslant 1, \tag{2.11}$$

where $a > 0$, and $\theta \in (0,1)$. A simple calculation yields that

$$\lambda_i(t) = \frac{1}{t}, \quad \lambda_{ij}(t) = \frac{1}{2t^2}, \quad \lambda_{123}(t) = \lambda_{321}(t) = \frac{1}{6t^3}.$$

Then the transformed equation is

$$y^{(4)}(t) + \frac{a}{\theta^3 t^4} y(\theta t) = 0, \quad t \geqslant 1,$$

which is clearly in canonical form. Further calculation shows that

$$\pi_1(t) \approx t, \quad \pi_2(t) \approx \frac{t^2}{2}, \quad \pi_3(t) \approx \frac{t^3}{6}, \quad \alpha_1(t) \approx t, \quad \alpha_2(t) \approx \frac{a}{6\theta^2 t^2}, \quad \alpha_3(t) \approx \frac{a}{6t^2}.$$

Now, condition (2.9) becomes

$$\liminf_{t\to\infty} t \int_t^\infty \frac{a}{6\theta^2 s^2} ds = \frac{a}{6\theta^2},$$

and condition (2.10) becomes

$$\liminf_{t\to\infty} t \int_t^\infty \frac{a}{6s^2} ds = \frac{a}{6}.$$

Therefore, by Theorem 3, equation (2.11) is oscillatory if $a > 1.5$. Note that equation (2.11) is oscillatory by Theorem 3 if $a > 1.5$ for any values of θ.

In particular case where $\theta = 0.8$, we see that $a > 1.5$ implies oscillation of (2.11) by Theorem 3. The oscillatory criterion obtained by different technique presented in [2] gives oscillation of (2.11) if $a > 1.963$, and [1] gives oscillation of (2.11) if $a > 7.193$, while [12, Corollary 2.4.1] requires $a > 4.5$, and [4, Theorem 3] requires $a > 2.0177507$.

On the other hand, for $\theta = 0.9$, Theorem 3 guarantees oscillation of (2.11) provided that $a > 1.5$, while [2, Theorem 2] requires $a > 1.77$ and the best criterion from [4] requires $a > 3.50$. Thus, the criterion given in Theorem 3 improves significantly that of in [1, 2, 4, 12].

3. Conclusion

This article examines the oscillatory behavior of solutions to a class of fourth-order strongly noncanonical delay differential equations. By transforming a noncanonical equation into its canonical form without adding any new conditions, the authors derived new oscillation criteria that simplify the analysis of its solutions effectively. This approach significantly streamlines the process of establishing oscillation conditions by reducing the number of classes of potential nonoscillatory solution types from 8 to 2. The improvements of our findings over existing results (see [1, 2, 4, 12]) are illustrated through a detailed example, highlighting the enhanced understanding of oscillation behavior in such equations. The criterion developed herein represents a significant advancement in the study of noncanonical differential equations, providing clearer insights applicable to both theory and practice in related mathematical fields. Finally, we would like to point out that the authors in [1, 2, 4, 12] had to assume that the delay function τ is increasing in order to prove their results, but this restriction is not necessary in obtaining our results.

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