EXISTENCE AND UNIQUENESS OF SOLUTIONS
FOR NONLINEAR FRACTIONAL DIFFERENTIAL
EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this paper, we study existence and uniqueness of solutions to nonlinear fractional
differential equations with integral boundary conditions in an ordered Banach space. We use the
Caputo fractional differential operator and the nonlinearity depends on the fractional derivative
of an unknown function. For the existence of solutions, we employ the nonlinear alternative
of Leray-Schauder and the Banach fixed point theorem. An example is included to show the
applicability of our results.

1. Introduction

The study of fractional differential equations has become a very important and
useful area of mathematics over the last few decades due to its numerous applications in
various areas of physics, chemistry and engineering such as viscoelasticity [5, 28, 29],
dynamical processes in self-similar structures [18], biosciences [19], signal processing
[23], systems control theory [32], electrochemistry [22] and diffusion processes [9, 20].
Further, fractional calculus has found many applications in classical mechanics [25]
and the calculus of variations [6] and is a very useful and simple means for obtaining
solutions to non-homogenous linear ordinary and partial differential equations. For
more details, we refer the reader to [21, 31].

There are several approaches to fractional derivatives such as Riemann-Liouville,
Caputo, Weyl, Hadamard and Grunwald-Letnikov, etc. Applied problems require those
definitions of a fractional derivative that allow the utilization of physically interpretable
initial and boundary conditions. The Caputo fractional derivative satisfies these de-
mands, while the Riemann-Liouville derivative is not suitable for mixed boundary con-
ditions.

Recently, the theory on existence and uniqueness of solutions of linear and non-
linear fractional differential equations has attracted the attention of many authors, see
for example, [1, 3, 4, 11, 15, 16, 17, 27, 30] and references therein. However, many
of the physical systems can better be described by integral boundary conditions. In-
tegral boundary conditions are encountered in various applications such as population
dynamics, blood flow models, chemical engineering and cellular systems. Moreover,
boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two-point, three-point, multi-point and nonlocal boundary value problems as special cases, see [3, 7, 13, 14] and references therein.

In this paper, we study existence and uniqueness of nonlinear fractional differential equations of the type

\[ cD^q_{0+} u(t) = f(t, u(t), cD^\sigma_{0+} u(t)), \quad \text{for } t \in [0, T], \]

subject to integral boundary conditions

\[ \alpha u(0) - \beta u'(0) = \int_0^T g(s, u) ds, \quad \gamma u(1) + \delta u'(1) = \int_0^T h(s, u) ds, \]

where \( 0 < \sigma < 1, 1 < q < 2, \alpha, \delta > 0, \beta, \gamma \geq 0 \) (or \( \alpha, \delta \geq 0, \beta, \gamma > 0 \)) and \( cD^q_{0+}, cD^\sigma_{0+} \) are the Caputo fractional derivatives. We use the nonlinear alternative of Leray-Schauder type and the Banach fixed point theorem to prove existence and uniqueness results. Our results allow \( f \) to depend on \( cD^\sigma_{0+} \), which leads to extra difficulties.

2. Preliminaries

We recall some basic definitions and lemmas from fractional calculus [12, 24]. Riemann’s modified form of Liouville’s fractional integral operator is a generalization of Cauchy’s iterated integral formula

\[ \int_a^t dt_1 \int_a^{t_1} dt_2 \ldots \int_a^{t_{n-1}} g(t_n) dt_n = \frac{1}{\Gamma(n)} \int_a^t g(s) \left( \frac{t}{s} \right)^{1-n} ds, \]

where \( \Gamma \) is Euler’s gamma function. Clearly, the right-hand side of equation (2.1) is meaningful for any positive real value of \( n \). Hence, it is natural to define the fractional integral as follows:

**DEFINITION 2.1.** If \( g \in C([a, b]) \) and \( \alpha > 0 \), then the Riemann-Liouville fractional integral is defined by

\[ I^\alpha_{a+} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t g(s) \left( \frac{t}{s} \right)^{\alpha-1} ds. \]  

For \( a = 0 \), the fractional integral (2.2) can be written as \( I^\alpha_{0+} h(t) = h(t) \ast \varphi_\alpha(t) \), where \( \varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \) for \( t > 0 \) and \( \varphi_\alpha(t) = 0 \) for \( t \leq 0 \).

**DEFINITION 2.2.** The Caputo fractional derivative of order \( \alpha > 0 \) of a continuous function \( g : (a, b) \to \mathbb{R} \) is defined by

\[ cD^\alpha_{a+} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{g^{(n)}(s)}{(t-s)^{n-\alpha+1}} ds, \]

where \( n = [\alpha] + 1 \), (the notation \( [\alpha] \) stands for the largest integer not greater than \( \alpha \)).
REMARK 2.3. Under natural conditions on $g(t)$, the Caputo fractional derivative becomes the conventional integer order derivative of the function $g(t)$ as $\alpha \to n$.

REMARK 2.4. Let $\alpha, \beta > 0$ and $n = [\alpha] + 1$, then the following relations hold:

$$cD^\alpha_{0+} t^{\beta - 1} = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} t^{\beta - 1}, \quad \beta > n$$

and

$$cD^\alpha_{0+} t^k = 0, \quad k = 0, 1, 2, \ldots, n - 1.$$

LEMMA 2.5. For $\alpha > 0$, $g(t) \in C(0,1) \cap \mathcal{L}(0,1)$, the homogenous fractional differential equation

$$cD^\alpha_{0+} g(t) = 0,$$

has a solution

$$g(t) = c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1},$$

where, $c_i \in \mathbb{R}$, $i = 0, \ldots, n$, and $n = [\alpha] + 1$.

LEMMA 2.6. Assume that $g(t) \in C(0,1) \cap \mathcal{L}(0,1)$, with derivative of order $n$ that belongs to $C(0,1) \cap \mathcal{L}(0,1)$, then

$$I^\sigma_{0+} cD^\alpha_{0+} g(t) = g(t) + c_1 t + c_2 t^2 + \cdots + c_n t^{n-1},$$

where, $c_i \in \mathbb{R}$, $i = 0, \ldots, n$, and $n = [\alpha] + 1$.

The following properties of fractional integrals and fractional differential operators will be useful for our further discussion.

LEMMA 2.7. \cite{2} Let $p, q \geq 0$, $f \in L^1[a,b]$. Then

$$I^p_{0+} I^q_{0+} f(t) = I^{p+q}_{0+} f(t) = I^q_{0+} I^p_{0+} f(t)$$

(2.3)

is satisfied almost everywhere on $[a,b]$. Moreover, if $f \in C[a,b]$ or $p + q \geq 1$, then (2.3) is true for all $t \in [a,b]$.

LEMMA 2.8. \cite{12} If $q > 0$, $f \in C[a,b]$, then $cD^q_{0+} I^q_{0+} f(t) = f(t)$ for all $t \in [a,b]$.

One of our main results is based on the following theorem.

LEMMA 2.9. \cite{8} (Nonlinear alternative of Leray-Schauder type) Let $X$ be a Banach space and $C$ be a nonempty convex subset of $X$ and $U$ be open in $C$ with $0 \in U$. Let $T : \overline{U} \to C$ be continuous and compact operator. Then either

(i) $T$ has a fixed point, or

(ii) there exists $u \in \partial U$ and $\lambda \in [0,1]$ with $u = \lambda T(u)$.

Define $X = \{ u : u \in C([0,T]) \text{ and } cD^\sigma_{0+} u \in C([0,T]), 0 < \sigma < 1 \}$ equipped with the norm $\|u\|_X = \max_{0 \leq t \leq 1} |u| + \max_{0 \leq t \leq 1} |cD^\sigma_{0+} u|$. The space $X$ is a Banach space \cite{30}.
3. Main Results

Lemma 3.1. Let $1 < \alpha \leq 2$ and $h, \phi, \psi \in (C([0,1]), \mathbb{R})$. Then the unique solution of the boundary value problem for fractional differential equation

\[ cD_0^\alpha u(t) = y(t), \quad t \in [0, T], \]  
\[ \alpha u(0) - \beta u'(0) = \int_0^T \phi(s)ds, \quad \gamma u(T) + \delta u'(T) = \int_0^T \psi(s)ds \]

is given by

\[ u(t) = \int_0^1 G(t,s)y(s)ds + \varphi(t), \]

where,

\[ G(t,s) = \begin{cases} 
\frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} - \frac{(\beta+\alpha)\gamma(T-s)^{\gamma-1}}{p\Gamma(\gamma)} - \frac{(\beta+\alpha)\delta(q-1)(T-s)^{\gamma-2}}{p\Gamma(\gamma)}, & 0 \leq s \leq t, \\
-\frac{(\beta+\alpha)\delta(q-1)(T-s)^{\gamma-2}}{p\Gamma(\gamma)}, & 0 \leq s \leq t, 
\end{cases} \]

\[ \varphi(t) = \delta + \gamma(T-t) \frac{1}{p} \int_0^T \phi(s)ds + \beta(1+t) \frac{1}{p} \int_0^T \psi(s)ds \quad \text{and} \quad p = \alpha(\delta + \gamma T) + \beta \gamma. \]

Proof. Assume that $u$ is a solution of the boundary value problem (3.1), (3.2), then using Lemma 2.6, we have

\[ u(t) = I_0^q y(t) + c_1 + c_2t, \quad c_1, c_2 \in \mathbb{R}. \]

From (3.2) and (3.5), we obtain

\[ \alpha c_1 - \beta c_2 = \int_0^T \phi(s)ds, \]
\[ \gamma c_1 + (\delta + \gamma T)c_2 = -\gamma I_0^q y(T) - \delta I_0^{q-1} y(T) + \int_0^T \psi(s)ds, \]
which implies that

\[ c_1 = -\frac{\beta}{p}(\gamma I_0^q y(T) + \delta I_0^{q-1} y(T)) + \frac{1}{p} \int_0^T ((\delta + \gamma T)\phi(s) + \beta \psi(s))ds, \]
\[ c_2 = -\frac{\alpha}{p}(\gamma I_0^q y(T) + \delta I_0^{q-1} y(T)) + \frac{1}{p} \int_0^T (\alpha \psi(s) - \gamma \phi(s))ds. \]

Using (3.6) and (3.7) in (3.5), we obtain

\[ u(t) = I_0^q y(t) - \frac{1}{p}(\gamma I_0^q y(T) + \delta I_0^{q-1} y(T))(\beta + \alpha t) \]
\[ + \frac{\delta + \gamma(T-t)}{p} \int_0^T \phi(s)ds + \frac{\beta + \alpha t}{p} \int_0^T \psi(s)ds, \]
which can be written as

\[ u(t) = \int_0^1 G(t,s)y(s)ds + \varphi(t), \]

where,

\[
G(t,s) = \begin{cases} 
\frac{(t-s)^{\frac{\gamma-1}{\Gamma(q)}} - (\beta+\alpha)t^{\frac{\gamma-1}{\Gamma(q)}}}{\Gamma(q)} \frac{(t-s)^{-\frac{\delta}{\Gamma(q)}} - (\beta+\alpha)t^{-\frac{\delta}{\Gamma(q)}}}{\Gamma(q)} - (\beta+\alpha)\delta(t-s)^{-\frac{\delta}{\Gamma(q)}} - (\beta+\alpha)(q-1)(t-s)^{-\frac{\gamma-2}{\Gamma(q)}}, & 0 \leq s \leq t, \\
\frac{\delta + \gamma(T-t)}{p} \int_0^T \phi(s)ds + \frac{(\beta+\alpha)}{p} \int_0^T \psi(s)ds. & 
\end{cases}
\]

and \( \varphi(t) = \frac{\delta + \gamma(T-t)}{p} \int_0^T \phi(s)ds + \frac{(\beta+\alpha)}{p} \int_0^T \psi(s)ds. \)

**Lemma 3.2.** Assume that \( f \in C([0,T]) \times \mathbb{R} \times \mathbb{R}, \) then \( u \in X \) is solution of fractional boundary value problem (3.1), (3.2) if and only if \( u \in X \) is solution of the fractional integral equation

\[
u(t) = \int_0^1 G(t,s)f(s,u(s),cD_{0+}^\alpha u(t))ds + \frac{\delta + \gamma(T-t)}{p} \int_0^T g(s,u(s))ds + \frac{(\beta+\alpha)}{p} \int_0^T h(s,u(s))ds. \tag{3.8}
\]

**Proof.** Let \( u \in X \) be a solution of the boundary value problem (3.1), (3.2), then by the same method as used in Lemma 3.1, we can prove that \( u \) is a solution of the fractional integral equation (3.8).

Conversely, let \( u \) satisfy (3.8) and denote the right hand side of equation (3.8) by \( w(t) \). Then, by Lemmas 2.7 and 2.8, we obtain

\[
w(t) = \int_0^1 G(t,s)f(s,u(s),cD_{0+}^\alpha u(s))ds + \frac{\delta + \gamma(T-t)}{p} \int_0^T g(s,u(s))ds + \frac{(\beta+\alpha)}{p} \int_0^T h(s,u(s))ds
\]

which implies that

\[
cD_{0+}^\alpha w(t) = cD_{0+}^\alpha \int_0^1 G(t,s)f(s,u(s),cD_{0+}^\alpha u(t))ds + \frac{\gamma}{p} cD_{0+}^\alpha I_{0+}^\alpha f(T,u(T),cD_{0+}^\alpha u(T))(\beta+\alpha t) - \frac{\delta}{p} I_{0+}^{\alpha-1} f(T,u(T),cD_{0+}^\alpha u(T))(\beta+\alpha t) + \frac{\delta + \gamma(T-t)}{p} \int_0^T g(s,u(s))ds + \frac{(\beta+\alpha)}{p} \int_0^T h(s,u(s))ds,
\]

which implies that

\[
cD_{0+}^\alpha w(t) = cD_{0+}^\alpha \int_0^1 G(t,s)f(s,u(s),cD_{0+}^\alpha u(t))ds + \frac{\gamma}{p} cD_{0+}^\alpha I_{0+}^\alpha f(T,u(T),cD_{0+}^\alpha u(T))(\beta+\alpha t) - \frac{\delta}{p} I_{0+}^{\alpha-1} f(T,u(T),cD_{0+}^\alpha u(T))(\beta+\alpha t) + f(t,u(t),cD_{0+}^\alpha u(t)).
\]
Hence, $u(t)$ is a solution of the fractional differential equation
\[
^cD_{0+}^\alpha u(t) = f(t, u(t), ^cD_{0+}^\beta u(t)).
\]
Also, it is easy to verify that
\[
\alpha u(0) - \beta u'(0) = \int_0^T g(s, u(s))ds, \quad \gamma u(T) + \delta u'(T) = \int_0^T h(s, u(s))ds.
\]

Now, define an operator $A : X \to X$ by
\[
Au(t) = \int_0^1 G(t, s)f(s, u(s), ^cD_{0+}^\alpha u(t))ds + \phi(t).
\]

Then, the boundary value problem (1.1), (1.2) is equivalent to the value problem
\[
\phi
\]
Hence,
\[
\phi
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functions
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\]
Also, it is easy to verify that
\[
\begin{aligned}
&H_1 \quad \text{The functions } f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \ g, h : [0, T] \times \mathbb{R} \to \mathbb{R} \text{ are continuous.} \\
&H_2 \quad \text{There exist continuous and nondecreasing functions } \psi_i^j : [0, \infty) \to (0, \infty), \ (i = 1, 2), \text{ such that} \\
&|f(t, u, v)| \leq \phi_j^1(t)\psi_i^1(|u|) + \phi_j^2(t)\psi_i^2(|v|), \quad \text{for } t \in [0, T], \ u, v \in \mathbb{R}. \\
&H_3 \quad \text{There exist a continuous and nondecreasing function } \psi_\delta : [0, \infty) \to (0, \infty), \text{ and a function } \phi_\delta \in L_1([0, T], (0, \infty)) \text{ such that} \\
&|g(t, u)| \leq \phi_\delta(t)\psi_\delta(|u|), \quad \text{for } t \in [0, T], \ u, v \in \mathbb{R}. \\
&H_4 \quad \text{There exist a continuous and nondecreasing function } \psi_h : [0, \infty) \to (0, \infty), \text{ and a function } \phi_h \in L_1([0, T], (0, \infty)) \text{ such that} \\
&|h(t, u)| \leq \phi_h(t)\psi_h(|u|), \text{ for } t \in [0, T], \ u, v \in \mathbb{R}. \\
\end{aligned}
\]

Define
\[
\begin{aligned}
a &= \int_0^T \phi_\delta(s)ds, \quad b = \int_0^T \phi_h(s)ds, \\
k_{r,q} := \psi_1^1(r)\|f_1^0\|_{L_1} + \psi_1^2(r)\|f_2^0\|_{L_1}, \\
b_i^r := \psi_1^r(\gamma b_1^0\phi_1^0(T)) + \delta b_2^0\phi_1^0(T), \quad (i = 1, 2), \\
M_r = \frac{1}{p}(\beta + \alpha \gamma T)(b_1^r + b_2^r) + \frac{a}{p}(\delta + 2 \alpha \gamma T)\psi_\delta(r) + \frac{b}{p}(\beta + \alpha \gamma T)\psi_h(r). \\
\end{aligned}
\]

(H5) There exists $r > 0$ such that
\[
\frac{r}{\Gamma(2-\sigma)k_{r,q} - 1} + k_{r,q} + \left(1 + \frac{r}{\Gamma(2-\sigma)}\right)M_r > 1.
\]

**Theorem 3.3.** Under the assumptions (H1) – (H5), the fractional boundary value problem (1.1), (1.2) has at least one solution on $[0, T]$. 
Proof. In view of the continuity of \( f, g \) and \( h \), the operator \( A \) is continuous. To show that \( A \) maps bounded sets into bounded sets in \( X \), choose \( \eta > 0 \) (fixed).

Let

\[
I \geq \max \left\{ \eta, \frac{T^{1-\sigma} k_{\alpha} q^{-1}}{\Gamma(2-\sigma)} + k_{\eta,q} + \left( 1 + \frac{T^{1-\sigma}}{\Gamma(2-\sigma)} \right) M_{\eta} \right\}
\]

and define

\[
U = \{ u \in X : \|u\|_{X} < \eta \} \quad \text{and} \quad C = \{ u \in X : \|u\|_{X} < I \}.
\]

For \( t \in [0,T] \) and \( u \in U \), we have

\[
|Au| = \left| \int_{0}^{T} G(t,s)f(s,u(s),cD_{0+}^{\sigma}u(s))ds \right|
\]

\[
+ \frac{1}{p} \int_{0}^{T} \left( ((\delta + \gamma(T-t))g(s,u) + (\beta + \alpha t)h(s,u)) \right) ds
\]

\[
\leq \int_{0}^{T} |G(t,s)||f(s,u(s),cD_{0+}^{\sigma}u(s))|ds + \frac{\delta + \gamma(T-t)}{p} \int_{0}^{T} |g(s,u)|ds
\]

\[
+ \frac{\beta + \alpha t}{p} \int_{0}^{T} |h(s,u)|ds.
\]

Using \( (H2) - (H4) \), we obtain

\[
|Au| \leq \int_{0}^{t} \left( \frac{(t-s)^{q-1}}{\Gamma(q)} (|f^{1}_{f}(s)|u|X| + |f^{1}_{f}(s)|u|X|)ds \right)
\]

\[
+ \frac{(\beta + \alpha t)}{p} \int_{1}^{T} \left( \frac{\gamma(T-s)^{q-1}}{\Gamma(q)} + \frac{\delta(T-s)^{q-2}}{\Gamma(q-1)} \right)
\]

\[
\times (|f^{1}_{f}(s)|u|X| + |f^{1}_{f}(s)|u|X|)ds
\]

\[
+ \frac{\delta + \gamma(T-t)}{p} \int_{0}^{T} |f_{g}(s)|u_{x}|X|ds + \frac{\beta + \alpha t}{p} \int_{0}^{T} |f_{h}(s)|u_{x}|X|ds
\]

\[
\leq \psi_{f}(\eta)I_{f1}^{q}f_{f}(\eta) + \psi_{f}(\eta)I_{f1}^{q}f_{f}(\eta)
\]

\[
+ \frac{\beta + \alpha T}{p} \left[ \psi_{f}(\eta)(\gamma I_{f1}^{q}f_{f}(T) + \delta I_{f1}^{q}f_{f}(T)) + \psi_{f}(\eta)(\gamma I_{f1}^{q}f_{f}(T) + \delta I_{f1}^{q}f_{f}(T)) \right]
\]

\[
+ \frac{(\delta + 2\gamma(T-t))}{p} \int_{0}^{T} |f_{g}(s)|ds + \frac{\beta + \alpha T}{p} \int_{0}^{T} |f_{h}(s)|ds
\]

\[
\leq k_{\eta,q} + \frac{1}{p}(\beta + \alpha T)(b_{\eta}^{1} + b_{\eta}^{2}) + \frac{a}{p}(\delta + 2\gamma(T-t))\psi_{f}(\eta) + \frac{b}{p}(\beta + \alpha T)\psi_{h}(\eta)
\]

\[
= k_{\eta,q} + M_{\eta}.
\]

Also,

\[
|(A(u))'| \leq \left| \int_{0}^{T} \frac{\partial}{\partial t} G(t,s)f(s,u(s),cD_{0+}^{\sigma}u(s))ds \right|
\]

\[
\leq \int_{0}^{T} \left| \frac{\partial}{\partial t} G(t,s) \right| |f(s,u(s),cD_{0+}^{\sigma}u(s))|ds + \frac{\gamma}{p} \int_{0}^{T} |g(s,u)|ds + \frac{\alpha}{p} \int_{0}^{T} |h(s,u)|ds
\]

\[
\leq \int_{0}^{t} \left( \frac{(t-s)^{q-2}}{\Gamma(q-1)} (|f^{1}_{f}(s)|u|X| + |f^{1}_{f}(s)|u|X|)ds \right)
\]
we take \( T - s \) \( \leq \frac{1}{\Gamma(q - 1)} \) \( \leq 36 \) R. A. KHAN, M. UR REHMAN AND J. HENDERSON

Hence, it follows that

\[
\begin{align*}
& \alpha \int_0^T \left( \frac{\gamma(T - s)^{q-1}}{\Gamma(q)} + \frac{\delta(T - s)^{q-2}}{\Gamma(q - 1)} \right) \\
& \times \left( |\phi_1^j(s)|u_2^j(\|u\|_X) + |\phi_2^j(s)|u_1^j(\|u\|_X) \right) ds \\
& + \gamma \int_0^T |\phi_3(s)|u_3^j(\|u\|_X) ds + \frac{\alpha}{p} \int_0^T |\phi_h(s)|u_4^j(\|u\|_X) ds \\
& \leq |\psi_j^1(\eta)| + \frac{1}{\Gamma(q + 1)} |\phi_1^j|_{L_1} + |\psi_j^2(\eta)| + \frac{1}{\Gamma(q + 1)} |\phi_2^j|_{L_1} \\
& + \frac{\alpha}{p} \left[ |\psi_j^3(\eta)| + \frac{1}{\Gamma(q + 1)} |\phi_3^j|_{L_1} + |\psi_j^4(\eta)| + \frac{1}{\Gamma(q + 1)} |\phi_4^j|_{L_1} \right] \\
& + \gamma \int_0^T |\phi_3(s)|u_3^j(\|u\|_X) ds + \frac{\alpha}{p} \int_0^T |\phi_h(s)|u_4^j(\|u\|_X) ds \\
& \leq k_{\eta, q - 1} + \frac{\alpha}{p} (b_1 + b_2^2) + \frac{a \gamma \psi_3(\eta)}{p} + \frac{b \alpha \psi_4(\eta)}{p} \\
\end{align*}
\]

Therefore, \( \|Au\|_X \leq l \), which implies that \( Au \in C \). Hence, \( A \) maps bounded sets into bounded sets in \( X \).

Now, we show that \( A \) maps bounded sets into equicontinuous sets of \( X \). For this, we take \( K = \max \{|f(t, u(t), ^cD_0^\sigma u(t))| : u \in U, t \in J\} \), \( L_1 = \max \{|g(t, u(t))| : u \in U, t \in J\} \) and \( L_2 = \max \{|h(t, u(t))| : u \in U, t \in J\} \). Choose \( t, \tau \in (0, T) \) such that \( t < \tau \) and \( u \in U \). Then,

\[
\begin{align*}
& |A(u)(\tau) - A(u)(t)| \\
& \leq \int_0^T |G(\tau, s) - G(t, s)| |f(s, u(s), ^cD_0^\sigma u(s))| ds \\
& + \frac{\gamma(\tau - t)}{p} \int_0^T |g(s, u)| ds + \frac{\alpha(\tau - t)}{p} \int_0^T |h(s, u)| ds \\
& \leq K \left[ \int_0^t |G(\tau, s) - G(t, s)| ds + \int_t^\tau |G(\tau, s) - G(t, s)| ds + \int_\tau^T |G(\tau, s) - G(t, s)| ds \right] \\
& + \frac{T}{p} (\gamma L_1 + \alpha L_2)(\tau - t)
\end{align*}
\]
\[
\leq K \left[ \int_0^t \left( \frac{(\tau-s)^{q-1}}{\Gamma(q)} + \left( \frac{\alpha \gamma (T-s)^{q-1}}{\Gamma(q)} + \frac{\alpha \delta (q-1)(T-s)^{q-2}}{p \Gamma(q)} \right) (\tau-t) \right) ds + \int_t^\tau \left( \frac{(\tau-s)^{q-1}}{\Gamma(q)} + \left( \frac{\alpha \gamma (T-s)^{q-1}}{\Gamma(q)} + \frac{\alpha \delta (q-1)(T-s)^{q-2}}{p \Gamma(q)} \right) (\tau-t) \right) ds \right. \\
+ \left. \int_t^T \left( \frac{\alpha \gamma (T-s)^{q-1}}{\Gamma(q)} + \frac{\alpha \delta (q-1)(T-s)^{q-2}}{p \Gamma(q)} \right) (\tau-t) ds \right] + \frac{T}{p} (\gamma L_1 + \alpha L_2) (\tau-t) \\
= K \alpha \left( \frac{\gamma T^q}{q} + \delta T^{q-1} \right) (\tau-t) + \frac{K (\tau^q - t^q)}{q \Gamma(q)} + \frac{T}{p} (\gamma L_1 + \alpha L_2) (\tau-t)
\]

and

\[
|^{\ast \ast}D_0^\sigma A(u)(\tau) - ^{\ast \ast}D_0^\sigma A(u)(t)| \\
= \frac{1}{\Gamma(1-\sigma)} \left[ \int_0^\tau (\tau-s)^{-\sigma}(Au)'(s) ds - \int_0^t (t-s)^{-\sigma}(Au)'(s) ds \right] \\
\leq \frac{1}{\Gamma(1-\sigma)} \left[ \int_0^\tau (\tau-s)^{-\sigma}(Au)'(s) ds - \int_0^t (t-s)^{-\sigma}(Au)'(s) ds \right. \\
+ \left. \frac{1}{\Gamma(1-\sigma)} \int_t^\tau (t-s)^{-\sigma}(Au)'(s) ds - \int_0^t (t-s)^{-\sigma}(Au)'(s) ds \right] \\
\leq \frac{1}{\Gamma(1-\sigma)} \left[ \int_\tau^\tau (\tau-s)^{-\sigma}|(Au)'(s)| ds + \int_t^\tau ((\tau-s)^{-\sigma} - (t-s)^{-\sigma})|(Au)'(s)| ds \right] \\
\leq \frac{1}{\Gamma(1-\sigma)} \left[ \int_0^\tau (\tau-s)^{-\sigma} \left( \int_0^\tau |\frac{\partial}{\partial s} G(s,z)||f(z,u(z),^{\ast \ast}D_0^\sigma u(z))|dz + \phi'(s) \right) ds \right. \\
+ \int_0^t ((\tau-s)^{-\sigma} - (t-s)^{-\sigma}) \left( \int_0^\tau |\frac{\partial}{\partial s} G(s,z)||f(z,u(z),^{\ast \ast}D_0^\sigma u(z))|dz + \phi'(s) \right) ds \right] \\
\leq \frac{K}{p \Gamma(q) \Gamma(1-\sigma)} ((\alpha \delta + p) T^{q-1} + \alpha \gamma T^q) \left[ \int_\tau^\tau (\tau-s)^{-\sigma} ds + \int_t^\tau ((\tau-s)^{-\sigma} - (t-s)^{-\sigma}) ds \right] \\
+ \frac{\gamma L_1 + \alpha L_2}{p \Gamma(q)} \left[ \int_\tau^\tau (\tau-s)^{-\sigma} ds + \int_t^\tau ((\tau-s)^{-\sigma} - (t-s)^{-\sigma}) ds \right] \\
\leq \frac{K}{p \Gamma(q) \Gamma(2-\sigma)} ((\alpha \delta + p) T^{q-1} + \alpha \gamma T^q) (\tau^{1-\sigma} - t^{1-\sigma}) + \frac{(\gamma L_1 + \alpha L_2) T}{p \Gamma(2-\sigma)} (\tau^{1-\sigma} - t^{1-\sigma}) \\
\leq \left( \frac{K}{p \Gamma(q) \Gamma(2-\sigma)} ((\alpha \delta + p) T^{q-1} + \alpha \gamma T^q) + \frac{(\gamma L_1 + \alpha L_2) T}{p \Gamma(2-\sigma)} \right) (\tau^{1-\sigma} - t^{1-\sigma}).
\]

Hence, \( |A(u)(\tau) - A(u)(t)| \to 0 \) and \( |^{\ast \ast}D_0^\sigma A(u)(\tau) - ^{\ast \ast}D_0^\sigma A(u)(t)| \to 0 \) as \( t \to \tau \). By Arzela-Ascoli, it follows that \( A : X \to X \) is completely continuous. Define

\[
U_1 = \{ u \in X : ||u||_X < r \}
\]
and assume that there exists \( u \in \partial U_1 \) such that \( u = \lambda A(u) \) for some \( \lambda \in (0, 1) \). Using (3.3), we obtain

\[
u(t) = \lambda \left( \int_0^T G(t,s)f(s,u(s),cD_{0+}^\alpha u(s))ds + \varphi(t) \right).
\]

In view \((H2) - (H4)\), we have

\[
|u(t)| \leq \int_0^T |G(t,s)||f(s,u(s),cD_{0+}^\alpha u(s))|ds + \frac{\delta + \gamma(T-t)}{p} \int_0^T |g(s,u)|ds
\]

\[
+ \frac{\beta + \alpha t}{p} \int_0^T |h(s,u)|ds
\]

\[
\leq \int_0^T \frac{(t-s)^{q-1}}{\Gamma(q)} \left( |\phi_j^1(s)|\psi_j^1(||u||_x) + |\phi_j^2(s)|\psi_j^2(||u||_x) \right)ds
\]

\[
+ \frac{(\beta + \alpha t)}{p} \int_0^T \left( \frac{\gamma(T-s)^{q-1}}{\Gamma(q)} + \frac{\delta(T-s)^q}{\Gamma(q-1)} \right) \times \left( |\phi_j^1(s)|\psi_j^1(||u||_x) + |\phi_j^2(s)|\psi_j^2(||u||_x) \right)ds
\]

\[
+ \frac{\delta + \gamma(T-t)}{p} \int_0^T |\phi_h(s)|\psi_h(||u||_x)ds + \frac{\beta + \alpha t}{p} \int_0^T |h(s,u)|ds
\]

\[
\leq k_{||u||_x,q} + \frac{1}{p}(\beta + \alpha T)(b_{||u||_x}^1 + b_{||u||_x}^2) + \frac{a}{p}(\delta + 2\gamma T)|\psi_j^1(||u||_x)|
\]

\[
+ \frac{b}{p}(\beta + \alpha T)|\psi_h(||u||_x)|.
\]

Also,

\[
|u'(t)| < \left| \int_0^T \frac{\partial}{\partial t} G(t,s)f(s,u(s),cD_{0+}^\alpha u(s))ds + \varphi'(t) \right|
\]

\[
\leq \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} \left( |\phi_j^1(s)|\psi_j^1(||u||_x) + |\phi_j^2(s)|\psi_j^2(||u||_x) \right)ds
\]

\[
+ \frac{\alpha}{p} \int_0^T \left( \frac{\gamma(T-s)^{q-1}}{\Gamma(q)} + \frac{\delta(T-s)^q}{\Gamma(q-1)} \right) \times \left( |\phi_j^1(s)|\psi_j^1(||u||_x) + |\phi_j^2(s)|\psi_j^2(||u||_x) \right)ds
\]

\[
+ \frac{\gamma}{p} \int_0^T |\phi_h(s)|\psi_h(||u||_x)ds + \frac{\alpha}{p} \int_0^T |h(s,u)|ds
\]

\[
\leq \psi_j^1(||u||_x)||I_{0+}^{q-1} \phi_j^1||_{L_1} + \psi_j^2(||u||_x)||I_{0+}^{q-1} \phi_j^2||_{L_1}
\]

\[
+ \frac{\alpha}{p} \left[ \psi_j^1(||u||_x)(\gamma I_{0+}^{q} \phi_j^1(T) + \delta I_{0+}^{q} \phi_j^1(T)) \right] + \frac{\alpha}{p} \left[ \psi_j^2(||u||_x)(\gamma I_{0+}^{q} \phi_j^2(T) + \delta I_{0+}^{q} \phi_j^2(T)) \right] + \frac{\gamma a|\psi_j^1(||u||_x)|}{p} + \frac{c a|\psi_h(||u||_x)|}{p}
\]

\[
\leq k_{||u||_x,q-1} + \frac{\alpha}{p}(b_{||u||_x}^1 + b_{||u||_x}^2) + \frac{a|\psi_j^1(||u||_x)|}{p} + \frac{b|\psi_h(||u||_x)|}{p}.
\]
Hence,
\[ |^cD_0^\sigma u(t)| < \frac{1}{\Gamma(1-\sigma)} \int_0^t (t-s)^{-\sigma} |u'(s)| ds \]
\[ \leq \frac{T^{1-\sigma}}{\Gamma(2-\sigma)} \left( k_1\|u\|_q + \frac{\alpha}{p} (b\|u\|_X + b_2^2\|u\|_X) + \frac{a\gamma\psi(\eta)}{p} + \frac{b\alpha\psi(\eta)}{p} \right). \]

Therefore, it follows that
\[ \frac{\|u\|_X}{\frac{T^{1-\sigma}k_1\|u\|_q + \frac{\alpha}{p} (b\|u\|_X + b_2^2\|u\|_X) + \frac{a\gamma\psi(\eta)}{p} + \frac{b\alpha\psi(\eta)}{p}}{1 + \frac{T^{1-\sigma}}{\Gamma(2-\sigma)} M\|u\|_X} < 1, \]
a contradiction to \((H5)\). Hence, \( u \neq \lambda A(u) \) for \( u \in \partial U_1, \lambda \in [0,1] \). By Theorem 2.9, the BVP \((1.1), (1.2)\) has at least one solution. \(\square\)

The uniqueness result is based on the Banach contraction principle.

**THEOREM 3.4.** Assume that:

\((H6)\) There exists a constant \( k > 0 \) such that
\[ |f(t,u,v) - f(t,\overline{u},\overline{v})| \leq k(|u - \overline{u}| + |v - \overline{v}|), \text{ for each } t \in [0,T] \text{ and all } u, \overline{u} \in \mathbb{R}. \]

\((H7)\) There exists a constant \( k_1 > 0 \) such that
\[ |g(t,u) - g(t,\overline{u})| \leq k_1|u - \overline{u}|, \text{ for each } t \in [0,T] \text{ and all } u, \overline{u} \in \mathbb{R}. \]

\((H8)\) There exists a constant \( k_2 > 0 \) such that
\[ |h(t,u) - h(t,\overline{u})| \leq k_2|u - \overline{u}|, \text{ for each } t \in [0,T] \text{ and all } u, \overline{u} \in \mathbb{R}. \]

If \( k < \frac{1}{N} \left( 1 + \frac{T^{1-\sigma}}{\Gamma(2-\sigma)} \right)^{-\sigma} \), where \( N = \left( \frac{T^q}{\Gamma(q)} + \frac{\beta + \alpha T}{pT^q} (\gamma T^q + \delta T^{q-1}) \right) \) and \( k_1 < \frac{p}{3T(\delta + 2\gamma T)} \), \( k_2 < \frac{p}{3T(\beta + \alpha T)} \), then the boundary value problem \((1.1), (1.2)\) has a unique solution.

**Proof.** We shall use Banach fixed point theorem. For this we need to verify that \( A \) is contraction. Let \( u, \overline{u} \in X \), then in view of \((H6) - (H8)\), for each \( t \in [0,T] \), we have
\[ |A(u)(t) - A(\overline{u})(t)| \leq \int_0^T G(t,s)|f(s,u(s),^cD_0^\sigma u(s)) - f(s,\overline{u},^cD_0^\sigma \overline{u}(s))| ds \]
\[ + \frac{\delta + \gamma(T-t)}{\alpha(\delta + \gamma T)} \int_0^T |g(s,u(s)) - g(s,\overline{u}(s))| ds \]
\[ + \frac{(\beta + \alpha T)}{\alpha(\delta + \gamma T)} \int_0^T |g(s,u(s)) - g(s,\overline{u}(s))| ds \]
\[ \leq \|u - \overline{u}\| \left( k \int_0^1 |G(t,s)| ds + \frac{T}{p} (k_1(\delta + \gamma(T-t)) + k_2(\beta + \alpha T)) \right) \]
\[ \leq \|u - \overline{u}\| \left( k \int_0^1 |G(t,s)| ds + \frac{T}{p} (k_1(\delta + 2\gamma T) + k_2(\beta + \alpha T)) \right). \]
From (3.4), we have
\[
\int_0^1 |G(t,s)|ds \leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} ds + \int_0^T (\beta + \alpha t) \left( \frac{\gamma(T-s)^{q-1}}{p \Gamma(q)} + \frac{\delta(q-1)(T-s)^{q-2}}{p \Gamma(q)} \right) ds \\
\leq \frac{t^q}{q \Gamma(q)} + \frac{\beta + \alpha t}{p \Gamma(q)} (\gamma T^q + \delta T^{q-1}) \leq \frac{T^q}{\Gamma(q)} + \frac{\beta + \alpha t}{p \Gamma(q)} (\gamma T^q + \delta T^{q-1}) = N,
\]
which implies that
\[
|A(u)(t) - A(\overline{u})(t)| \leq \|u - \overline{u}\| \left[ kN + \frac{T}{p} (k_1 (\delta + 2\gamma T) + k_2 (\beta + \alpha T)) \right].
\]

Hence,
\[
|^{c}D^{\sigma}_{0+}(Au)(t) - ^{c}D^{\sigma}_{0+}(A\overline{u})(t)| \\
= \left| \frac{1}{\Gamma(1-\sigma)} \int_0^t (t-s)^{-\sigma} ((Au)'(s) - (A\overline{u})'(s)) ds \right| \\
\leq \frac{1}{\Gamma(1-\sigma)} \int_0^t (t-s)^{-\sigma} \left( \int_0^1 \frac{\partial}{\partial s} G(s,z) \middle| f(z,u(z),^{c}D^{\sigma}_{0+}u(z) - f(z,\overline{u}(z),^{c}D^{\sigma}_{0+}\overline{u}(z)) ds \right) ds \\
\leq \frac{k\|u - \overline{u}\|}{\Gamma(1-\sigma)} \int_0^t (t-s)^{-\sigma} \left( \int_0^1 \frac{\partial}{\partial s} G(s,z) \right) ds.
\]

Again, from (3.4), we have
\[
\int_0^1 \left| \frac{\partial}{\partial t} G(t,s) \right| ds \leq \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} ds + \frac{\alpha}{p \Gamma(q)} \int_0^T (\gamma(T-s)^{q-1} + \delta(q-1)(T-s)^{q-2}) ds \\
\leq \frac{t^{q-1}}{q \Gamma(q)} + \frac{\gamma \alpha T^q}{pq \Gamma(q)} + \frac{\delta \alpha T^{q-1}}{pq \Gamma(q)} \leq \frac{T^{q-1}}{q \Gamma(q)} + \frac{\alpha}{p \Gamma(q)} (\gamma T^q + \delta T^{q-1}).
\]

Thus, we have
\[
|^{c}D^{\sigma}_{0+}(Au)(t) - ^{c}D^{\sigma}_{0+}(A\overline{u})(t)| \leq \frac{kT^{-\sigma}}{\Gamma(2-\sigma)} \left( \frac{T^q}{\Gamma(q)} + \frac{\beta + \alpha T}{p \Gamma(q)} (\gamma T^q + \delta T^{q-1}) \right).
\]

Therefore, \( \|Au - A\overline{u}\| \leq L \|u - \overline{u}\| \) where \( L = k(1 + \frac{T^{-\sigma}}{\Gamma(2-\sigma)}) N + \frac{T}{p} (k_1 (\delta + 2\gamma T) + k_2 (\beta + \alpha T)) < 1 \). By the contraction mapping principle, the BVP (1.1), (1.2) has a unique solution. □

**Example 3.5.** Consider the following fractional BVP,
\[
^{c}D^{3\sigma}_{0+}u(t) = \frac{e^{-\alpha t} (|u| + |^{c}D^{1\sigma}_{0+}u(t)|)}{124 \sqrt{\pi} + e^{-\alpha t} (1 + |u|)}, \quad t \in [0,1] \tag{3.10}
\]
\[
u(0) - u'(0) = \int_0^1 \frac{|u(s)|e^{-s}}{9(1 + |u(s)|)} ds, \tag{3.11}
\]

\[
\]
By simple calculations we have,

\[ u(1) + u'(1) = \int_0^1 \frac{|u(s)| \sin s}{36} ds. \] (3.12)

Set \( f(t, u, v) = \frac{e^{-ct}u(t) + v(t)}{(24\sqrt{\pi} + e^{-ct})(1 + u(t) + v(t))} \), \( g(t, u) = \frac{u(t)e^{-t}}{9(1 + u(t))} \) and \( h(t, u) = \frac{u(t)\sin t}{36} \) for \( t \in [0, 1] \) and \( u, v \in [0, \infty) \).

Let \( t \in [0, 1] \) and \( u, \overline{u}, v, \overline{v} \in [0, \infty) \), then we have

\[
|f(t, u, v) - f(t, \overline{u}, \overline{v})| = \frac{e^{-ct}}{(24\sqrt{\pi} + e^{-ct})} \frac{|u(t) + v(t)|}{1 + u(t) + v(t)} \frac{|u(t) + \overline{u}(t) + \overline{v}(t)|}{1 + u(t) + \overline{u}(t) + \overline{v}(t)} \]
\[
\leq \frac{e^{-ct}(|u(t) - \overline{u}(t)| + |v(t) - \overline{v}(t)|)}{(24\sqrt{\pi} + e^{-ct})(1 + u(t) + v(t))(1 + \overline{u}(t) + \overline{v}(t))} \]
\[
\leq \frac{1}{124\sqrt{\pi}} \frac{e^{-ct}}{1 + e^{-ct}} (|u(t) - \overline{u}(t)| + |v(t) - \overline{v}(t)|). \]

By simple calculations we have, \( p = 3, N = \frac{14}{3\sqrt{\pi}}, \frac{1}{3N} \left( 1 + \frac{T^{\sigma}}{122-\sigma} \right)^{-1} = \frac{\pi}{14(\sqrt{\pi}+2)}. \)

Here \( k = \frac{1}{124\sqrt{\pi}} < \frac{\pi}{14(\sqrt{\pi}+2)}. \) Hence condition (H6) is satisfied.

Now, for \( u, \overline{u} \in [0, \infty) \) we have

\[
|g(t, u) - g(t, \overline{u})| = \frac{e^{-t}}{9} \frac{|u(t)|}{1 + u(t)} \frac{|u(t) - \overline{u}(t)|}{1 + \overline{u}(t)} \]
\[
\leq \frac{e^{-t} |u(t) - \overline{u}(t)|}{9(1 + u(t))(1 + \overline{u}(t))} \leq \frac{1}{9} |u(t) - \overline{u}(t)|, \]

where \( k_1 = \frac{1}{9} < \frac{p}{5(\beta+2\gamma)} = \frac{1}{3}. \) Hence (H7) is satisfied. Also,

\[
|h(t, u) - h(t, \overline{u})| = \frac{\sin t}{36} |u(t)| \leq \frac{1}{36} |u(t) - \overline{u}(t)|, \]

where \( k_2 = \frac{1}{36} < \frac{p}{5(\beta+\sigma)} = \frac{1}{3}. \) All conditions of the Theorem 3.4 are satisfied. Therefore the fractional BVP (3.10)–(3.12) has a unique solution.

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(Received August 3, 2009)

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