Differential
Equations
& Applications
Volume 18, Number 2 (2026), 147–159
doi:10.7153/dea-2026-18-10

EXISTENCE RESULT FOR A SYSTEM OF TWO SEMILINEAR COUPLED POISSON EQUATIONS WITH ASYMPTOTICALLY LINEAR NONLINEARITIES

ABLANVI SONGO

(*Communicated by D. Kang*)

Abstract. We establish the existence of at least one solution to a system of two semilinear coupled Poisson equations with asymptotically linear nonlinearities, without imposing the Ambrosetti-Rabinowitz condition or any of its refinements. The proof relies on a generalized Saddle Point Theorem due to Colin and the author [2].

1. Introduction

In this paper, we study the following system of two semilinear coupled Poisson equations

$$\begin{cases} -\Delta u + u = g_1(x) f_1(v), & x \in \mathbb{R}^N, \\ -\Delta v + v = g_2(x) f_2(u), & x \in \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N). \end{cases} \tag{1.1}$$

We are interested in the asymptotically linear case

$$\lim_{|v|\to\infty} \frac{f_1(v)}{v} = \lim_{|u|\to\infty} \frac{f_2(u)}{u} = 0. \tag{1.2}$$

If, we define in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ the functional

$$J(u,v) = \int_{\mathbb{R}^N} \Big(\nabla u \nabla v + uv\Big) dx - \int_{\mathbb{R}^N} \Big(g_1(x) F_1(v) + g_2(x) F_2(u)\Big) dx, \tag{1.3}$$

where

$$F_1(v) = \int_0^{v(x)} f_1(\zeta) d\zeta, \quad F_2(u) = \int_0^{u(x)} f_2(\zeta) d\zeta,$$

Mathematics subject classification (2020): 35A15, 35J61, 58E05.
Keywords and phrases: Semilinear coupled Poisson equations, strongly indefinite functionals, generalized Saddle Point Theorem.

© ELEMENT, Zagreb
Paper DEA-18-10

then, it is well known that the first part of the functional J is strongly indefinite operator and under suitable assumptions, J is of class $\mathscr{C}^1$ on $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, and its critical points are weak solutions of (1.1). Precisely, the functional J has the form

$$\frac{\|Q(u,v)\|^2}{2}-\frac{\|P(u,v)\|^2}{2}-\varphi(x,u,v),$$

for every $(u,v)\in\Big(X=Y\oplus Z:=H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N),\|\cdot\|\Big)$, where Y and Z are infinite-dimensional subspaces of X, P and Q are, respectively, the orthogonal projections of X into Y and Z.

There is a vast literature devoted to the study of the existence of solutions to semilinear coupled Poisson equations, both in the whole space and in bounded or unbounded domains; see for example [1, 3–5, 9, 12] and the references therein. In [1, 5, 9, 12], the following condition on the nonlinearities near 0 is assumed:

$$\lim_{v\to 0}\frac{f_1(v)}{v}=\lim_{u\to 0}\frac{f_2(u)}{u}=0$$

and the authors obtained nontrivial solutions by means of variational arguments in the spirit of the generalized linking theorem of Kryszewski and Szulkin [7].

To the best of our knowledge, no results are available in the literature concerning the existence of solutions to semilinear coupled Poisson equations of the form (1.1) when the nonlinearities f_1, f_2 satisfy only the asymptotically linear condition (1.2).

The aim of this paper is to prove the existence of at least one solution to problem (1.1). The main difficulty arises from the fact that the energy functional (1.3) associated with (1.1) possesses a strongly indefinite quadratic part. Consequently, the proofs of our main results cannot rely on classical min-max arguments. It is therefore of particular interest to investigate this strongly indefinite case. By applying a generalized Saddle Point Theorem due to the author and Colin [2], we establish an existence result for problem (1.1).

The paper is organized as follows. In Section 2, we recall some classical results and present additional preliminary material needed in the sequel. In Section 3, we state and prove our main result.

2. Kryszewski-Szulkin degree theory and a generalized Saddle Point Theorem

2.1. Kryszewski-Szulkin degree theory

In this subsection, we are following the presentation of the degree theory of Kryszewski and Szulkin given in [11] by Michel Willem.

Let Y be a real separable Hilbert space endowed with inner product $(\cdot,\cdot)$ and the associated norm $\|\cdot\|$.

On Y we consider the σ-topology introduced by Kryszewski and Szulkin; that is, the topology generated by the norm

$$|u|_\sigma:=\sum_{k=0}^{\infty}\frac{1}{2^{k+1}}|(u,e_k)|,\quad u\in Y, \tag{2.1}$$

where $(e_k)_{k\geqslant 0}$ is a total orthonormal sequence in Y.

REMARK 1. By the Cauchy-Schwarz inequality, one can show that $|u|_\sigma \leqslant \|u\|$ for every $u \in Y$. Moreover, if (u_n) is a bounded sequence in Y then

$$u_n \rightharpoonup u \Longleftrightarrow u_n \xrightarrow{\sigma} u,$$

where $\rightharpoonup$ denotes the weak convergence and $\xrightarrow{\sigma}$ denotes the convergence in the σ-topology.

Let U be an open bounded subset of Y such that its closure $\overline{U}$ is σ-closed.

DEFINITION 1. ([11]) A map $f : \overline{U} \to Y$ is σ-admissible (admissible for short) if

(1) f is σ-continuous,

(2) each point $u \in U$ has a σ-neighborhood N_u in Y such that $(\mathrm{id} - f)(N_u \cap U)$ is contained in a finite-dimensional subspace of Y.

DEFINITION 2. ([11]) A map $h : [0,1] \times \overline{U} \to Y$ is an admissible homotopy if

(1) $0 \notin h([0,1] \times \partial U)$,

(2) h is σ-continuous, that is $t_n \to t$ and $u_n \xrightarrow{\sigma} u$ implies $h(t_n, u_n) \xrightarrow{\sigma} h(t,u)$,

(3) h is σ-locally finite-dimensional. That is, for any $(t,u) \in [0,1] \times U$ there is a neighborhood $N_{(t,u)}$ in the product topology of $[0,1]$ and (X,σ) such that $\{v - h(s,v) \mid (s,v) \in N_{(t,u)} \cap ([0,1] \times U)\}$ is contained in a finite-dimensional subspace of Y.

Then, for an admissible map f such that $0 \notin f(\partial U)$ we have (see [11, Theorem 6.6])

$$deg(h(0,.),U) = deg(h(1,.),U),$$

where deg is the topological degree of f (about 0). Such a degree (called Kryszewski and Szulkin degree, see [11, Definition 6.3]) possesses the usual properties; in particular, if $f : \overline{U} \to Y$ is an admissible map with $0 \notin f(\partial U)$ and $deg(f,U) \neq 0$, then there exists $u \in U$ such that $f(u) = 0$.

Now, let $X = X^- \oplus X^+$, where X^- is closed and $X^+ = (X^-)^\perp$, be a real separable Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$. Let $(e_k)_{k\geqslant 0}$ be an orthogonal basis of X^-. We set on X, a new norm defined by

$$|u|_\tau := \max\left(\sum_{k=0}^{\infty} \frac{1}{2^{k+1}} |\langle Pu, e_k \rangle|, \|Qu\| \right), \quad u \in X.$$

Here, P and Q denote the orthogonal projections of X onto X^- and X^+, respectively; τ denotes the topology generated by this norm. The topology τ was introduced by Kryszewski and Szulkin [7].

REMARK 2. For every $u \in X$, we have $\|Qu\| \leqslant \|u\|_\tau$ and $|Pu|_\sigma \leqslant \|u\|_\tau$. Moreover, if (u_n) is a bounded sequence in X then

$$u_n \xrightarrow{\tau} u \Longleftrightarrow Pu_n \rightharpoonup Pu \text{ and } Qu_n \to Qu, \tag{2.2}$$

where $\rightharpoonup$ denotes the weak convergence and $\xrightarrow{\tau}$ denotes the convergence in the τ-topology.

2.2. Generalized Saddle Point Theorem

DEFINITION 3. Let $J \in \mathscr{C}^1(X,\mathbb{R})$.

1. We say that the functional J is τ-upper semi-continuous if $u_n \xrightarrow{\tau} u$ implies
$$J(u) \geqslant \overline{\lim_{n\to\infty}} J(u_n).$$
2. We say that ∇J is weakly sequentially continuous if
$$u_n \rightharpoonup u \text{ implies } \nabla J(u_n) \rightharpoonup \nabla J(u).$$
3. The functional J is said to satisfy the $(PS)_c$ condition (or the Palais-Smale condition at level c) if any sequence $(u_n) \subset X$ such that
$$J(u_n) \to c \text{ and } J'(u_n) \to 0, \text{ as } n \to \infty,$$
has a convergent subsequence.

We consider the class of $\mathscr{C}^1$-functionals $J : X \to \mathbb{R}$ such that
(A) J is τ-upper semi-continuous and ∇J is weakly sequentially continuous.
We recall the following result from [2] that will play a key role in the proof of our main result; see [2, Theorem 2.1].

For $\rho > 0$, set
$$M = \Big\{ u \in X^- \mid \|u\| \leqslant \rho \Big\}, \tag{2.3}$$
Then M is a sub-manifold of X^- with boundary ∂M.
Let us define $\text{int}(M) := \Big\{ u \in Y \mid \|u\| < \rho \Big\}$.

THEOREM 1. (Generalized Saddle Point Theorem, [2]) *Assume that $J \in \mathscr{C}^1(X,\mathbb{R})$ satisfies (A), that is, J is τ-upper semi-continuous and ∇J is weakly sequentially continuous. We suppose that*

$$b := \inf_{u\in X^+} J(u) > a := \sup_{u\in\partial M} J(u). \tag{2.4}$$

Let $c \in \mathbb{R}$ be characterized as

$$c := \inf_{\gamma\in\Gamma} \sup_{u\in M} J(\gamma(u)),$$

where

$$\Gamma := \Big\{ \gamma : M \to X \;\Big|\; \gamma \quad \textit{is } \tau\textit{-continuous, } \gamma\big|_{\partial M} = \mathrm{id} \textit{ and}$$
$$\textit{every } u \in \mathrm{int}(M) \textit{ has a } \tau\textit{-neighborhood } N_u \textit{ in } X \textit{ such that } (\mathrm{id}-\gamma)\Big(N_u \cap \mathrm{int}(M)\Big)$$
$$\textit{is contained in a finite-dimensional subspace of } X \Big\}.$$

Then, there exists $(u_n) \subset X$ *such that*

$$J(u_n) \to c, \;\; J'(u_n) \to 0, \;\; \textit{as} \;\; n \to \infty.$$

3. Main result

Consider the Hilbert space

$$H^1(\mathbb{R}^N) := \Big\{ u \in L^2(\mathbb{R}^N) \;\Big|\; |\nabla u| \in L^2(\mathbb{R}^N) \Big\},$$

endowed with the inner product

$$(u,v)_1 = \int_{\mathbb{R}^N} \Big(\nabla u \cdot \nabla v + uv \Big) dx,$$

and the associated norm $\|.\|_1$ given by

$$\|u\|_1^2 := \int_{\mathbb{R}^N} \Big(|\nabla u|^2 + |u|^2 \Big) dx.$$

We denote the product space

$$X = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N),$$

the Hilbert space endowed with the inner product

$$\langle (u,v), (\phi,\psi) \rangle := \int_{\mathbb{R}^N} \Big(\nabla u \nabla \phi + u\phi \Big) dx + \int_{\mathbb{R}^N} \Big(\nabla v \nabla \psi + v\psi \Big) dx, \;\; \text{for every } (\phi,\psi) \in X, \tag{3.1}$$

and the corresponding norm $\|\cdot\|$, given by

$$\|(u,v),(u,v)\|^2 = \|u\|_1^2 + \|v\|_1^2.$$

If we define

$$X^- := \Big\{ (-v,v) \in X \Big\} \;\text{ and }\; X^+ := \Big\{ (u,u) \in X \Big\},$$

since we can write (u,v) as

$$(u,v) = \frac{1}{2}(u+v, u+v) + \frac{1}{2}(-v+u, v-u),$$

then, $X = X^- \oplus X^+$.

Let us denote by P the projection of X onto X^- and by Q the projection of X onto X^+. We have (see [12] or [9])

$$\int_{\mathbb{R}^N}\Big(\nabla u\nabla v+uv\Big)dx=\frac{\|Q(u,v)\|^2}{2}-\frac{\|P(u,v)\|^2}{2}. \tag{3.2}$$

Finally, let us define the functional $J:X\to\mathbb{R}$

$$\begin{aligned}J(u,v)&:=\int_{\mathbb{R}^N}\Big(\nabla u\nabla v+uv\Big)dx-\int_{\mathbb{R}^N}\Big(g_1(x)F_1(v)+g_2(x)F_2(u)\Big)dx \\ &=\frac{\|Q(u,v)\|^2}{2}-\frac{\|P(u,v)\|^2}{2}-\varphi(x,u,v),\end{aligned} \tag{3.3}$$

where

$$\varphi(x,u,v):=\int_{\mathbb{R}^N}\Big(g_1(x)F_1(v)+g_2(x)F_2(u)\Big)dx.$$

DEFINITION 4. We say that (u,v) is a weak solution of problem (1.1) if $(u,v)\in X$, and satisfies for any $(\phi,\psi)\in X$:

$$\int_{\mathbb{R}^N}\Big(\nabla u\nabla\psi+u\psi\Big)dx+\int_{\mathbb{R}^N}\Big(\nabla v\nabla\phi+v\phi\Big)dx-\int_{\mathbb{R}^N}\Big(g_1(x)f_1(v)\psi+g_2(x)f_2(u)\phi\Big)dx=0.$$

Throughout this section, $|\cdot|_p$ stands for the L^p-norm; $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively.

Our assumptions on (1.1) are the following:

(f) The functions $f_1,f_2\in\mathscr{C}(\mathbb{R},\mathbb{R})$ satisfy (1.2), $tf_1(t)\geqslant 0$, $tf_2(t)\geqslant 0$ for all $t\in\mathbb{R}$.

(g) The functions $g_1,g_2\in L^2(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, $g_1(x)\geqslant 0$, $g_2(x)\geqslant 0$ for every $x\in\mathbb{R}^N$.

REMARK 3. We would like to mention that

1. These conditions coincide with those considered in [6].
2. Since assumption (f) implies that $f_1(0)=0$ and $f_2(0)=0$, then $(u,v)=(0,0)$ is a trivial solution of problem (1.1).

EXAMPLE 1. The set of functions f_i and g_i, $i\in\{1,2\}$, that satisfy the above conditions (f) and (g) is nonempty. Indeed, define

$$f_1(t)=\frac{t}{1+t^2},\ \ f_2(t)=\frac{t}{1+t^4},\ \text{ for every } t\in\mathbb{R};$$

$$g_1(x)=g_2(x)=e^{-|x|^2},\ \text{ for every } x\in\mathbb{R}^N.$$

Then, the functions f_i and g_i, $i\in\{1,2\}$ satisfy assumptions (f) and (g).

The main result of this paper is as follows:

THEOREM 2. *Under assumptions* (f) *and* (g), *problem* (1.1) *admits at least one nontrivial solution.*

Under assumptions (f) and (g), we have

$$g_1(x)F_1(v) \in L^1(\mathbb{R}^N) \text{ and } g_2(x)F_2(u) \in L^1(\mathbb{R}^N), \text{ for every } (u,v) \in X.$$

Therefore, the functional $J(u,v)$ given in (3.3) is well defined. Furthermore, using variational standard arguments, the functional $J(u,v)$ is of class $\mathscr{C}^1$ in X and

$$\begin{aligned} J'(u,v)(\phi,\psi) = &\int_{\mathbb{R}^N} \Big(\nabla u \nabla \psi + u\psi\Big)dx + \int_{\mathbb{R}^N} \Big(\nabla v \nabla \phi + v\phi\Big)dx \\ &- \int_{\mathbb{R}^N} \Big(g_1(x)f_1(v)\psi + g_2(x)f_2(u)\phi\Big)dx, \text{ for every } (\phi,\psi) \in X. \quad (3.4) \end{aligned}$$

Hence, the weak solutions of problem (1.1) are exactly the critical points of $J(u,v)$ in X (see Definition 4).

To prepare the proof of Theorem 2, we first establish several lemmas.

LEMMA 1. *The functional* $J(u,v)$ *given in* (3.3) *satisfies* (A), *that is,* $J(u,v)$ *is* τ*-upper semi-continuous and* ∇J *is weakly sequentially continuous.*

Proof. 1. The functional J is τ-upper semi-continuous: for every $c \in \mathbb{R}$, let show that the set

$$\Big\{(u,v) \in X \,|\, J(u,v) \geqslant c\Big\} \text{ is } \tau\text{-closed.}$$

Let $(u_n,v_n) \subset X$ such that

$$(u_n,v_n) \xrightarrow{\tau} (u,v) \text{ in } X \text{ and } c \leqslant J(u_n,v_n).$$

Then, (u_n,v_n) is bounded. Indeed, by Remark 2, $\Big(\|Q(u_n,v_n)\|\Big)$ is bounded; since

$$\varphi(x,u,v) = \int_{\mathbb{R}^N} \Big(g_1(x)F_1(v) + g_2(x)F_2(u)\Big)dx \geqslant 0,$$

then

$$\begin{aligned} \|P(u_n,v_n)\|^2 &= \|Q(u_n,v_n)\|^2 - 2J(u_n,v_n) - 2\varphi(x,u_n,v_n) \\ &\leqslant \|Q(u_n,v_n)\|^2 - 2c. \end{aligned}$$

It follows that $\Big(\|P(u_n,v_n)\|\Big)$ is also bounded. Thus, there is a subsequence of (u_n,v_n) that we still denote (u_n,v_n) such that

$$(u_n,v_n) \rightharpoonup (u,v) \text{ in } X.$$

Up to a subsequence, we also have

$$(u_n,v_n) \to (u,v), \text{ almost everywhere in } \mathbb{R}^N.$$

Since $g_1(x)F_1(v_n) \geqslant 0$ and $g_2(x)F_2(u_n) \geqslant 0$, then by the Fatou Lemma, we have

$$\int_{\mathbb{R}^N} g_1(x)F_1(v)dx = \int_{\mathbb{R}^N} \underline{\lim}_{n\to\infty} g_1(x)F_1(v_n)dx \leqslant \underline{\lim}_{n\to\infty} \int_{\mathbb{R}^N} g_1(x)F_1(v_n)dx.$$

In the same way, we have

$$\int_{\mathbb{R}^N} g_2(x)F_2(u)dx = \int_{\mathbb{R}^N} \underline{\lim}_{n\to\infty} g_2(x)F_2(u_n)dx \leqslant \underline{\lim}_{n\to\infty} \int_{\mathbb{R}^N} g_2(x)F_2(u_n)dx.$$

Since $\|\cdot\|$ is weak lower semi-continuous, we have

$$\|P(u,v)\|^2 \leqslant \underline{\lim}_{n\to\infty} \|P(u_n,v_n)\|^2.$$

Moreover, since

$$\|Q(u_n,v_n)\|^2 \to \|Q(u,v)\|^2, \quad \text{as} \quad n\to\infty,$$

we have

$$\overline{\lim}_{n\to\infty}\Big(-\|Q(u_n,v_n)\|^2\Big) = -\|Q(u,v)\|^2 = \underline{\lim}_{n\to\infty}\Big(-\|Q(u_n,v_n)\|^2\Big).$$

Hence,

$$\begin{aligned}
-J(u,v) &= \frac{1}{2}\Big(\|P(u,v)\|^2 - \|Q(u,v)\|^2\Big) + \varphi(x,u,v)\\
&\leqslant \underline{\lim}_{n\to\infty}\left(\frac{1}{2}\Big(\|P(u_n,v_n)\|^2 - \|Q(u_n,v_n)\|^2\Big) + \varphi(x,u_n,v_n)\right)\\
&= \underline{\lim}_{n\to\infty}(-J(u_n,v_n))\\
&= -\overline{\lim}_{n\to\infty} J(u_n,v_n)\\
&\leqslant -c.
\end{aligned}$$

2. Now, let us show that ∇J is weakly sequentially continuous. Let $(u_n,v_n) \subset X$ such that

$$(u_n,v_n) \rightharpoonup (u,v) \quad \text{in} \quad X.$$

We have to show that

$$\nabla J(u_n,v_n) \rightharpoonup \nabla J(u,v).$$

For every $(w,z) \in \mathscr{C}_c^\infty(\mathbb{R}^N) \times \mathscr{C}_c^\infty(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} \Big(\nabla u_n \nabla z + u_n z + \nabla v_n w + v_n w\Big)dx \to \int_{\mathbb{R}^N} \Big(\nabla u \nabla z + uz + \nabla v \nabla w + vw\Big)dx.$$

For every compact subset $\Omega \subset \mathbb{R}^N$ such that $supp\{(w,z)\} \subset \Omega \times \Omega$, we have

$$\int_{\mathbb{R}^N} \Big(g_1(x)f_1(v_n)z + g_2(x)f_2(u_n)w\Big)dx = \int_{\Omega} \Big(g_1(x)f_1(v_n)z + g_2(x)f_2(u_n)w\Big)dx.$$

By compact Sobolev embedding theorem, up to a subsequence, since

$$(u_n, v_n) \rightharpoonup (u, v) \text{ in } \Omega \times \Omega,$$

we have

$$(u_n, v_n) \to (u, v) \text{ in } L^2(\Omega) \times L^2(\Omega), \text{ as } n \to \infty,$$

$$\Big(u_n(x), v_n(x)\Big) \to \Big(u(x), v(x)\Big), \text{ almost everywhere, in } \Omega \times \Omega, \text{ as } n \to \infty.$$

The functions f_i, $i \in \{1,2\}$ are continuous. So,

$$f_1(v_n(x)) \to f_1(v(x)), \text{ almost everywhere, in } \Omega, \text{ as } n \to \infty,$$

$$f_2(u_n(x)) \to f_2(u(x)), \text{ almost everywhere, in } \Omega, \text{ ;as } n \to \infty.$$

Moreover, by (1.2), there exist $c > 0$ and $c_0 > 0$ such that

$$|f_i(\zeta)| \leqslant \frac{c}{|g_i|_\infty}|\zeta| + c_0, \text{ for every } \zeta \in H^1(\mathbb{R}^N).$$

It follows that

$$g_1(x)f_1(v_n)z \text{ and } g_2(x)f_2(u_n)w \text{ are in } L^1(\Omega).$$

The Lebesgue dominated convergence theorem implies that

$$\int_\Omega \Big(g_1(x)f_1(v_n)z + g_2(x)f_2(u_n)w\Big)dx \to \int_\Omega \Big(g_1(x)f_1(v)z + g_2(x)f_2(u)w\Big)dx, \text{ as } n \to \infty.$$

Thus, by (3.4), we have

$$\Big|J'(u_n, v_n)(w,z) - J'(u,v)(w,z)\Big| \to 0, \text{ as } n \to \infty.$$

Therefore, for every $(w,z) \in \mathscr{C}_c^\infty(\mathbb{R}^N) \times \mathscr{C}_c^\infty(\mathbb{R}^N)$,

$$\langle \nabla J(u_n, v_n), (w,z)\rangle \to \langle \nabla J(u,v), (w,z)\rangle.$$

Hence, $\nabla J(u_n, v_n) \rightharpoonup \nabla J(u,v)$. □

The next result shows that the functional J given in (3.3) satisfies the geometric assumption (2.4) of Theorem 2.

LEMMA 2. *There exists* $\rho > 0$ *such that*

$$b := \inf_{(u,u)\in X^+} J(u,u) > a := \sup_{(-u,u)\in \partial M} J(-u,u).$$

Proof. Since the space $H^1(\mathbb{R}^N)$ embeds continuously into $L^2(\mathbb{R}^N)$, there exists $S > 0$ such that

$$\text{for all } t \in H^1(\mathbb{R}^N), \quad S|t|_2^2 \leqslant \|t\|_1^2. \tag{3.5}$$

By (f), there is $c_0 > 0$ such that for every $(u,v) \in X$,

$$|f_1(v)| \leqslant \frac{S}{4|g_1|_\infty}|v| + c_0, \quad |F_1(v)| \leqslant \frac{S}{4|g_1|_\infty}|v|^2 + c_0|v|, \tag{3.6}$$

$$|f_2(u)| \leqslant \frac{S}{4|g_2|_\infty}|u| + c_0, \quad |F_2(u)| \leqslant \frac{S}{4|g_2|_\infty}|u|^2 + c_0|u|. \tag{3.7}$$

On X^+, by (3.5) and Hölder inequality, we have

$$\begin{aligned}
J(u,u) &= \frac{1}{2}\|Q(u,u)\|^2 - \varphi(x,u,u) \\
&\geqslant \|u\|_1^2 - \int_{\mathbb{R}^N} |g_1(x)|\Big(\frac{Su^2}{4|g_1|_\infty} + c_0|u|\Big)dx - \int_{\mathbb{R}^N} |g_2(x)|\Big(\frac{Su^2}{4|g_2|_\infty} + c_0|u|\Big)dx \\
&\geqslant \|u\|_1^2 - \frac{S}{4}|u|_2^2 - c_0\int_{\mathbb{R}^N}\Big(|g_1(x)|\,|u|\Big)dx - \frac{S}{4}|u|_2^2 - c_0\int_{\mathbb{R}^N}\Big(|g_2(x)|\,|u|\Big)dx \\
&\geqslant \|u\|_1^2 - \frac{S}{2}|u|_2^2 - c_0|g_1|_2|u|_2 - c_0|g_2|_2|u|_2 \\
&\geqslant \|u\|_1^2 - \frac{1}{2}\|u\|_1^2 - c_0S^{-\frac{1}{2}}\Big(|g_1|_2 + |g_2|_2\Big)\|u\|_1 \\
&= \frac{1}{2}\|u\|_1^2 - c_0S^{-\frac{1}{2}}\Big(|g_1|_2 + |g_2|_2\Big)\|u\|_1.
\end{aligned}$$

So, $\inf_{X^+} J > -\infty$.

On X^-, since $g_1(x)F_1(v) \geqslant 0$ and $g_2(x)F_2(u) \geqslant 0$, we have

$$\begin{aligned}
J(-u,u) &= -\frac{1}{2}\|P(-u,u)\|^2 - \varphi(x,u,u) \\
&\leqslant -\frac{1}{2}\|P(-u,u)\|^2 \\
&= -\|u\|_1^2 \\
&\leqslant -\|u\|_1^2 + c_0S^{-\frac{1}{2}}\Big(|g_1|_2 + |g_2|_2\Big)\|u\|_1.
\end{aligned}$$

So, for ρ large enough, we obtain the result. □

Proof of Theorem 2. The functional J given in (3.3), associated with problem (1.1), is of $\mathscr{C}^1$, τ-upper semi-continuous, and its gradient ∇J is weakly sequentially continuous by Lemma 1. Hence, J satisfies assumption (A) of Theorem 1. Moreover, the geometric assumption (2.4) of Theorem 1 is fulfilled by Lemma 2.

By Theorem 1, for some $c \in \mathbb{R}$, there exists $\Big\{(u_n, v_n)\Big\} \subset X$ such that

$$J(u_n, v_n) \to c, \;\; J'(u_n, v_n) \to 0, \;\; \text{as} \;\; n \to \infty.$$

For $u \in X$, we denote by $u^\pm$ the orthogonal projection of u on $X^\pm$.

Let

$$\Big(\overline{u}_n, \overline{v}_n\Big) = \Big(u_n^+ - u_n^-, v_n^+ - v_n^-\Big).$$

Then,

$$\begin{aligned}\|(\overline{u}_n,\overline{v}_n)\|^2 &= \|(u_n^+-u_n^-,v_n^+-v_n^-)\|^2\\ &= \|u_n^+-u_n^-\|_1^2+\|v_n^+-v_n^-\|_1^2\\ &= \|u_n^+\|_1^2+\|u_n^-\|_1^2+\|v_n^+\|_1^2+\|v_n^-\|_1^2\\ &= \|u_n\|_1^2+\|v_n\|_1^2\\ &= \|(u_n,v_n)\|^2.\end{aligned}$$

For sufficiently large $n\in\mathbb{N}$, we have $\|J'(u_n,v_n)\|\leqslant 1$. Hence, for such n

$$\|(u_n,v_n)\| = \|(\overline{u}_n,\overline{v}_n)\| \geqslant J'(u_n,v_n)(\overline{u}_n,\overline{v}_n),$$

where by (3.4),

$$\begin{aligned}J'(u_n,v_n)(\overline{u}_n,\overline{v}_n) = &\int_{\mathbb{R}^N}\Big(\nabla u_n\nabla\overline{v}_n+u_n\overline{v}_n\Big)dx+\int_{\mathbb{R}^N}\Big(\nabla v_n\nabla\overline{u}_n+v_n\overline{u}_n\Big)dx\\ &-\int_{\mathbb{R}^N}\Big(g_1(x)f_1(v_n)\overline{v}_n+g_2(x)f_2(u_n)\overline{u}_n\Big)dx.\end{aligned}$$

On the one hand, we have

$$\begin{aligned}\int_{\mathbb{R}^N}\Big(\nabla u_n\nabla\overline{v}_n+u_n\overline{v}_n\Big)dx = &\int_{\mathbb{R}^N}\Big(\nabla u_n^+\nabla v_n^++u_n^+v_n^+\Big)dx-\int_{\mathbb{R}^N}\Big(\nabla u_n^-\nabla v_n^-+u_n^-v_n^-\Big)dx\\ &-\int_{\mathbb{R}^N}\Big(\nabla u_n^+\nabla v_n^-+u_n^+v_n^-\Big)dx+\int_{\mathbb{R}^N}\Big(\nabla u_n^-\nabla v_n^++u_n^-v_n^+\Big)dx,\end{aligned}\tag{3.8}$$

$$\begin{aligned}\int_{\mathbb{R}^N}\Big(\nabla v_n\nabla\overline{u}_n+v_n\overline{u}_n\Big)dx = &\int_{\mathbb{R}^N}\Big(\nabla v_n^+\nabla u_n^++v_n^+u_n^+\Big)dx-\int_{\mathbb{R}^N}\Big(\nabla v_n^-\nabla u_n^-+v_n^-u_n^-\Big)dx\\ &+\int_{\mathbb{R}^N}\Big(\nabla u_n^+\nabla v_n^-+u_n^+v_n^-\Big)dx-\int_{\mathbb{R}^N}\Big(\nabla u_n^-\nabla v_n^++u_n^-v_n^+\Big)dx.\end{aligned}\tag{3.9}$$

Combining equations (3.8) and (3.9), we get

$$\begin{aligned}&\int_{\mathbb{R}^N}\Big(\nabla u_n\nabla\overline{v}_n+u_n\overline{v}_n\Big)dx+\int_{\mathbb{R}^N}\Big(\nabla v_n\nabla\overline{u}_n+v_n\overline{u}_n\Big)dx\\ &=2\int_{\mathbb{R}^N}\Big(\nabla u_n^+\nabla v_n^++u_n^+v_n^+\Big)dx-2\int_{\mathbb{R}^N}\Big(\nabla u_n^-\nabla v_n^-+u_n^-v_n^-\Big)dx\end{aligned}\tag{3.10}$$

But then, by equation (3.2),

$$\begin{aligned}\int_{\mathbb{R}^N}\Big(\nabla u_n^+\nabla v_n^++u_n^+v_n^+\Big)dx &= \frac{\|(u_n^+,v_n^+)\|^2}{2},\\ \int_{\mathbb{R}^N}\Big(\nabla u_n^-\nabla v_n^-+u_n^-v_n^-\Big)dx &= -\frac{\|(u_n^-,v_n^-)\|^2}{2}.\end{aligned}$$

Il follows that

$$\int_{\mathbb{R}^N}\Big(\nabla u_n\nabla\overline{v}_n+u_n\overline{v}_n\Big)dx+\int_{\mathbb{R}^N}\Big(\nabla v_n\nabla\overline{u}_n+v_n\overline{u}_n\Big)dx=\|(u_n,v_n)\|^2. \tag{3.11}$$

On the other hand, since $|\overline{v}_n|\leqslant|v_n^+|+|v_n^-|$ and $|v_n^+|_2+|v_n^-|_2\leqslant 2|v_n|_2$, by (3.6), we have

$$\begin{aligned}\int_{\mathbb{R}^N}\Big(g_1(x)f_1(v_n)\overline{v}_n\Big)dx&\leqslant\int_{\mathbb{R}^N}|g_1(x)|\left(\frac{S}{4|g_1|_\infty}|v_n|+c_0\right)\Big(|v_n^+|+|v_n^-|\Big)dx\\&\leqslant\int_{\mathbb{R}^N}\left(\frac{S}{4}|v_n|+c_0|g_1(x)|\right)\Big(|v_n^+|+|v_n^-|\Big)dx\\&\leqslant\left(\frac{S}{4}|v_n|_2+c_0|g_1|_2\right)\Big(|v_n^+|_2+|v_n^-|_2\Big)\quad\text{(by Hölder inequality)}\\&\leqslant\left(\frac{S}{4}|v_n|_2+c_0|g_1|_2\right)\Big(2|v_n|_2\Big)\\&\leqslant\frac{1}{2}\|v_n\|_1^2+2c_0|g_1|_2S^{-\frac{1}{2}}\|v_n\|_1\quad\text{(by (3.5))}.\end{aligned}$$

Similarly, we have

$$\int_{\mathbb{R}^N}\Big(g_2(x)f_2(u_n)\overline{u}_n\Big)dx\leqslant\frac{1}{2}\|u_n\|_1^2+2c_0|g_2|_2S^{-\frac{1}{2}}\|u_n\|_1.$$

That is,

$$\begin{aligned}&\int_{\mathbb{R}^N}\Big(g_1(x)f_1(v_n)\overline{v}_n\Big)dx+\int_{\mathbb{R}^N}\Big(g_2(x)f_2(u_n)\overline{u}_n\Big)dx\\&\leqslant\frac{1}{2}\Big(\|u_n\|_1^2+\|v_n\|_1^2\Big)+2c_0S^{-\frac{1}{2}}\Big(|g_2|_2\|u_n\|_1+|g_1|_2\|v_n\|_1\Big)\\&\leqslant\frac{1}{2}\|(u_n,v_n)\|^2+2c_0S^{-\frac{1}{2}}\Big(|g_1|_2+|g_2|_2\Big)\|(u_n,v_n)\|.\end{aligned} \tag{3.12}$$

Equations (3.11) and (3.12), taken together, yield

$$\|(u_n,v_n)\|\geqslant\|(u_n,v_n)\|^2-\frac{1}{2}\|(u_n,v_n)\|^2-2c_0S^{-\frac{1}{2}}\Big(|g_1|_2+|g_2|_2\Big)\|(u_n,v_n)\|.$$

That is,

$$\|(u_n,v_n)\|\geqslant\frac{1}{2}\|(u_n,v_n)\|^2-2c_0S^{-\frac{1}{2}}\Big(|g_1|_2+|g_2|_2\Big)\|(u_n,v_n)\|.$$

It follows that $\Big\{(u_n,v_n)\Big\}$ is bounded. There exists a subsequence of $\Big\{(u_n,v_n)\Big\}$, still denoted $\Big\{(u_n,v_n)\Big\}$, and there exists $(u,v)\in X$ such that

$$(u_n,v_n)\to(u,v)\ \text{ weakly in } X,\ \text{ as } n\to\infty,$$

$$(u_n, v_n) \to (u, v), \text{ in } L^2_{loc}(\mathbb{R}^N) \times L^2_{loc}(\mathbb{R}^N), \text{ as } n \to \infty,$$

$$\big(u_n(x), v_n(x)\big) \to \big(u(x), v(x)\big), \text{ almost everywhere, in } \mathbb{R}^N, \text{ as } n \to \infty.$$

We deduce that, for every $(w, z) \in \mathscr{C}^\infty_c(\mathbb{R}^N) \times \mathscr{C}^\infty_c(\mathbb{R}^N)$,

$$J'(u, v)(w, z) = \lim_{n \to \infty} J'(u_n, v_n)(w, z) = 0.$$

We conclude that $J'(u, v) = 0$. That is, (u, v) is a solution of (1.1).
The proof of Theorem 2 is thus complete. □

REFERENCES

[1] F. Colin and M. Frigon, *Systems of coupled Poisson equations with critical growth in unbounded domains*, Nonlinear Differential Equations and Applications **13** (2006), 369–384.

[2] F. Colin and A. Songo, *An infinite dimensional saddle point theorem and application*, Boundary Value Problems **2026**, 50 (2026).

[3] D. G. de Figueiredo, J. M. do Ó, and B. Ruf, *Critical and subcritical elliptic systems in dimension two*, Indiana University Mathematics Journal **53**, 4 (2004), 1037–1054.

[4] D. G. de Figueiredo and J. Yang, *Decay, symmetry and existence of solutions of semilinear elliptic systems*, Nonlinear Analysis: Theory, Methods and Applications **33**, 3 (1998), 211–234.

[5] G. Li and J. Yang, *Asymptotically linear elliptic systems*, Communications in Partial Differential Equations **29**, 5–6 (2004), 925–954.

[6] S. Liu and Z. Shen, *Generalized saddle point theorem and asymptotically linear problems with periodic potential*, Nonlinear Anal. **86** (2013), 52–57.

[7] W. Kryszewski and A. Szulkin, *Generalized linking theorem with an application to a semilinear Schrödinger equation*, Adv. Differential Equations **3** (1998), 441–472.

[8] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics, Published for the Conference Board of the Mathematical Sciences by American Mathematical Sciences **65**, Washington DC, 1986.

[9] A. Songo and F. Colin, *An infinite-dimensional Linking theorem with application to a system of coupled Poisson equations*, J. Math. Anal. Appl. **566** (2027), no. 131036, Issue 1, https://doi.org/10.1016/j.jmaa.2026.131036.

[10] A. Songo and F. Colin, *On the Fountain Theorem for Continuous Functionals and Its Application to a Semilinear Elliptic Problem in* $\mathbb{R}^2$, https://doi.org/10.48550/arXiv.2509.16059, preprint (2025).

[11] M. Willem, *Minimax Theorems, Progress in Nonlinear Differential Equations and their Applications*, Birkhäuser, Boston, 1996.

[12] G. Zhang and S. Liu, *Existence Result for a Class of Elliptic Systems with Indefinite Weights*, Boundary Value Problems **2008** (2008), 1–10.

(Received January 28, 2026)

Ablanvi Songo
Département de mathématiques
Université de Sherbrooke
Sherbrooke, 2500 boul. de l'Université, J1K 2R1, Québec, Canada
e-mail: ablanvi.songo@usherbrooke.ca