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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this paper, we study existence and uniqueness of solutions to nonlinear fractional differential equations with integral boundary conditions in an ordered Banach space. We use the Caputo fractional differential operator and the nonlinearity depends on the fractional derivative of an unknown function. For the existence of solutions, we employ the nonlinear alternative of Leray-Schauder and the Banach fixed point theorem. An example is included to show the applicability of our results.

1. Introduction

The study of fractional differential equations has become a very important and useful area of mathematics over the last few decades due to its numerous applications in various areas of physics, chemistry and engineering such as viscoelasticity [5, 28, 29], dynamical processes in self-similar structures [18], biosciences [19], signal processing [23], systems control theory [32], electrochemistry [22] and diffusion processes [9, 20]. Further, fractional calculus has found many applications in classical mechanics [25] and the calculus of variations [6] and is a very useful and simple means for obtaining solutions to non-homogenous linear ordinary and partial differential equations. For more details, we refer the reader to [21, 31].

There are several approaches to fractional derivatives such as Riemann-Liouville, Caputo, Weyl, Hadamard and Grunwald-Letnikov, etc. Applied problems require those definitions of a fractional derivative that allow the utilization of physically interpretable initial and boundary conditions. The Caputo fractional derivative satisfies these demands, while the Riemann-Liouville derivative is not suitable for mixed boundary conditions.

Recently, the theory on existence and uniqueness of solutions of linear and nonlinear fractional differential equations has attracted the attention of many authors, see for example, [1, 3, 4, 11, 15, 16, 17, 27, 30] and references therein. However, many of the physical systems can better be described by integral boundary conditions. Integral boundary conditions are encountered in various applications such as population dynamics, blood flow models, chemical engineering and cellular systems. Moreover,

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boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two-point, three-point, multi-point and nonlocal boundary value problems as special cases, see [3, 7, 13, 14] and references therein.

In this paper, we study existence and uniqueness of nonlinear fractional differential equations of the type

$$ {}^cD^q_{0+}u(t) = f(t,u(t),{}^cD^\sigma_{0+}u(t)), \text{ for } t \in [0,T], \tag{1.1}$$

subject to integral boundary conditions

$$ \alpha u(0) - \beta u'(0) = \int_0^T g(s,u)ds, \ \gamma u(1) + \delta u'(1) = \int_0^T h(s,u)ds, \tag{1.2}$$

where $0<\sigma<1$, $1<q<2$, $\alpha,\delta>0$, $\beta,\gamma\geqslant 0$ (or $\alpha,\delta\geqslant 0$, $\beta,\gamma>0$) and ${}^cD^q_{0+}$, ${}^cD^\sigma_{0+}$ are the Caputo fractional derivatives. We use the nonlinear alternative of Leray-Schauder type and the Banach fixed point theorem to prove existence and uniqueness results. Our results allow f to depend on ${}^cD^\sigma_{0+}$, which leads to extra difficulties.

2. Preliminaries

We recall some basic definitions and lemmas from fractional calculus [12, 24]. Riemann's modified form of Liouville's fractional integral operator is a generalization of Cauchy's iterated integral formula

$$ \int_a^t dt_1 \int_a^{t_1} dt_2 \dots \int_a^{t_{n-1}} g(t_n)dt_n = \frac{1}{\Gamma(n)} \int_a^t \frac{g(s)}{(t-s)^{1-n}}ds, \tag{2.1}$$

where Γ is Euler's gamma function. Clearly, the right-hand side of equation (2.1) is meaningful for any positive real value of n. Hence, it is natural to define the fractional integral as follows:

DEFINITION 2.1. If $g \in C([a,b])$ and $\alpha>0$, then the Riemann-Liouville fractional integral is defined by

$$ I^\alpha_{a+}g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g(s)}{(t-s)^{1-\alpha}}ds. \tag{2.2}$$

For $a=0$, the fractional integral (2.2) can be written as $I^\alpha_{0+}h(t) = h(t)*\varphi_\alpha(t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t>0$ and $\varphi_\alpha(t)=0$ for $t\leqslant 0$.

DEFINITION 2.2. The Caputo fractional derivative of order $\alpha>0$ of a continuous function $g:(a,b)\to\mathbb{R}$ is defined by

$$ {}^cD^\alpha_{a+}g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds, $$

where $n=[\alpha]+1$, (the notation $[\alpha]$ stands for the largest integer not greater than α).

REMARK 2.3. Under natural conditions on $g(t)$, the Caputo fractional derivative becomes the conventional integer order derivative of the function $g(t)$ as $\alpha \to n$.

REMARK 2.4. Let $\alpha, \beta > 0$ and $n = [\alpha] + 1$, then the following relations hold:

$$ {}^cD_{0+}^{\alpha} t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} t^{\beta-1}, \ \beta > n \text{ and } {}^cD_{0+}^{\alpha} t^k = 0, \ k = 0, 1, 2, \cdots, n-1. $$

LEMMA 2.5. *For $\alpha > 0$, $g(t) \in C(0,1) \cap Ł(0,1)$, the homogenous fractional differential equation*

$$ {}^cD_{0+}^{\alpha} g(t) = 0, $$

has a solution

$$ g(t) = c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1}, $$

where, $c_i \in \mathbb{R}$, $i = 0, ..., n$, and $n = [\alpha] + 1$.

LEMMA 2.6. *Assume that $g(t) \in C(0,1) \cap Ł(0,1)$, with derivative of order n that belongs to $C(0,1) \cap Ł(0,1)$, then*

$$ I_{0+}^{\alpha} {}^cD_{0+}^{\alpha} g(t) = g(t) + c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1}, $$

where, $c_i \in \mathbb{R}$, $i = 0, ..., n$, and $n = [\alpha] + 1$.

The following properties of fractional integrals and fractional differential operators will be useful for our further discussion.

LEMMA 2.7. [2] *Let $p, q \geqslant 0$, $f \in L_1[a,b]$. Then*

$$ I_{0+}^{p} I_{0+}^{q} f(t) = I_{0+}^{p+q} f(t) = I_{0+}^{q} I_{0+}^{p} f(t) \tag{2.3} $$

is satisfied almost everywhere on $[a,b]$. Moreover, if $f \in C[a,b]$ or $p + q \geqslant 1$, then (2.3) *is true for all $t \in [a,b]$.*

LEMMA 2.8. [12] *If $q > 0$, $f \in C[a,b]$, then ${}^cD_{0+}^{q} I_{0+}^{q} f(t) = f(t)$ for all $t \in [a,b]$.*

One of our main results is based on the following theorem.

LEMMA 2.9. [8] (Nonlinear alternative of Leray-Schauder type) *Let X be a Banach space and C be a nonempty convex subset of X and U be open in C with $0 \in U$. Let $T : \overline{U} \to C$ be continuous and compact operator. Then either*

(i) T has a fixed point, or

(ii) there exists $u \in \partial U$ and $\lambda \in [0,1]$ with $u = \lambda T(u)$.

Define $X = \{u : u \in C([0,T]) \text{ and } {}^cD_{0+}^{\sigma} u \in C([0,T]), 0 < \sigma < 1\}$ equipped with the norm $\|u\|_X = \max\limits_{0 \leqslant t \leqslant 1} |u| + \max\limits_{0 \leqslant t \leqslant 1} |{}^cD_{0+}^{q} u|$. The space X is a Banach space [30].

3. Main Results

LEMMA 3.1. *Let* $1 < \alpha \leqslant 2$ *and* $h, \phi, \psi \in (C([0,1]), \mathbb{R})$. *Then the unique solution of the boundary value problem for fractional differential equation*

$$ {}^cD^q_{0+}u(t) = y(t),\ t \in [0,T], \tag{3.1} $$

$$ \alpha u(0) - \beta u'(0) = \int_0^T \phi(s)ds,\ \ \gamma u(T) + \delta u'(T) = \int_0^T \psi(s)ds \tag{3.2} $$

is given by

$$ u(t) = \int_0^1 G(t,s)y(s)ds + \varphi(t), \tag{3.3} $$

where,

$$ G(t,s) = \begin{cases} \frac{(t-s)^{q-1}}{\Gamma(q)} - \frac{(\beta+\alpha t)\gamma(T-s)^{q-1}}{p\Gamma(q)} - \frac{(\beta+\alpha t)\delta(q-1)(T-s)^{q-2}}{p\Gamma(q)}, & 0 \leqslant s \leqslant t, \\ -\frac{(\beta+\alpha t)\gamma(T-s)^{q-1}}{p\Gamma(q)} - \frac{(\beta+\alpha t)\delta(q-1)(T-s)^{q-2}}{p\Gamma(q)}, & 0 \leqslant s \leqslant t, \end{cases} \tag{3.4} $$

$$ \varphi(t) = \frac{\delta+\gamma(T-t)}{p}\int_0^T \phi(s)ds + \frac{\beta(1+t)}{p}\int_0^T \psi(s)ds \text{ and } p = \alpha(\delta+\gamma T) + \beta\gamma. $$

Proof. Assume that u is a solution of the boundary value problem (3.1), (3.2), then using Lemma 2.6, we have

$$ u(t) = I^q_{0+}y(t) + c_1 + c_2 t,\ c_1, c_2 \in \mathbb{R}. \tag{3.5} $$

From (3.2) and (3.5), we obtain

$$ \alpha c_1 - \beta c_2 = \int_0^T \phi(s)ds, $$

$$ \gamma c_1 + (\delta+\gamma T)c_2 = -\gamma I^q_{0+}y(T) - \delta I^{q-1}_{0+}y(T) + \int_0^T \psi(s)ds, $$

which implies that

$$ c_1 = -\frac{\beta}{p}(\gamma I^q_{0+}y(T) + \delta I^{q-1}_{0+}y(T)) + \frac{1}{p}\int_0^T ((\delta+\gamma T)\phi(s) + \beta\psi(s))ds, \tag{3.6} $$

$$ c_2 = -\frac{\alpha}{p}(\gamma I^q_{0+}y(T) + \delta I^{q-1}_{0+}y(T)) + \frac{1}{p}\int_0^T (\alpha\psi(s) - \gamma\phi(s))ds. \tag{3.7} $$

Using (3.6)and (3.7) in (3.5), we obtain

$$ \begin{aligned} u(t) =& I^q_{0+}y(t) - \frac{1}{p}(\gamma I^q_{0+}y(T) + \delta I^{q-1}_{0+}y(T))(\beta+\alpha t) \\ &+ \frac{\delta+\gamma(T-t)}{p}\int_0^T \phi(s)ds + \frac{(\beta+\alpha t)}{p}\int_0^T \psi(s)ds, \end{aligned} $$

which can be written as

$$u(t)=\int_0^1 G(t,s)y(s)ds+\varphi(t),$$

where,

$$G(t,s)=\begin{cases}\frac{(t-s)^{q-1}}{\Gamma(q)}-\frac{(\beta+\alpha t)\gamma(T-s)^{q-1}}{p\Gamma(q)}-\frac{(\beta+\alpha t)\delta(q-1)(T-s)^{q-2}}{p\Gamma(q)}, & 0\leqslant s\leqslant t,\\ -\frac{(\beta+\alpha t)\gamma(T-s)^{q-1}}{p\Gamma(q)}-\frac{(\beta+\alpha t)\delta(q-1)(T-s)^{q-2}}{p\Gamma(q)}, & 0\leqslant s\leqslant t,\end{cases}$$

and $\varphi(t)=\frac{\delta+\gamma(T-t)}{p}\int_0^T\phi(s)ds+\frac{(\beta+\alpha t)}{p}\int_0^T\psi(s)ds.$ □

LEMMA 3.2. *Assume that $f\in C([0,T])\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, then $u\in X$ is solution of fractional boundary value problem* (3.1)*,* (3.2) *if and only if $u\in X$ is solution of the fractional integral equation*

$$u(t)=\int_0^1 G(t,s)f(s,u(s),{}^cD_{0+}^{\sigma}u(t))ds+\frac{\delta+\gamma(T-t)}{p}\int_0^T g(s,u(s))ds+\frac{(\beta+\alpha t)}{p}\int_0^T h(s,u(s))ds. \tag{3.8}$$

Proof. Let $u\in X$ be a solution of the boundary value problem (3.1), (3.2), then by the same method as used in Lemma 3.1, we can prove that u is a solution of the fractional integral equation (3.8).

Conversely, let u satisfy (3.8) and denote the right hand side of equation (3.8) by $w(t)$. Then, by Lemmas 2.7 and 2.8, we obtain

$$\begin{aligned}w(t)=&\int_0^1 G(t,s)f(s,u(s),{}^cD_{0+}^{\sigma}u(s))ds+\frac{\delta+\gamma(T-t)}{p}\int_0^T g(s,u(s))ds\\&+\frac{(\beta+\alpha t)}{p}\int_0^T h(s,u(s))ds\\=&I_{0+}^q f(t,u(t),{}^cD_{0+}^{\sigma}u(t))-\frac{\gamma}{p}I_{0+}^q f(T,u(T),{}^cD_{0+}^{\sigma}u(T))(\beta+\alpha t)\\&-\frac{\delta}{p}I_{0+}^{q-1}f(T,u(T),{}^cD_{0+}^{\sigma}u(T))(\beta+\alpha t)+\frac{\delta+\gamma(T-t)}{p}\int_0^T g(s,u(s))(s)ds\\&+\frac{(\beta+\alpha t)}{p}\int_0^T h(s,u(s))(s)ds,\end{aligned}$$

which implies that

$$\begin{aligned}{}^cD_{0+}^q w(t)=&{}^cD_{0+}^q I_{0+}^q f(t,u(t),{}^cD_{0+}^{\sigma}u(t))-\frac{\gamma}{p}{}^cD_{0+}^q I_{0+}^q f(T,u(T),{}^cD_{0+}^{\sigma}u(T))(\beta+\alpha t)\\&-\frac{\delta}{p}{}^cD_{0+}^q I_{0+}^{q-1}f(T,u(T),{}^cD_{0+}^{\sigma}u(T))(\beta+\alpha t)\\=&f(t,u(t),{}^cD_{0+}^{\sigma}u(t)).\end{aligned}$$

Hence, $u(t)$ is a solution of the fractional differential equation

$$^cD^q_{0+}u(t) = f(t,u(t),{}^cD^\sigma_{0+}u(t)).$$

Also, it is easy to verify that

$$\alpha u(0) - \beta u'(0) = \int_0^T g(s,u(s))ds, \quad \gamma u(T) + \delta u'(T) = \int_0^T h(s,u(s))ds. \qquad \square$$

Now, define an operator $A : X \to X$ by

$$Au(t) = \int_0^1 G(t,s)f(s,u(s),{}^cD^q_{0+}u(t))ds + \varphi(t). \tag{3.9}$$

Then, the boundary value problem (1.1), (1.2) is equivalent to the fixed point problem $Au = u$. In what follows, we establish an existence result using the nonlinear alternative of Leray-Schauder type by imposing growth conditions on f, g and h.

Assume that the following hold:

$(H1)$ The functions $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g,h : [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous.

$(H2)$ There exist continuous and nondecreasing functions $\psi^i_f : [0,\infty) \to (0,\infty)$, and functions $\phi^i_f \in L_1([0,T],(0,\infty))$, $(i = 1,2)$, such that

$$|f(t,u,v)| \leqslant \phi^1_f(t)\psi^1_f(|u|) + \phi^2_f(t)\psi^2_f(|v|), \quad \text{for } t \in [0,T],\ u,v \in \mathbb{R}.$$

$(H3)$ There exist a continuous and nondecreasing function $\psi_g : [0,\infty) \to (0,\infty)$, and a function $\phi_g \in L_1([0,T],(0,\infty))$ such that

$$|g(t,u)| \leqslant \phi_g(t)\psi_g(|u|), \quad \text{for } t \in [0,T],\ u,v \in \mathbb{R}.$$

$(H4)$ There exist a continuous and nondecreasing function $\psi_h : [0,\infty) \to (0,\infty)$, and a function $\phi_h \in L_1([0,T],(0,\infty))$ such that $|h(t,u)| \leqslant \phi_h(t)\psi_h(|u|)$, for $t \in [0,T]$, $u,v \in \mathbb{R}$. Define

$$\begin{aligned}
&a = \int_0^T \phi_g(s)ds,\ b = \int_0^T \phi_h(s)ds,\\
&k_{r,q} := \psi^1_f(r)\|I^q_{+0}\phi^1_f\|_{L_1} + \psi^2_f(r)\|I^q_{+0}\phi^2_f\|_{L_1},\\
&b^i_r := \psi^i_f(r)(\gamma I^q_{+0}\phi^i_f(T)) + \delta I^q_{+0}\phi^i_f(T)),\ (i = 1,2),\\
&M_r = \frac{1}{p}(\beta + \alpha T)(b^1_r + b^2_r) + \frac{a}{p}(\delta + 2\gamma T)\psi_g(r) + \frac{b}{p}(\beta + \alpha T)\psi_h(r).
\end{aligned}$$

$(H5)$ There exists $r > 0$ such that

$$\frac{r}{\frac{T^{1-\sigma}k_{r,q-1}}{\Gamma(2-\sigma)} + k_{r,q} + \left(1 + \frac{T^{-\sigma}}{\Gamma(2-\sigma)}\right)M_r} > 1.$$

THEOREM 3.3. *Under the assumptions* $(H1) - (H5)$, *the fractional boundary value problem* (1.1), (1.2) *has at least one solution on* $[0,T]$.

Proof. In view of the continuity of f, g and h, the operator A is continuous. To show that A maps bounded sets into bounded sets in X, choose $\eta > 0$ (fixed).
Let $l \geqslant \max\{\eta, \frac{T^{1-\sigma}k_{\eta,q-1}}{\Gamma(2-\sigma)} + k_{\eta,q} + \left(1+\frac{T^{-\sigma}}{\Gamma(2-\sigma)}\right)M_\eta\}$ and define

$$U = \{u \in X : \|u\|_X < \eta\} \text{ and } C = \{u \in X : \|u\|_X < l\}.$$

For $t \in [0,T]$ and $u \in U$, we have

$$\begin{aligned}|Au| &= \left|\int_0^T G(t,s)f(s,u(s),{}^cD^\sigma_{0+}u(s))ds \right.\\ &\quad \left. + \frac{1}{p}\int_0^T ((\delta+\gamma(T-t))g(s,u) + (\beta+\alpha t)h(s,u))\,ds\right|\\ &\leqslant \int_0^T |G(t,s)||f(s,u(s),{}^cD^\sigma_{0+}u(s))|ds + \frac{\delta+\gamma(T-t)}{p}\int_0^T |g(s,u)|ds\\ &\quad + \frac{\beta+\alpha t}{p}\int_0^T |h(s,u)|ds.\end{aligned}$$

Using $(H2)-(H4)$, we obtain

$$\begin{aligned}|Au| &\leqslant \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}(|\phi_f^1(s)|\psi_f^1(\|u\|_X) + |\phi_f^2(s)|\psi_f^2(\|u\|_X))ds\\ &\quad + \frac{(\beta+\alpha t)}{p}\int_1^T \left(\frac{\gamma(T-s)^{q-1}}{\Gamma(q)} + \frac{\delta(T-s)^{q-2}}{\Gamma(q-1)}\right)\\ &\quad \times (|\phi_f^1(s)|\psi_f^1(\|u\|_X) + |\phi_f^2(s)|\psi_f^2(\|u\|_X))ds\\ &\quad + \frac{\delta+\gamma(T-t)}{p}\int_0^T |\phi_g(s)|\psi_g(\|u\|_X)ds + \frac{\beta+\alpha t}{p}\int_0^T |\phi_h(s)|\psi_h(\|u\|_X)ds\\ &\leqslant \psi_f^1(\eta)\|I^q_{+0}\phi_f^1\|_{L_1} + \psi_f^2(\eta)\|I^q_{+0}\phi_f^2\|_{L_1}\\ &\quad + \frac{(\beta+\alpha T)}{p}\left[\psi_f^1(\eta)(\gamma I^q_{+0}\phi_f^1(T) + \delta I^q_{+0}\phi_f^1(T)) + \psi_f^2(\eta)(\gamma I^q_{+0}\phi_f^2(T) + \delta I^q_{+0}\phi_f^2(T))\right]\\ &\quad + \frac{(\delta+2\gamma T)\psi_g(\eta)}{p}\int_0^T \phi_g(s)ds + \frac{(\beta+\alpha T)\psi_h(\eta)}{p}\int_0^T \phi_h(s)ds\\ &\leqslant k_{\eta,q} + \frac{1}{p}(\beta+\alpha T)(b_\eta^1 + b_\eta^2) + \frac{a}{p}(\delta+2\gamma T)\psi_g(\eta) + \frac{b}{p}(\beta+\alpha T)\psi_h(\eta)\\ &= k_{\eta,q} + M_\eta.\end{aligned}$$

Also,

$$\begin{aligned}|(A(u))'| &\leqslant \left|\int_0^T \frac{\partial}{\partial t}G(t,s)f(s,u(s),{}^cD^\sigma_{0+}u(s))ds + \varphi'(t)\right|\\ &\leqslant \int_0^T \left|\frac{\partial}{\partial t}G(t,s)\right||f(s,u(s),{}^cD^\sigma_{0+}u(s))|ds + \frac{\gamma}{p}\int_0^T |g(s,u)|ds + \frac{\alpha}{p}\int_0^T |h(s,u)|ds\\ &\leqslant \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)}(|\phi_f^1(s)|\psi_f^1(\|u\|_X) + |\phi_f^2(s)|\psi_f^2(\|u\|_X))ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha}{p}\int_0^T \left(\frac{\gamma(T-s)^{q-1}}{\Gamma(q)}+\frac{\delta(T-s)^{q-2}}{\Gamma(q-1)}\right)\\
&\times(|\phi_f^1(s)|\psi_f^1(\|u\|_X)+|\phi_f^2(s)|\psi_f^2(\|u\|_X))ds\\
&+\frac{\gamma}{p}\int_0^T |\phi_g(s)|\psi_g(\|u\|_X)ds+\frac{\alpha}{p}\int_0^T |\phi_h(s)|\psi_h(\|u\|_X)ds\\
\leqslant\ &\psi_f^1(\eta)\|I_{+0}^{q-1}\phi_f^1\|_{L_1}+\psi_f^2(\eta)\|I_{+0}^{q-1}\phi_f^2\|_{L_1}\\
&+\frac{\alpha}{p}\left[\psi_f^1(\eta)(\gamma I_{+0}^q\phi_f^1(T)+\delta I_{+0}^q\phi_f^1(T))+\psi_f^2(\eta)(\gamma I_{+0}^q\phi_f^2(T)+\delta I_{+0}^q\phi_f^2(T))\right]\\
&+\frac{\gamma\psi_g(\eta)}{p}\int_0^T \phi_g(s)d+\frac{\alpha\psi_h(\eta)}{p}\int_0^T \phi_h(s)ds\\
\leqslant\ &k_{\eta,q-1}+\frac{\alpha}{p}(b_\eta^1+b_\eta^2)+\frac{a\gamma\psi_g(\eta)}{p}+\frac{b\alpha\psi_h(\eta)}{p}.
\end{aligned}
$$

Hence, it follows that

$$
\begin{aligned}
|{}^cD_{0+}^\sigma A(u)(t)| &= \frac{1}{\Gamma(1-\sigma)}\left|\int_0^t (t-s)^{-\sigma}(Au)'(s)ds\right|\\
&\leqslant \frac{1}{\Gamma(1-\sigma)}\int_0^t (t-s)^{-\sigma}|(Au)'(s)|ds\\
&\leqslant \frac{T^{1-\sigma}}{\Gamma(2-\sigma)}(k_{q-1}+\frac{\alpha}{p}(b_\eta^1+b_\eta^2)+\frac{a\gamma\psi_g(\eta)}{p}+\frac{b\alpha\psi_h(\eta)}{p})\\
&\leqslant \frac{T^{-\sigma}}{\Gamma(2-\sigma)}(M_\eta+Tk_{\eta,q-1}).
\end{aligned}
$$

Therefore, $\|Au\|_X \leqslant l$, which implies that $Au \in C$. Hence, A maps bounded sets into bounded sets in X.

Now, we show that A maps bounded sets into equicontinuous sets of X. For this, we take $K=\max\{|f(t,u(t),{}^cD_{0+}^\sigma u(t))| : u\in U, t\in J\}$, $L_1=\max\{|g(t,u(t)| : u\in U, t\in J\}$ and $L_2=\max\{|h(t,u(t)| : u\in U, t\in J\}$. Choose $t,\tau\in(0,T]$ such that $t<\tau$ and $u\in U$. Then,

$$
\begin{aligned}
&|A(u)(\tau)-A(u)(t)|\\
\leqslant &\int_0^T |G(\tau,s)-G(t,s)||f(s,u(s),{}^cD_{0+}^\sigma u(s))|ds\\
&+\frac{\gamma(\tau-t)}{p}\int_0^T |g(s,u)|ds+\frac{\alpha(\tau-t)}{p}\int_0^T |h(s,u)|ds\\
\leqslant &K\left[\int_0^t |G(\tau,s)-G(t,s)|ds+\int_t^\tau |G(\tau,s)-G(t,s)|ds+\int_\tau^T |G(\tau,s)-G(t,s)|ds\right]\\
&+\frac{T}{p}(\gamma L_1+\alpha L_2)(\tau-t)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant K\left[\int_0^t \left(\frac{(\tau-s)^{q-1}-(t-s)^{q-1}}{\Gamma(q)}+\left(\frac{\alpha\gamma(T-s)^{q-1}}{\Gamma(q)}+\frac{\alpha\delta(q-1)(T-s)^{q-2})}{p\Gamma(q)}\right)(\tau-t)\right)ds\right.\\
&\quad+\int_t^\tau \left(\frac{(\tau-s)^{q-1}}{\Gamma(q)}+\left(\frac{\alpha\gamma(T-s)^{q-1}}{\Gamma(q)}+\frac{\alpha\delta(q-1)(T-s)^{q-2})}{p\Gamma(q)}\right)(\tau-t)\right)ds\\
&\quad\left.+\int_\tau^T \left(\frac{\alpha\gamma(T-s)^{q-1}}{\Gamma(q)}+\frac{\alpha\delta(q-1)(T-s)^{q-2})}{p\Gamma(q)}\right)(\tau-t)ds\right]+\frac{T}{p}(\gamma L_1+\alpha L_2)(\tau-t)\\
&\leqslant K\left[\int_0^T \left(\frac{\alpha\gamma(T-s)^{q-1}}{\Gamma(q)}+\frac{\alpha\delta(q-1)(T-s)^{q-2})}{p\Gamma(q)}\right)(\tau-t)ds\right]+\frac{T}{p}(\gamma L_1+\alpha L_2)(\tau-t)\\
&=\frac{K\alpha}{p\Gamma(q)}\left(\frac{\gamma T^q}{q}+\delta T^{q-1}\right)(\tau-t)+\frac{K(\tau^q-t^q)}{q\Gamma(q)}+\frac{T}{p}(\gamma L_1+\alpha L_2)(\tau-t)
\end{aligned}
$$

and

$$
\begin{aligned}
&|{}^cD_{0+}^\sigma A(u)(\tau)-{}^cD_{0+}^\sigma A(u)(t)|\\
&=\frac{1}{\Gamma(1-\sigma)}\left|\int_0^\tau (\tau-s)^{-\sigma}(Au)'(s)ds-\int_0^t (t-s)^{-\sigma}(Au)'(s)ds\right|\\
&\leqslant\frac{1}{\Gamma(1-\sigma)}\left|\int_0^\tau (\tau-s)^{-\sigma}(Au)'(s)ds-\int_0^t (\tau-s)^{-\sigma}(Au)'(s)ds\right|\\
&\quad+\frac{1}{\Gamma(1-\sigma)}\left|\int_0^t (\tau-s)^{-\sigma}(Au)'(s)ds-\int_0^t (t-s)^{-\sigma}(Au)'(s)ds\right|\\
&\leqslant\frac{1}{\Gamma(1-\sigma)}\left(\int_t^\tau (\tau-s)^{-\sigma}|(Au)'(s)|ds+\int_0^t ((\tau-s)^{-\sigma}-(t-s)^{-\sigma})|(Au)'(s)|ds\right)\\
&\leqslant\frac{1}{\Gamma(1-\sigma)}\left[\int_t^\tau (\tau-s)^{-\sigma}\left(\int_0^T |\frac{\partial}{\partial s}G(s,z)||f(z,u(z),{}^cD_{0+}^\sigma u(z))|dz+\varphi'(s)\right)ds\right.\\
&\quad\left.+\int_0^t ((\tau-s)^{-\sigma}-(t-s)^{-\sigma})\left(\int_0^T |\frac{\partial}{\partial s}G(s,z)||f(z,u(z),{}^cD_{0+}^\sigma u(z))|dz+\varphi'(s)\right)ds\right]\\
&\leqslant\frac{K}{p\Gamma(q)\Gamma(1-\sigma)}((\alpha\delta+p)T^{q-1}+\alpha\gamma T^q)\left[\int_t^\tau (\tau-s)^{-\sigma}ds+\int_0^t ((\tau-s)^{-\sigma}-(t-s)^{-\sigma})ds\right]\\
&\quad+\frac{(\gamma L_1+\alpha L_2)T}{p\Gamma(1-\sigma)}\left[\int_t^\tau (\tau-s)^{-\sigma}ds+\int_0^t ((\tau-s)^{-\sigma}-(t-s)^{-\sigma})ds\right]\\
&\leqslant\frac{K}{p\Gamma(q)\Gamma(2-\sigma)}((\alpha\delta+p)T^{q-1}+\alpha\gamma T^q)(\tau^{1-\sigma}-t^{1-\sigma})+\frac{(\gamma L_1+\alpha L_2)T}{p\Gamma(2-\sigma)}(\tau^{1-\sigma}-t^{1-\sigma})\\
&\leqslant\left(\frac{K}{p\Gamma(q)\Gamma(2-\sigma)}((\alpha\delta+p)T^{q-1}+\alpha\gamma T^q)+\frac{(\gamma L_1+\alpha L_2)T}{p\Gamma(2-\sigma)}\right)(\tau^{1-\sigma}-t^{1-\sigma}).
\end{aligned}
$$

Hence, $|A(u)(\tau)-A(u)(t)|\to 0$ and $|{}^cD_{0+}^\sigma A(u)(\tau)-{}^cD_{0+}^\sigma A(u)(t)|\to 0$ as $t\to\tau$. By Arzela-Ascoli, it follows that $A:X\to X$ is completely continuous. Define

$$
U_1=\{u\in X:\|u\|_X<r\}
$$

and assume that there exists $u \in \partial U_1$ such that $u = \lambda A(u)$ for some $\lambda \in (0,1)$. Using (3.3), we obtain

$$u(t) = \lambda \left(\int_0^T G(t,s) f(s,u(s), {}^c D_{0+}^{\sigma} u(s)) ds + \varphi(t) \right).$$

In view $(H2)-(H4)$, we have

$$\begin{aligned}
|u(t)| <& \int_0^T |G(t,s)||f(s,u(s), {}^c D_{0+}^{\sigma} u(s))| ds + \frac{\delta + \gamma(T-t)}{p} \int_0^T |g(s,u)| ds \\
&+ \frac{\beta + \alpha t}{p} \int_0^T |h(s,u)| ds \\
\leqslant& \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} (|\phi_f^1(s)| \psi_f^1(\|u\|_X) + |\phi_f^2(s)| \psi_f^2(\|u\|_X)) ds \\
&+ \frac{(\beta + \alpha t)}{p} \int_1^T \left(\frac{\gamma(T-s)^{q-1}}{\Gamma(q)} + \frac{\delta(T-s)^{q-2}}{\Gamma(q-1)} \right) \\
&\times (|\phi_f^1(s)| \psi_f^1(\|u\|_X) + |\phi_f^2(s)| \psi_f^2(\|u\|_X)) ds \\
&+ \frac{\delta + \gamma(T-t)}{p} \int_0^T |\phi_g(s)| \psi_g(\|u\|_X) ds + \frac{\beta + \alpha t}{p} \int_0^T |\phi_h(s)| \psi_h(\|u\|_X) ds \\
\leqslant& k_{\|u\|_X, q} + \frac{1}{p} (\beta + \alpha T)(b^1_{\|u\|_X} + b^2_{\|u\|_X}) + \frac{a}{p} (\delta + 2\gamma T) \psi_g(\|u\|_X) \\
&+ \frac{b}{p} (\beta + \alpha T) \psi_h(\|u\|_X).
\end{aligned}$$

Also,

$$\begin{aligned}
|u'(t)| <& \left| \int_0^T \frac{\partial}{\partial t} G(t,s) f(s,u(s), {}^c D_{0+}^{\sigma} u(s)) ds + \varphi'(t) \right| \\
\leqslant& \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} (|\phi_f^1(s)| \psi_f^1(\|u\|_X) + |\phi_f^2(s)| \psi_f^2(\|u\|_X)) ds \\
&+ \frac{\alpha}{p} \int_0^T \left(\frac{\gamma(T-s)^{q-1}}{\Gamma(q)} + \frac{\delta(T-s)^{q-2}}{\Gamma(q-1)} \right) \\
&\times (|\phi_f^1(s)| \psi_f^1(\|u\|_X) + |\phi_f^2(s)| \psi_f^2(\|u\|_X)) ds \\
&+ \frac{\gamma}{p} \int_0^T |\phi_g(s)| \psi_g(\|u\|_X) ds + \frac{\alpha}{p} \int_0^T |\phi_h(s)| \psi_h(\|u\|_X) ds \\
\leqslant& \psi_f^1(\|u\|_X) \|I_{+0}^{q-1} \phi_f^1\|_{L_1} + \psi_f^2(\|u\|_X) \|I_{+0}^{q-1} \phi_f^2\|_{L_1} \\
&+ \frac{\alpha}{p} \left[\psi_f^1(\|u\|_X)(\gamma I_{+0}^q \phi_f^1(T) + \delta I_{+0}^q \phi_f^1(T)) \right. \\
&\left. + \psi_f^2(\|u\|_X)(\gamma I_{+0}^q \phi_f^2(T) + \delta I_{+0}^q \phi_f^2(T)) \right] + \frac{\gamma a \psi_g(\|u\|_X)}{p} + \frac{\alpha b \psi_h(\|u\|_X)}{p} \\
\leqslant& k_{\|u\|_X, q-1} + \frac{\alpha}{p} (b^1_{\|u\|_X} + b^2_{\|u\|_X}) + \frac{a\gamma \psi_g(\|u\|_X)}{p} + \frac{b\alpha \psi_h(\|u\|_X)}{p}.
\end{aligned}$$

Hence,

$$\begin{aligned}|{}^cD^{\sigma}_{0+}u(t)| &< \frac{1}{\Gamma(1-\sigma)}\int_0^t (t-s)^{-\sigma}|u'(s)|ds \\ &\leqslant \frac{T^{1-\sigma}}{\Gamma(2-\sigma)}\left(k_{\|u\|_X,q-1}+\frac{\alpha}{p}(b^1_{\|u\|_X}+b^2_{\|u\|_X})+\frac{a\gamma\psi_g(\eta)}{p}+\frac{b\alpha\psi_h(\eta)}{p}\right).\end{aligned}$$

Therefore, it follows that

$$\frac{\|u\|_X}{\frac{T^{1-\sigma}k_{\|u\|_X,q-1}}{\Gamma(2-\sigma)}+k_{\|u\|_X,q}+\left(1+\frac{T^{-\sigma}}{\Gamma(2-\sigma)}\right)M_{\|u\|_X}}<1,$$

a contradiction to $(H5)$. Hence, $u\neq\lambda A(u)$ for $u\in\partial U_1$, $\lambda\in[0,1]$. By Theorem 2.9, the BVP (1.1), (1.2) has at least one solution. □

The uniqueness result is based on the Banach contraction principal.

THEOREM 3.4. *Assume that:*

(H6) There exists a constant $k>0$ such that

$$|f(t,u,v)-f(t,\overline{u},\overline{v})),|\leqslant k(|u-\overline{u}|+|v-\overline{v}|),\ \textit{for each } t\in[0,T] \textit{ and all } u,\overline{u}\in\mathbb{R}.$$

(H7) There exists a constant $k_1>0$ such that

$$|g(t,u)-g(t,\overline{u})|\leqslant k_1|u-\overline{u}|,\ \textit{for each } t\in[0,T] \textit{and all } u,\overline{u}\in\mathbb{R}.$$

(H8) There exists a constant $k_2>0$ such that

$$|h(t,u)-h(t,\overline{u})|\leqslant k_2|u-\overline{u}|,\ \textit{for each } t\in[0,T] \textit{ and all } u,\overline{u}\in\mathbb{R}.$$

If $k<\frac{1}{3N}\left(1+\frac{T^{-\sigma}}{\Gamma(2-\sigma)}\right)^{-1}$, where $N=\left(\frac{T^q}{\Gamma(q)}+\frac{\beta+\alpha T}{p\Gamma(q)}(\gamma T^q+\delta T^{q-1})\right)$ and $k_1<\frac{p}{3T(\delta+2\gamma T)}$, $k_2<\frac{p}{3T(\beta+\alpha T)}$, then the boundary value problem (1.1)*,* (1.2) *has a unique solution.*

Proof. We shall use Banach fixed point theorem. For this we need to verify that A is contraction. Let $u,\overline{u}\in X$, then in view of $(H6)-(H8)$, for each $t\in[0,T]$, we have

$$\begin{aligned}|A(u)(t)-A(\overline{u})(t)| &\leqslant \int_0^1 G(t,s)|f(s,u(s),{}^cD^{\sigma}_{0+}u(s))-f(s,\overline{u},{}^cD^{\sigma}_{0+}\overline{u}(s))|ds \\ &\quad+\frac{\delta+\gamma(T-t)}{\alpha(\delta+\gamma T)+\beta\gamma}\int_0^T|g(s,u(s))-g(s,\overline{u}(s))|ds \\ &\quad+\frac{(\beta+\alpha t)}{\alpha(\delta+\gamma T)+\beta\gamma}\int_0^T|g(s,u(s))-g(s,\overline{u}(s))|ds \\ &\leqslant\|u-\overline{u}\|\left(k\int_0^1|G(t,s)|ds+\frac{T}{p}(k_1(\delta+\gamma(T-t))+k_2(\beta+\alpha t)\right) \\ &\leqslant\|u-\overline{u}\|\left(k\int_0^1|G(t,s)|ds+\frac{T}{p}(k_1(\delta+2\gamma T)+k_2(\beta+\alpha T))\right).\end{aligned}$$

From (3.4), we have

$$\int_0^1 |G(t,s)|ds \leqslant \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}ds + \int_0^T (\beta+\alpha t)\left(\frac{\gamma(T-s)^{q-1}}{p\Gamma(q)} + \frac{\delta(q-1)(T-s)^{q-2}}{p\Gamma(q)}\right)ds$$
$$\leqslant \frac{t^q}{q\Gamma(q)} + \frac{\beta+\alpha t}{p\Gamma(q)}(\gamma T^q + \delta T^{q-1}) \leqslant \frac{T^q}{\Gamma(q)} + \frac{\beta+\alpha t}{p\Gamma(q)}(\gamma T^q + \delta T^{q-1}) = N,$$

which implies that

$$|A(u)(t) - A(\overline{u})(t)| \leqslant \|u-\overline{u}\| \left[kN + \frac{T}{p}(k_1(\delta+2\gamma T) + k_2(\beta+\alpha T))\right].$$

Hence,

$$|{}^cD^{\sigma}_{0+}(Au)(t) - {}^cD^{\sigma}_{0+}(A\overline{u})(t)|$$
$$= \left|\frac{1}{\Gamma(1-\sigma)}\int_0^t (t-s)^{-\sigma}((Au)'(s) - (A\overline{u})'(s))ds\right|$$
$$\leqslant \frac{1}{\Gamma(1-\sigma)}\int_0^t (t-s)^{-\sigma}\left(\int_0^1 \left|\frac{\partial}{\partial s}G(s,z)\right| |f(z,u(z),{}^cD^{\sigma}_{0+}u(z) - f(z,\overline{u}(z),{}^cD^{\sigma}_{0+}\overline{u}(z)|dz\right)ds$$
$$\leqslant \frac{k\|u-\overline{u}\|}{\Gamma(1-\sigma)}\int_0^t (t-s)^{-\sigma}\left(\int_0^1 \left|\frac{\partial}{\partial s}G(s,z)\right|dz\right)ds.$$

Again, from (3.4), we have

$$\int_0^1 |\frac{\partial}{\partial t}G(t,s)|ds \leqslant \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)}ds + \frac{\alpha}{p\Gamma(q)}\int_0^T (\gamma(T-s)^{q-1} + \delta(q-1)(T-s)^{q-2})ds$$
$$\leqslant \frac{t^{q-1}}{q\Gamma(q)} + \frac{\gamma\alpha T^q}{pq\Gamma(q)} + \frac{\delta\alpha T^{q-1}}{p\Gamma(q)} \leqslant \frac{T^{q-1}}{q\Gamma(q)} + \frac{\alpha}{p\Gamma(q)}(\gamma T^q + \delta T^{q-1}).$$

Thus, we have

$$|{}^cD^{\sigma}_{0+}(Au)(t) - {}^cD^{\sigma}_{0+}(A\overline{u})(t)| \leqslant \frac{kT^{-\sigma}}{\Gamma(2-\sigma)}\left(\frac{T^q}{\Gamma(q)} + \frac{\beta+\alpha T}{p\Gamma(q)}(\gamma T^q + \delta T^{q-1})\right).$$

Therefore, $\|Au - A\overline{u}\| \leqslant L\|u-\overline{u}\|$ where $L = k(1+\frac{T^{-\sigma}}{\Gamma(2-\sigma)})N + \frac{T}{p}(k_1(\delta+2\gamma T) + k_2(\beta+\alpha T)) < 1$. By the contraction mapping principle, the BVP (1.1), (1.2) has a unique solution. □

EXAMPLE 3.5. Consider the following fractional BVP,

$${}^cD^{\frac{3}{2}}_{0+}u(t) = \frac{e^{-ct}(|u| + |{}^cD^{\frac{1}{2}}_{0+}u(t)|)}{(124\sqrt{\pi} + e^{-ct})(1+|u|)},\ t \in [0,1] \tag{3.10}$$

$$u(0) - u'(0) = \int_0^1 \frac{|u(s)|e^{-s}}{9(1+|u(s)|)}ds, \tag{3.11}$$

$$u(1)+u'(1)=\int_0^1 \frac{|u(s)|\sin s}{36}ds. \tag{3.12}$$

Set $f(t,u,v)=\frac{e^{-ct}(u(t)+v(t))}{(24\sqrt{\pi}+e^{-ct})(1+u(t)+v(t))}$, $g(t,u)=\frac{u(t)e^{-t}}{9(1+u(t))}$ and $h(t,u)=\frac{u(t)\sin t}{36}$ for $t\in[0,1]$ and $u,v\in[0,\infty)$.

Let $t\in[0,1]$ and $u,\overline{u},v,\overline{v}\in[0,\infty)$, then we have

$$\begin{aligned}|f(t,u,v)-f(t,\overline{u},\overline{v})| &= \frac{e^{-ct}}{(124\sqrt{\pi}+e^{-ct})}\left|\frac{u(t)+v(t)}{1+u(t)+v(t)}-\frac{\overline{u}(t)+\overline{v}(t)}{1+\overline{u}(t)+\overline{v}(t)}\right| \\ &\leqslant \frac{e^{-ct}(|u(t)-\overline{u}(t)|+|v(t)-\overline{v}(t)|)}{(124\sqrt{\pi}+e^{-ct})(1+u(t)+v(t))(1+\overline{u}(t)+\overline{v}(t))} \\ &\leqslant \frac{e^{-ct}(|u(t)-\overline{u}(t)|+|v(t)-\overline{v}(t)|)}{124\sqrt{\pi}+e^{-ct}} \\ &\leqslant \frac{1}{124\sqrt{\pi}}(|u(t)-\overline{u}(t)|+|v(t)-\overline{v}(t)|).\end{aligned}$$

By simple calculations we have, $p=3$, $N=\frac{14}{3\sqrt{\pi}}$, $\frac{1}{3N}\left(1+\frac{T^{-\sigma}}{\Gamma(2-\sigma)}\right)^{-1}=\frac{\pi}{14(\sqrt{\pi}+2)}$.

Here $k=\frac{1}{124\sqrt{\pi}}<\frac{\pi}{14(\sqrt{\pi}+2)}$. Hence condition $(H6)$ is satisfied.
Now, for u, $\overline{u}\in[0,\infty)$ we have

$$\begin{aligned}|g(t,u)-g(t,\overline{u})| &= \frac{e^{-t}}{9}\left|\frac{u(t)}{1+u(t)}-\frac{\overline{u}(t)}{1+\overline{u}(t)}\right| \\ &\leqslant \frac{e^{-t}|u(t)-\overline{u}(t)|}{9(1+u(t))(1+\overline{u}(t))}\leqslant \frac{1}{9}|u(t)-\overline{u}(t)|,\end{aligned}$$

where $k_1=\frac{1}{9}<\frac{p}{3T(\delta+2\gamma T)}=\frac{1}{3}$. Hence $(H7)$ is satisfied. Also,

$$|h(t,u)-h(t,\overline{u})|=\frac{\sin t}{36}|u(t)-\overline{u}(t)|\leqslant\frac{1}{36}|u(t)-\overline{u}(t)|,$$

where $k_2=\frac{1}{36}<\frac{p}{3T(\beta+\alpha T)}=\frac{1}{3}$. All conditions of the Theorem 3.4 are satisfied. Therefore the fractional BVP (3.10)–(3.12) has a unique solution.

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