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# PATH PROBABILITY OF RANDOM FRACTIONAL SYSTEMS DEFINED BY WHITE NOISES IN COARSE–GRAINED TIME. APPLICATION OF FRACTIONAL ENTROPY

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*To the memory of*
*Professor Mandelbrot*

*Abstract.* One considers a class of fractional random processes defined as non-random dynamics subject to Gaussian white noises in coarse-grained time, according to Maruyama's notation. After some prerequisites on modified Riemann-Liouville fractional derivative, fractional Taylor's series and integration with respect to $(dx)^{\alpha}$, one displays the main results which are as follows: firstly, a general scheme to obtain the path probability density (in Feynman's sense) of some fractional stochastic dynamics; secondly an approximation, via Itô's lemma, for their characteristic functions, therefore approximate expressions for their path probability density; and thirdly, an approach via the maximum entropy principle (MEP) which holds when the dynamical equations of the state moments are available. One first uses the MEP combined with Shannon entropy, and then one applies the MEP with a new concept of fractional entropy which takes account of defects in observation. As a last application, one uses an optimization of distributed entropy based on fractional Fokker-Planck equation. All the paper is based on the modified Riemann-Liouville derivative and the generalization of the Maruyama notation for Brownian motion, and the mathematics so involved is customarily referred to as physical mathematics or engineering mathematics.

## 1. Introduction

*On fractional Brownian motion*

Loosely speaking, a fractional Brownian motion $b(t,\alpha)$, $t \in \Re$, is a self-similar stochastic process which satisfies the conditions

$$E\left(db(t,\alpha)\right) = 0, \tag{1.1}$$

and

$$\operatorname{cov}\left(b(t_1,\alpha), b(t_2,\alpha)\right) = 2^{-1}\left(|t_1|^{2\alpha} + |t_2|^{2\alpha} - |t_1 - t_2|^{2\alpha}\right) \operatorname{Var} b(1,\alpha), \tag{1.2}$$

where $E(b)$, which is written also $\langle b \rangle$ in the physical literature, denotes the mathematical expectation. It appears that this kind of noise is more and more relevant in

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the modeling and the analysis of many practical phenomena (physics in coarse-grained space, randomness in mathematical finance, for instance), and in quite a natural way one has been led to generalize the Itô's stochastic differential equation

$$dx = f(x,t)dt + g(x,t)db(t) \tag{1.3}$$

in the form

$$dx = f(x,t)dt + g(x,t)db(t,\alpha). \tag{1.4}$$

As $b(t,\alpha)$ is highly discontinuous, it is very hard to soundly define the solution of this equation, it is likely that several families of solutions are possible, and consequently approximate results will be welcome on a practical standpoint.

Assume that we are not interested in the trajectory generated by $x(t)$, but instead we are concerned with the probability density $p(x,t)$ of $x$ at time $t$. To get its value, we can try to solve the corresponding Fokker-Planck equation, but the latter is a fractional partial differential equation which can be solved in some special cases only. Otherwise, numerical approximations are necessary.

*White noise in coarse-grained time versus fractional Brownian motion*

In order to try to circumvent, or likely to avoid the framework of fractional stochastic differential equations which is highly mathematical, we rather propose to use dynamical systems involving coarse-grained time as follows.

(i) Consider the (non-random) system

$$dx = f(x,t)dt, \tag{1.5}$$

and assume that time is coarse-grained in such a manner that the differential increment of time is not $dt$, but rather $(dt)^\alpha$, $0 < \alpha < 1$, $(dt)^\alpha > dt$. Then the above equation turns to be

$$dx = f(x,t)(dt)^\alpha, \tag{1.6}$$

and in fractional calculus, one can show that this equation can be re-written in the form

$$x^{(\alpha)}(t) = K(\alpha)f(x,t), \tag{1.7}$$

where $x^{(\alpha)}(t)$ is the $\alpha th$-fractional derivative of $x(t)$ and $K(\alpha)$ denotes a constant which depends upon $\alpha$.

(ii) Assume now that $f(x,t)$ is a Gaussian white noise $w(t)$, then we obtain the equation

$$dx = w(t)(dt)^\alpha, \tag{1.8}$$

which can be considered as defining a Brownian motion running with coarse-grained time (It is not a fractional Brownian motion in the copyrighted sense of this term). Equation (1.8) is a direct extension of the Maruyama's notation $w(t)\sqrt{dt}$ for (standard) Brownian motion [32].

In this way of thought, we are then led to consider fractional stochastic processes as picturing the dynamics of non-random fractional dynamical systems driven by Gaussian white noise. By combining this point of view with the fractional Taylor's series derived from the so-called modified Riemann-Liouville fractional derivative (as we named it), we have been able to get some general results. Here, we shall use this framework to obtain some approximations for the probability density $p(x,t)$ of the process.

The paper is organized as follows. For the convenience of the reader, we first gives a short background on modified Riemann-Liouville fractional derivative (Section 2), fractional Taylor's series of non-differentiable functions (Section 3) and integration with respect to $(dx)^{\alpha}$ (Section 4). Then, with "pedagogical purpose" in mind, we show how one can simply obtain approximation for the path probability density and the characteristic function of some (standard) Markovian processes (Section 5). Then, after recalling our representation of fractional stochastic processes (Section 6) we shall show how one can obtain the path probability and the characteristic functions of these processes (Sections 7 and 8). Lastly, we shall show how one can use Jaynes' maximum entropy principle to tackle this problem. When the dynamical equations of the state moments are available, the problem is rather simple (Section 9), but in the more general case, we shall use an optimization procedure involving the fractional partial differential equation which defines the probability density of the system (Section 10). We take this opportunity to show why informational entropy of fractional order, seems to be quite meaningful in this problem (Section 9).

*All this paper is based on the modified Riemann-Liouville derivative and the generalization of a noise modeling firstly introduced by Maruyama to represent Brownian motion. The mathematics so involved are essentially applied mathematics or engineering mathematics.*

*Warning to the reader.* At date, fractional stochastic differential equation is a highly controversial topic, and to the best of our knowledge, we have not yet sound theory. Nevertheless in many practical problems like mathematical finance for instance, we are facing this topic. The practitioner needs at least approximate technique to handle this kind of equations, and the present paper is in this wake of thought. It is parallel to the literature on fractional stochastic differential equations, and this is the reason why it refers to Brownian motion in coarse-grained time instead of fractional Brownian motion (which is copyrighted)

In the following, we shall denote the mathematical expectation of the random variable $X$ by $\langle X\rangle$ (customary notation in physics) instead of $E\{X\}$ as usual in mathematics, to avoid confusing with the Mittag-Leffler function $E_{\alpha}(x)$.

## 2. Background on Fractional Derivative (Revisited)

### 2.1. Fractional derivative via fractional difference

DEFINITION 2.1. Let $f:\Re\rightarrow\Re$, $x\rightarrow f(x)$, denote a continuous (but not necessarily differentiable) function, and let $h>0$ denote a constant discretizing span. Define the forward operator $FW(h)$ by the equality (the symbol := means that the left side is defined by the right side)

$$FW(h)f(x):=f(x+h); \tag{2.1}$$

then the fractional difference of order $\alpha$, $0<\alpha<1$, of $f(x)$ is defined by the expression [11, 14–19].

$$\begin{aligned}\Delta^{\alpha} f(x) &:= (FW-1)^{\alpha} f(x) \\ &= \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f[x+(\alpha-k)h], \end{aligned} \tag{2.2}$$

and its fractional derivative of order $\alpha$ is defined by the limit

$$f^{(\alpha)}(x) = \lim_{h \downarrow 0} \frac{\Delta^{\alpha} f(x)}{h^{\alpha}}. \tag{2.3}$$

This definition (which is slightly different from some other ones in the literature) is close to the standard definition of derivative (calculus for beginners), and as a direct result, the $\alpha$-th derivative of a constant is zero.

In the following we shall display the counterpart of this definition in terms of integral.

## 2.2. Modified fractional Riemann-Liouville derivative (via integral)

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*An alternative to the Riemann-Liouville definition of fractional derivative*

In order to circumvent some defects involved in the classical Riemann-Liouville definition, we use the following alternative referred to as modified Riemann-Liouville derivative [16].

PROPOSITION 2.1. (Riemann-Liouville definition revisited [17]) *Refer to the function $f(x)$ above:*

*(i) Assume that $f(x)$ is a constant $K$. Then its fractional derivative of order $\alpha$ is*

$$D_x^{\alpha} K = \frac{K}{\Gamma(1-\alpha)} x^{-\alpha}, \quad \alpha \leqslant 0, \tag{2.4}$$

$$= 0, \quad \alpha > 0. \tag{2.5}$$

*(ii) When $f(x)$ is not a constant, then one will set*

$$f(x) = f(0) + (f(x) - f(0)),$$

*and its fractional derivative will be defined by the expression*

$$f^{(\alpha)}(x) = D_x^{\alpha} f(0) + D_x^{\alpha} (f(x) - f(0))$$

*in which, for negative $\alpha$, one has*

$$D_x^{\alpha} (f(x) - f(0)) := \frac{1}{\Gamma(-\alpha)} \int_0^x (x-\xi)^{-\alpha-1} f(\xi)\, d\xi, \quad \alpha < 0. \tag{2.6}$$

*whilst for positive* $\alpha$, $0 < \alpha < 1$, *one will set*

$$D_x^\alpha (f(x) - f(0)) = D_x^\alpha f(x) = \left(f^{(\alpha-1)}(x)\right)' \\ = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha} (f(\xi) - f(0)) \, d\xi. \qquad (2.7)$$

*When* $n \leqslant \alpha < n+1$, *one will set*

$$f^{(\alpha)}(x) := \left(f^{(\alpha-n)}(x)\right)^{(n)}, \quad n \leqslant \alpha < n+1, \quad n \geqslant 1. \qquad (2.8)$$

*Proof.* A way to obtain this result is to show that the Laplace's transforms of (2.3) and (2.6) respectively are the same, and to this end we shall simplify the writing by setting $\tilde{f}(x) := f(x) - f(0)$. We can then write the Laplace's transform of (2.2) as

$$L\{\Delta^\alpha \tilde{f}(x)\} = \sum_{k=1}^{\infty} (-1)^k \binom{\alpha}{k} e^{-(k-\alpha)hs} L\{\tilde{f}(x)\} - e^{\alpha h} \int_0^{\alpha h} e^{-sx} \tilde{f}(x) dx \\ = \left(1 - e^{-hs}\right)^\alpha L\{\tilde{f}(x)\} - e^{\alpha h} \int_0^{\alpha h} e^{-sx} \tilde{f}(x) dx \\ = h^\alpha s^\alpha \left(L\{f(x)\} - s^{-1} f(0)\right) - \alpha h e^{\alpha h} e^{sx(\theta \alpha h)} \tilde{f}(\theta \alpha h)$$

therefore, on letting $h \downarrow 0$, we obtain the transform of (2.3) which is

$$L\left\{f^{(\alpha)}(x)\right\} = s^\alpha L\{f(x)\} - s^{\alpha-1} f(0). \qquad (2.9)$$

This being the case, taking the Laplace's transform of the convolution in (2.5) direct yields

$$L\left\{f^{(\alpha)}(x)\right\} = \frac{s}{\Gamma(1-\alpha)} \left(\frac{\Gamma(1-\alpha)}{s^{1-\alpha}} L\{f(x)\}\right) - \frac{s}{\Gamma(1-\alpha)} \left(\frac{\Gamma(1-\alpha)}{s^{1-\alpha}} \frac{1}{s}\right) f(0)$$

that is to say exactly (2.7).

We shall refer to this fractional derivative as to the *modified Riemann Liouville derivative*, and it is of order to point out that it is strictly equivalent to the definition 2.1, via the equation (2.2).

*Further remark*

The fact that we drop the initial value of $f(x)$ in (2.7) amounts to consider only those functions satisfying the condition $f(0) = 0$, and to some extent, this is not too much surprising at all, as far as fractional derivative and self-similarity exhibit some relations, and that a self-similar function takes the value zero at the origin.

The following complements will be useful for our purpose.

### 2.3. On sequences of cascaded fractional derivatives

*On the order of cascaded derivatives*

Assume that we want to calculate $D^{\alpha+\theta}f(x)$, $0<\alpha,\theta<1$, by applying $D^{\alpha}$ and $D^{\theta}$ in any order. At first glance, one could use either $D^{\alpha}D^{\theta}f(x)$ or $D^{\theta}D^{\alpha}f(x)$, but the results so obtained are sensibly different, since then in terms of Laplace's transform (see Equ. (2.9)) one has

$$L\left\{D^{\theta}D^{\alpha}f(x)\right\}=s^{\alpha+\theta}F(s)-s^{\alpha+\theta-1}f(0)-s^{\theta-1}f^{(\alpha)}(0), \tag{2.10}$$

and

$$L\left\{D^{\alpha}D^{\theta}f(x)\right\}=s^{\alpha+\theta}F(s)-s^{\alpha+\theta-1}f(0)-s^{\alpha-1}f^{(\theta)}(0). \tag{2.11}$$

The same problem occurs when $\theta$, for instance, is a positive integer $n$, and here again one has $D^{n}D^{\alpha}f(x)\neq D^{\alpha}D^{n}f(x)$. For instance, when $f(x)=x^2$, $n=3$ and $\alpha=0.5$; one obtains

$$D^{0.5}D^{3}(x^2)=0$$

and

$$D^{3}D^{0.5}(x^2)=KD^{3}(x^{1.5})=-1.5(0.5)^2Kx^{-1.5},$$

with $K$ denoting a constant.

Once more, we are facing the same problem when we try to define $D^{\alpha}f(x)$ with $n<\alpha<n+1$, in which case we have to set either $D^{\alpha}:=D^{n}D^{\alpha-n}$ or $D^{\alpha}:=D^{\alpha-n}D^{n}$.

As a result, we have to select a model, and we suggest the following

DEFINITION 2.2. *Principle of derivative increasing orders.* The fractional derivative of fractional order $D^{\alpha+\theta}$ expressed in terms of $D^{\alpha}$ and $D^{\theta}$ is defined by the equality

$$D^{\alpha+\theta}f(x):=D^{\max(\alpha,\theta)}\left(D^{\min(\alpha,\theta)}f(x)\right). \tag{2.12}$$

On doing so, we merely follows the practical rule in accordance of which we increase the derivation order rather than the opposite. Or again, we start from low order derivative to define large order derivative.

*On the decomposition of fractional derivatives*

Let be $\alpha$ positive, and assume that $0<3\alpha<1$. There are two different manners to obtain $D^{3\alpha}f(x)$. One can calculate $D^{\alpha}D^{\alpha}D^{\alpha}f(x)$ to obtain the Laplace's transform

$$L\{D^{\alpha}D^{\alpha}D^{\alpha}f(x)\}=s^{3\alpha}F(s)-s^{3\alpha-1}f(0)-s^{2\alpha-1}f^{(\alpha)}(0)-s^{\alpha-1}f^{(2\alpha)}(0).$$

or else calculate $D^{3\alpha}f(x)$ to obtain

$$L\left\{D^{3\alpha}f(x)\right\}=s^{3\alpha}F(s)-s^{3\alpha-1}f(0),$$

in such a manner that one will have

$$D^{\alpha}D^{\alpha}D^{\alpha}f(x)\neq D^{3\alpha}f(x),\quad 0<3\alpha<1.$$

For instance $f(x) = x^{2\alpha}$ yields

$$D^{\alpha}D^{\alpha}D^{\alpha}(x^{2\alpha}) = 0$$

and

$$D^{3\alpha}(x^{2\alpha}) = \frac{\Gamma(1+\alpha)}{\Gamma(1-2\alpha)}x^{-2\alpha}.$$

This pitfall can be easily circumvented if we carefully define the framework. When the problem which we are dealing with involves $D^{\alpha}$ as the basic derivative, then we shall necessarily refer to $D^{\alpha}D^{\alpha}D^{\alpha}$. Otherwise, if the smaller derivative so involved in the problem is $D^{3\alpha}$, then we shall use the modified Riemann-Liouville expression for the later.

For further reading on fractional calculus, regarding history and complements, see for instance [3, 4, 21, 24, 26, 27, 30, 31, 33, 35–41].

## 3. Background on Taylor's Series of Fractional Order

### 3.1. Main definition

A generalized Taylor expansion of fractional order which applies to non-differentiable functions (F-Taylor series in the following) reads as follows [11, 14–19].

PROPOSITION 3.1. *Assume that the continuous function $f : \Re \rightarrow \Re,\ x \rightarrow f(x)$ has fractional derivative of order $k\alpha$, for any positive integer $k$ and any $\alpha, 0 < \alpha < 1$, then the following equality holds, which is*

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{\alpha k}}{\Gamma(1+\alpha k)} f^{(\alpha k)}(x), \quad 0 < \alpha \leqslant 1. \tag{3.1}$$

*where $f^{(\alpha k)}(x)$ is the modified Riemann-Liouville derivative of order $\alpha k$, i.e. $D^{\alpha}D^{\alpha}...D^{\alpha}$, $k$ times, of $f(x)$.*

*With the notation*

$$\Gamma(1+\alpha k) =: (\alpha k)!,$$

*one has the formula*

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{\alpha k}}{(\alpha k)!} f^{(\alpha k)}(x), \quad 0 < \alpha \leqslant 1$$

*which looks like the classical one.*

*Alternatively, in a more compact form, one can write*

$$f(x+h) = E_{\alpha}(h^{\alpha}D_x^{\alpha})f(x), \tag{3.2}$$

*where $D_x$ is the derivative operator with respect to $x$ and $E_{\alpha}(y)$ denotes the Mittag-Leffler function defined as*

$$E_{\alpha}(y) := \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(1+\alpha k)}.$$

$$E_\alpha(u) = \sum_{k=0}^{\infty} \frac{u^k}{(\alpha k)!}. \tag{3.3}$$

*Proof. (Step 1) Lemma 3.1* One first notices that the solution of the fractional differential equation

$$y^{(\alpha)}(x) = \lambda y(x), \quad 0 < \alpha < 1, \quad y(0) = y_0, \tag{3.4}$$

where $\lambda$ is a real valued parameter is

$$y(x) = y_0 E_\alpha(\lambda x^\alpha). \tag{3.5}$$

*Proof.* Indeed, let us seek a solution in the form

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k\alpha}.$$

We substitute this series into (3.4) and we use the relations

$$D^\alpha x^{\alpha k} = \frac{(\alpha k)!}{(\alpha k - \alpha)!} x^{\alpha(k-1)} \tag{3.6}$$

together with

$$D^\alpha x^0 = 0, \tag{3.7}$$

to obtain the recursive equation

$$a_{k+1} = \frac{(\alpha k)!}{(\alpha(k+1))!} a_k,$$

therefore the result.

*(Step 2)* This being the case, the purpose of the proof of (3.1) is to show that one has the formal operational equation

$$D_h^\alpha F_w(h) = F_w(h) D_x^\alpha, \tag{3.8}$$

of which the solution is

$$F_w(h) = E_\alpha(h^\alpha D_x^\alpha). \tag{3.9}$$

To this end, it is sufficient to write successively

$$\begin{aligned} \Delta_h^\alpha F_w(h) &= \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x + h + (\alpha - k)H) \\ &= F_w(h) \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x + (\alpha - k)H). \end{aligned}$$

and on dividing both sides by $H^\alpha$ and taking the limit as $h$ tends to zero yields the result.

*Further remarks.* (i) The fractional Taylor's series provides the approximation

$$\frac{f(x+h)-f(x)}{h} \cong \frac{f^{(\alpha)}(x)}{h^{1-\alpha}\alpha!}, \quad 0<\alpha\leqslant 1,$$

and as a result, when $f^{(\alpha)}(x)$ exists and is bounded with $\alpha<1$, then $f(x)$ is not differentiable

(ii) An argument which could help the reader to take this fractional Taylor's series for granted is as follows. It is easy to check this series applies to the Mittag-Leffler function. Given this remark, it is sufficient to consider functions which can be approximated by linear combinations of Mittag-Leffler functions.

COROLLARY 3.1. *Assume that* $m<\alpha\leqslant m+1$, $m\in N-\{0\}$ *and that* $f(x)$ *has derivatives of order* $k$ *(integer),* $1\leqslant k\leqslant m$. *Assume further that* $f^{(m)}(x)$ *has a fractional Taylor's series of order* $\alpha-m=:\beta$ *provided by the expression*

$$f^{(m)}(x+h)=\sum_{k=0}^{\infty}\frac{h^{k(\alpha-m)}}{\Gamma[1+k(\alpha-m)]}\,D^{k(\alpha-m)}f^{(m)}(x), \quad m<\alpha\leqslant m+1. \tag{3.10}$$

*Then, integrating this series with respect to* $h$ *yields*

$$f(x+h)=\sum_{k=o}^{m}\frac{h^k}{k!}f^{(k)}(x)+\sum_{k=1}^{\infty}\frac{h^{(k\beta+m)}}{\Gamma(k\beta+m+1)}\,f^{(k\beta+m)}(x), \quad \beta:=\alpha-m. \tag{3.11}$$

*In the special case when* $m=1$, *one has*

$$f(x+h)=f(x)+hf'(x)+\sum_{k=1}^{\infty}\frac{h^{k\beta+1}}{\Gamma(k\beta+2)}f^{(k\beta+1)}(x), \quad \beta:=\alpha-1. \tag{3.12}$$

The order of the derivation in $f^{(k\beta+m)}(x)$ is of paramount importance and should be understood as $D^{k\beta}f^{(m)}(x)$, since we start with the fractional Taylor's series of $f^{(m)}(x)$.

*Mc-Laurin series of fractional order*

Let us make the substitution $h\leftarrow x$ and $x\leftarrow 0$ into (3.1), we so obtain the fractional Mc-Laurin series

$$f(x)=\sum_{k=0}^{\infty}\frac{x^{\alpha k}}{\Gamma(1+\alpha k)}f^{(\alpha k)}(0), \quad 0<\alpha\leqslant 1. \tag{3.13}$$

EXAMPLE 3.1. Let us consider the function

$$f(x)=x^{\alpha N}, \qquad 0<\alpha N<1$$

with $N$ denoting a positive integer. A simple calculation yields

$$f^{(\alpha k)}(x)=\frac{(\alpha N)!}{(\alpha N-\alpha k)!}x^{\alpha(N-k)}, \quad 0\leqslant k\leqslant N$$

$$f^{(\alpha k)}(x) = 0, \quad k \geqslant N+1$$

therefore the equality

$$(x+h)^{\alpha N} = \sum_{k=0}^{N} \frac{(\alpha N)!}{(\alpha k)!(\alpha N - \alpha k)!} \, x^{\alpha(N-k)} h^{\alpha k}.$$

### 3.2. Fractional Taylor's series for multivariable functions

On generalizing (3.2), we shall write the fractional Taylor's series for a two-variable function in the form

$$\begin{aligned} f(x+h, y+l) &= E_\alpha\left((hD_x + lD_y)^\alpha\right) f(x,y) \\ &\cong E_\alpha(h^\alpha D_x^\alpha) E_\alpha(l^\alpha D_y^\alpha) f(x,y), \quad 0 < \alpha < 1. \end{aligned} \tag{3.14}$$

therefore the differential

$$df(x,y) = \Gamma^{-1}(1+\alpha)\left(f_x^{(\alpha)}(x,y)(dx)^\alpha + f_y^{(\alpha)}(x,y)(dy)^\alpha\right), \quad 0 < \alpha < 1. \tag{3.15}$$

For larger values of $\alpha$, one will have

$$\begin{aligned} df = f_x dx + f_y dy + \Gamma^{-1}(1+\alpha)\left(f_x^{(\alpha)}(dx)^\alpha + f_y^{(\alpha)}(dy)^\alpha\right) \\ + \Gamma^{-2}(1+\alpha) f_{xy}^{(2\alpha)}, \quad 1 < \alpha < 2\alpha \end{aligned} \tag{3.16}$$

### 3.3. Some useful relations

First of all, the equation (3.1) provides the useful differential relation

$$d^\alpha f \cong \Gamma(1+\alpha)\, df, \quad 0 < \alpha < 1, \tag{3.17}$$

or in terms of fractional difference, $\Delta^\alpha f \cong \alpha! \Delta f$, which, in accordance of the second remark above, applies to non-differentiable functions only. This being the case, one has the following

COROLLARY 3.2. *The following equalities hold, which are*

$$D^\alpha x^\gamma = \Gamma(\gamma+1)\Gamma^{-1}(\gamma+1-\alpha) x^{\gamma-\alpha}, \quad \gamma > 0, \tag{3.18}$$

*or, what amounts to the same (we set* $\alpha = n + \theta$*)*

$$D^{n+\theta} x^\gamma = \Gamma(\gamma+1)\Gamma^{-1}(\gamma+1-n-\theta) x^{\gamma-n-\theta}, \quad 0 < \theta < 1,$$

$$(u(x)v(x))^{(\alpha)} = u^{(\alpha)}(x)v(x) + u(x)v^{(\alpha)}(x), \tag{3.19}$$

$$(f[u(x)])^{(\alpha)} = f'_u(u) u^{(\alpha)}(x), \tag{3.20}$$

$$= f_u^{(\alpha)}(u)(u'_x)^\alpha. \tag{3.21}$$

*$u(x)$ is non-differentiable in (3.19) and (3.20) and differentiable in (3.21), $v(x)$ is non-differentiable, and $f(u)$ is differentiable in (3.20) and non-differentiable in (3.21).*

The proof of (3.19) is based on the equality

$$d(uv) = v(du) + u(dv)$$

which yields

$$\alpha!d(uv) = v(\alpha!du) + u(\alpha!dv)$$

that is to say

$$d^\alpha(uv) = v(d^\alpha u) + u(d^\alpha v)$$

These last two formulae (3.20) and (3.21) can be obtained by using the chain rules

$$\frac{d^\alpha f(u)}{dx^\alpha} = \frac{d^\alpha f}{du}\frac{du}{dx^\alpha} = \frac{\alpha!df}{du}\frac{du}{dx^\alpha} = \frac{df}{du}\frac{\alpha!du}{dx^\alpha} = \frac{df}{du}\frac{d^\alpha u}{dx^\alpha}$$

and

$$\frac{d^\alpha f(u)}{dx^\alpha} = \frac{\alpha!df}{dx^\alpha} = \frac{\alpha!df}{du^\alpha}\left(\frac{du}{dx}\right)^\alpha = \frac{d^\alpha f}{du^\alpha}\left(\frac{du}{dx}\right)^\alpha.$$

COROLLARY 3.3. *Assume that $f(x)$ and $x(t)$ are two $\Re \to \Re$ functions which both have derivatives of order $\alpha$, $0 < \alpha < 1$, then one has the chain rule*

$$f_t^{(\alpha)}(x(t)) = \Gamma(2-\alpha)x^{\alpha-1}f_x^{(\alpha)}(x)x^{(\alpha)}(t). \tag{3.22}$$

*Proof.* The $\alpha$-th derivative of $x$ provides the equality

$$d^\alpha x = \frac{1}{(1-\alpha)!}x^{1-\alpha}(dx)^\alpha. \tag{3.23}$$

which allows us to write

$$d^\alpha f = f_x^{(\alpha)}(dx)^\alpha = f_x^{(\alpha)}(x)(1-\alpha)!x^{\alpha-1}d^\alpha x$$

whereby the result.

### 3.4. Further results and remarks

*On the suitable fractional derivative definition to be selected*

(i) With the modified Riemann-Liouville derivative, the solution of the equation (and this is why we introduced this modified Riemann-Liouville fractional derivative!)

$$D_t^\alpha x(t) = x(t), \quad x(0) = x_0, \quad x(0) = x_0. \tag{3.24}$$

is exactly the Mittag-Leffler function, and this can be obtained easily on looking for a solution in the form

$$x(t) = \sum_{k=0}^{\infty} x_k (t^\alpha)^k. \tag{3.25}$$

An alternative is to take the Laplace transform of the equation (3.17) to obtain, with the modified Riemann-Liouville derivative,

$$s^{\alpha}X(s)-s^{\alpha-1}x(0)=-\lambda X(s),$$

where $X(s)$ , which is the Laplace transform of $x(t)$, yields the Mittag-Leffler function.

*On the differentiability of f(x)*

(ii) As it is obvious, the series (3.1) applies to nondifferentiable functions, whilst (3.12) refers to differentiable functions.

(iii) Assume that $\alpha = 1/N$, $N$ integer, in the F-Taylor series (3.1); then when $k = N$ we come across the first derivative. Nevertheless this does not mean that there is some inconsistency somewhere, but rather it is the meaning of these equations which must be clarified. Indeed, because of the presence of $h^{\alpha}$, $h$ is restricted to be positive, $h > 0$; and as a result, *all the derivatives involved in the F-Taylor series (3.1), either they are fractional or not, are derivatives on the right.*

*Modeling irreversibility of time*

Assume that $f(.)$ is a function $f(t)$ of time; then according to the above comments, the F-Taylor series of $f(t+\Delta t)$ holds for positive $\Delta t$ only. This property can be thought of as a practical describing of the irreversibility of time.

*Relation with previous results in the literature*

(iv) Osler [37] has previously proposed a generalization of Taylor's series in the complex plane, in the form

$$f(z)=\alpha\sum_{k=-\infty}^{k=+\infty}\frac{f^{(\alpha k)}(z_0)}{\Gamma(1+\alpha k)}(z-z_0)^{\alpha k}, \tag{3.26}$$

which provides the fractional Mc Laurin's series

$$f(x)=\alpha\sum_{k=-\infty}^{k=+\infty}\frac{f^{(\alpha k)}(0)}{\Gamma(1+\alpha k)}x^{\alpha k}. \tag{3.27}$$

In order to enlighten the discrepancy between this series and our's, we proceed as follows. On taking the expression (3.27) in terms of modified Riemann-Liouville derivative on the one hand, and identifying $f(x)$ with $E_{\alpha}(x^{\alpha})$ on the other hand, we would obtain the equality

$$E_{\alpha}(x^{\alpha})=\alpha E_{\alpha}(x^{\alpha})+\alpha\sum_{k=1}^{\infty}\frac{x^{-\alpha k}}{\Gamma(1-\alpha k)},$$

therefore

$$E_{\alpha}(x^{\alpha})=\frac{\alpha}{1-\alpha}\sum_{k=1}^{\infty}\frac{x^{-\alpha k}}{\Gamma(1-\alpha k)},$$

in such a manner that here we would have $E_{\alpha}(0)=\infty$.

We think that the basic reason for this difference is due to the fact that the definition of fractional derivative is not quite the same in the two series.

(vi) More recently Kolwankar and Gangal [24,25] proved the so-called "local fractional Taylor expansion" (or fractional Rolle's formula)

$$f(x+h)=\sum_{k=0}^{m}\frac{h^k}{k!}f^{(k)}(x)+\frac{f^{(\alpha)}(x)}{\Gamma(1+\alpha)}h^{\alpha}+R_{\alpha}(h),\quad m<\alpha<m+1, \tag{3.28}$$

where $R_{\alpha}(h)$ is a remainder, which is negligible when compared with the other terms. This is exactly our series (3.4), but here we give an explicit expression for $R_{\alpha}(h)$, namely

$$R_{\alpha}(h)=\sum_{k=2}^{\infty}\frac{h^{(k\beta+m)}}{\Gamma(k\beta+m+1)}f^{(k\beta+m)}(x),\quad \beta:=\alpha-m. \tag{3.29}$$

Nevertheless, it is relevant to point out that these authors do not use the Riemann-Liouville expression of derivative as we did it, but rather define the later as the limit of a quotient involving the increment of the function on the one hand, and a so-called coarse grained mass or $\alpha$-mass of a subset which is generally fractal, exactly $df/(dx)^{\alpha}$. Loosely speaking the function is fractal because it is defined on a set which itself is fractal.

We can go a step farther. Assume that we are moving in a space in which the points (the pixels) have a thickness $\delta x$.Then the corresponding infinitesimal increment is not $dx$ but $\delta x>dx$, and a possible modeling is to put $\delta x=(dx)^{\alpha}$, $0<\alpha<1$.

## 4. Integration with respect to $(dx)^{\alpha}$

The integral with respect to $(dx)^{\alpha}$ is defined as the solution of the fractional differential equation

$$dy=f(x)(dx)^{\alpha},\quad x\geqslant 0,\quad y(0)=0, \tag{4.1}$$

which is provided by the following result:

DEFINITION 4.1. Let $f(x)$ denote a continuous function, then the solution $y(x)$, $y(0)=0$, of the equation (4.1) is defined by the equality

$$\begin{aligned} y&=\int_0^x f(\xi)(d\xi)^{\alpha}\\ &=\alpha\int_0^x (x-\xi)^{\alpha-1}f(\xi)d\xi,\quad 0<\alpha<1. \end{aligned}\tag{4.2}$$

*Derivation*

On multiplying both sides of (4.1) by $\alpha!$, and on taking account of (3.17), we have the equality

$$y^{(\alpha)}(x)=\alpha!f(x)$$

which provides

$$y(x) = \alpha! D^{-\alpha} f(x), \\ = \frac{\alpha!}{\Gamma(1+\alpha)} \int_0^x (x-\xi)^{\alpha-1} f(\xi) d\xi. \tag{4.3}$$

As a result, the integration of (4.1) can be written also in the form

$$D^{-\alpha} f(x) = \frac{1}{\alpha!} \int_0^x f(\xi)(d\xi)^{\alpha}. \tag{4.4}$$

or in a like manner

$$f(x) = \frac{1}{\alpha!} \frac{d^{\alpha}}{dx^{\alpha}} \int_0^x f(\xi)(d\xi)^{\alpha}. \tag{4.5}$$

LEMMA 4.1. *As a direct consequence of (4.2), one has the equality*

$$\int_0^x f(\xi)(d\xi)^{\alpha+\beta} = \frac{\alpha+\beta}{\alpha} \int_0^x (x-\xi)^{\beta} f(\xi)(d\xi)^{\alpha}, \quad 0 < \alpha+\beta < 1 \tag{4.6}$$

$$= \frac{\alpha+\beta}{\beta} \int_0^x (x-\xi)^{\alpha} f(\xi)(d\xi)^{\beta}, \quad 0 < \alpha+\beta < 1.$$

The proof is a result of the equality

$$\int_0^x f(\xi)(d\xi)^{\alpha+\beta} = (\alpha+\beta) \int_0^x (x-\xi)^{\alpha+\beta-1} f(\xi) d\xi \\ = (\alpha+\beta) \int_0^x (x-\xi)^{\alpha-1} \left((x-\xi)^{\beta} f(\xi)\right) d\xi.$$

LEMMA 4.2. (Fractional integration by part.) *The formula reads*

$$\int_a^b u^{(\alpha)}(x)v(x)(dx)^{\alpha} = \alpha! \, [u(x)v(x)]_a^b - \int_a^b u(x)v^{(\alpha)}(x)(dx)^{\alpha}, \tag{4.7}$$

*and can be obtained easily by combining* (4.1) *with* (4.3)*.*

LEMMA 4.3. (Transformation of variable.) *Consider the new variable* $y$ *defined by the equation* $y = g(x)$. *When* $g(x)$ *is differentiable, then one has*

$$\int f(y)\,(dy)^{\alpha} = \int f(g(x))\,\left(g'(x)\right)^{\alpha}(dx)^{\alpha}, \quad 0 < \alpha < 1, \tag{4.8}$$

*and when* $g(x)$ *has a fractional derivative of order* $\beta$, $0 < \alpha, \beta < 1$, *one has*

$$\int f(y)\,(dy)^{\alpha} = \Gamma^{-\alpha}(1+\beta) \int f(g(x)) \left(g^{(\beta)}(x)\right)^{\alpha} (dx)^{\alpha\beta}, \quad 0 < \alpha, \beta < 1.$$

This is a direct consequence of the fact that $dy = g'(x)dx$ in the first case, while in the second one, we have $dy = (\beta!)^{-1} g^{(\beta)}(x)(dx)^{\beta}$.

*Some examples*

(i) On making $f(x) = x^{\gamma}$ in (4.1) one obtains

$$\int_0^x \xi^{\gamma}(d\xi)^{\alpha} = \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}, \quad 0 < \alpha \leqslant 1 \tag{4.9}$$

and, more especially one has

$$\int_0^x (d\xi)^{\alpha} = x^{\alpha}, \quad 0 < \alpha \leqslant 1.$$

(ii) Assume now that $f(x)$ is the Dirac delta generalized function $\delta(x)$, then one has

$$\int_0^x \delta(\xi)(d\xi)^{\alpha} = \alpha x^{\alpha-1}, \quad 0 < \alpha \leqslant 1. \tag{4.10}$$

*Application to the fractional derivative of the Dirac delta function*

On using the equation (3.19) on the one hand, and extending well known definition on the other hand, we shall define the fractional derivative of the Dirac delta function by the equality

$$\int \delta^{(\alpha)}(\xi) f(\xi)(d\xi)^{\alpha} = -\int \delta(\xi) f^{(\alpha)}(\xi)(d\xi)^{\alpha}, \quad 0 < \alpha \leqslant 1 \tag{4.11}$$

and the equation (4.2), direct will yields

$$\int \delta^{(\alpha)}(\xi) f(\xi)(d\xi)^{\alpha} = -\alpha x^{\alpha-1} f^{(\alpha)}(0), \quad 0 < \alpha \leqslant 1 \tag{4.12}$$

In the next section we shall summarize some results on Markovian processes, which will be duplicated latter to deal with fractional stochastic.

## 5. Background on Probability Functionals of Markov Processes

### 5.1. Path probability of some Markov processes

Let us consider a scalar valued continuous random process $x(t)$ defined as the solution of the nonlinear stochastic differential equation which, with the Maruyama's notation, can be written in the form

$$dx = f(x,t)dt + g(x,t)w\sqrt{dt}, \tag{5.1}$$

where $w(t)$ is a Gaussian white noise (in the usual engineering sense of this term) with zero mean and the constant variance $\sigma^2$. Our purpose is to obtain the value of the path probability density defined as

$$p_r(\{x(t)\}, t \in [0,T]) := \lim_{N\uparrow\infty} p_r(x(t_1), x(t_2), ..., x(t_N)), \tag{5.2}$$

where $t_1 = 0 < t_2 < ... < t_N = T$. This limit is not a stochastic limit (probability, meansquare, almost sure) but merely the limit in the usual sense of this term, that is the limit of $p(x_n, t)$ as $x_n \to x$. The probability so defined by (5.2) is the probability density of the trajectory of $x(t)$ on $[0, T]$.

*First preliminary remark.* We first remark that the probability density of $w(t)$ is

$$p(w) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{w^2}{2\sigma^2}\right)$$

therefore we conclude that the probability density of the variable $Y := w/\sqrt{dt}$ is

$$p(y) = \left(\frac{\sigma}{\sqrt{dt}}\sqrt{2\pi}\right)^{-1} \exp\left(-\frac{y^2}{2\sigma^2}dt\right). \tag{5.3}$$

*Second preliminary remark.* Assume that a random variable has the probability density $q(y)$. Assume further that we make the transformation $Y = \phi(X)$ where $\phi(.)$ is a differentiable function, then the probability $p(x)$ of $X$ is

$$p(x) = q(\phi(x)) \left|\phi'(x)\right|. \tag{5.4}$$

*Derivation of the path probability density.* Given this prerequisite, we assume that the process $x(t)$ in the equation (5.1) is observed on a finite time interval $[0, T]$ of time, we select a time increment $\tau$, and we set

$$t_{k+1} - t_k =: \tau, \quad k = 1, 2, ..., N-1,$$

$$\tau = \frac{T}{N-1}, \quad t_1 = 0, \quad t_N = T. \tag{5.5}$$

This being the case, we re-write the equation (5.1) in the form

$$\frac{dx}{dt} = f(x,t) + g(x,t)\frac{w}{\sqrt{dt}}$$

therefore we derive the discrete approximation scheme

$$\frac{x_{k+1} - x_k}{\tau} = f(x_k, t_k) + g(x_k, t_k)\frac{w_{k+1}}{\sqrt{\tau}}$$

which provides

$$\frac{w_{k+1}}{\sqrt{\tau}} = \frac{1}{g(x_k, t_k)}\left(\frac{x_{k+1} - x_k}{\tau} - f(x_k, t_k)\right). \tag{5.6}$$

This equation can be considered as a transformation from $w_{k+1}/\sqrt{\tau}$ to $x_{k+1}$, and then using (5.3) and (5.4) we have the probability density of $X_{k+1}$ in the form

$$p(x_{k+1}) = \frac{1}{\sigma\sqrt{2\pi\tau}} \exp\left(-\frac{1}{2\sigma^2 g^2(x_k, t_k)}\left(\frac{x_{k+1} - x_k}{\tau} - f(x_k, t_k)\right)^2 \tau\right). \tag{5.7}$$

As a result one have the probability density

$$p(x_1, x_2, ..., x_N) = \left(\frac{1}{\sigma\sqrt{2\pi\tau}}\right)^N \exp\left(-\frac{\tau}{2\sigma^2}\sum_1^N \frac{1}{g^2(x_k,t_k)}\left(\frac{\Delta x_k}{\tau} - f(x_k,t_k)\right)^2\right).$$

On taking the limit as $\tau \downarrow 0$ and $N \uparrow \infty$, we find that

$$p_r(\{x(t)\}, t \in [0,T]) := K \exp\left(-\frac{1}{2\sigma^2}\int_0^T g^{-2}(x(t),t)\,(\dot{x}(t) - f(x(t),t))^2\, dt\right), \quad (5.8)$$

where K is normalizing constant of which the value is to be determined.

The fact that the value of $K$ is unknown should not be troublesome, because, on a practical standpoint, this probability density will be used mainly to compare stochastic trajectories.

EXAMPLE 5.1. Let us consider the stochastic differential equation

$$dx = rxdt + xw\sqrt{dt}, \quad (5.9)$$

which is basic in mathematical biology and mathematical finance. One has the identification

$$f(x,t) \equiv rx \quad \text{and} \quad g(x) \equiv x,$$

in such a manner that (5.8) provides

$$p_r(\{x(t)\}, t \in [0,T]) := K \exp\left(-\frac{1}{2\sigma^2}\int_0^T \left(\frac{\dot{x}}{x} - r\right)^2 dt\right). \quad (5.10)$$

$\dot{x}/x = \rho$ is the actual increase rate of the actual trajectory $x(t)$. The formula (5.10) clearly says that the more $\rho(t)$ deviates from $r$, the less the corresponding trajectory is likely to occur. Loosely speaking, $\rho(t)$ will evolve around $r$.

EXAMPLE 5.2. For the standard Brownian motion $b(t)$ (or order $1/2$)

$$db(t) = w(t)\sqrt{dt}, \quad (5.11)$$

one has

$$f(x,t) = 0 \quad \text{and} \quad g(x) = 1,$$

therefore the expression

$$p_r(\{b(t)\}, t \in [0,T]) := K \exp\left(-\frac{1}{2\sigma^2}\int_0^T (\dot{x}(t))^2\, dt\right). \quad (5.12)$$

which can be found in some applications in physics.

EXAMPLE 5.3. Let us consider the coloured noise or Ornstein-Uhlenback process defined by the equation

$$dx = -\rho(xdt - w(t)\sqrt{dt}), \quad \rho > 0. \tag{5.13}$$

Here one has

$$f(x,t) = -\rho x \quad and \quad g(x) = \rho$$

therefore

$$p_r(\{x(t)\}, t \in [0,T]) := K \exp\left(-\frac{1}{2\sigma^2}\int_0^T \left(\frac{\dot{x}}{\rho} + x\right)^2 dt\right). \tag{5.14}$$

In physics, the practical meaning of this expression is straightforward since $(\dot{x})^2$ can be thought of as the kinetic energy of a particle with unit mass.

## 5.2. Characteristic function estimate for some Markov processes

A useful technique to obtain the probability density of a random variable is to determine its characteristic function defined as the mathematical expectation $\Phi(u) := E(e^{iux}) \equiv E(\varphi) \equiv \langle\varphi\rangle$. Here, $x(t)$ is defined by the stochastic differential equation (5.1). According to the Itô's lemma, one has the differential

$$d\varphi = \left(iuf - \frac{1}{2}u^2 g^2 \sigma^2\right)\varphi\, dt + iu\varphi g w\sqrt{dt}, \tag{5.15}$$

therefore

$$d(\ln\varphi) = \left(iuf - \frac{1}{2}u^2 g^2 \sigma^2\right) dt + iugw\sqrt{dt}. \tag{5.16}$$

In order to obtain an estimate $\widehat{\Phi}(u)$ of $\Phi(u)$, we use the approximation

$$\langle \ln\varphi\rangle = \ln\hat{\Phi} \cong \ln\langle\varphi\rangle,$$

which yields

$$d\left(\ln\widehat{\Phi}(u,t)\right) \cong E\left(iuf(x,t) - \frac{1}{2}u^2 g^2(x,t)\sigma^2\right) dt.$$

$$= \left(iu\langle f(x,t)\rangle - \frac{1}{2}u^2\left\langle g^2(x,t)\right\rangle\sigma^2\right) dt$$

therefore

$$\hat{\Phi}(u,t) \cong \Phi(u,0)\exp\left(\int_0^t \left(iu\langle f(x,s)\rangle - \frac{1}{2}u^2\left\langle g^2(x,s)\right\rangle\sigma^2\right) ds\right), \tag{5.17}$$

with

$$\Phi_0(u) := \left\langle e^{ix(0)}\right\rangle.$$

Remark that we would obtain the same result by taking the approximate mathematical expectation of (5.13) to obtain the equality

$$d\langle\hat{\varphi}\rangle = \left(iu\langle f\rangle - \frac{1}{2}u^2\left\langle g^2\right\rangle\sigma^2\right)\langle\hat{\varphi}\rangle\,dt.$$

We bear in mind that $\Phi(u)$ is the Fourier's transform of $p(x,t)$ in such a manner that we can define the latter as the inverse transform of $\Phi(u)$.

EXAMPLE 5.3. Once more we consider the equation (5.9). A direct simple calculation yields the equations

$$d\langle x\rangle = r\langle x\rangle\,dt,$$
$$d\left\langle x^2\right\rangle = \left(2r+\sigma^2\right)\left\langle x^2\right\rangle dt,$$

therefore

$$\langle x\rangle = x_0 e^{rt}, \tag{5.18}$$

$$\left\langle x^2\right\rangle = (x_0)^2\exp\left((2r+\sigma^2)t\right), \tag{5.19}$$

and, on substituting into (5.13),

$$\hat{\Phi}(u) = \Phi(u,0)\exp\left(iu\frac{x_0}{r}e^{rt} - \frac{(x_0)^2}{2\left(2r+\sigma^2\right)}u^2\exp\left((2r+\sigma^2)t\right)\right). \tag{5.20}$$

EXAMPLE 5.4. For the Brownian motion $db = w\sqrt{dt}$, one has

$$\langle x\rangle = 0 \quad \textit{and} \quad \left\langle g^2\right\rangle = 1$$

therefore the well known result

$$\hat{\Phi}(u,t) = \Phi(u,0)\exp\left(-\frac{1}{2}u^2\sigma^2 t\right). \tag{5.21}$$

In the following we shall use these techniques to fractional stochastic processes, but before we need to define the framework

## 6. Stochastic Processes Defined by Coarse-grained Time

### 6.1. Modeling via stochastic differential equations of fractional order

As we pointed out in the introduction, the presence of a coarse-grained phenomenon in time causes that practical physical differential increment of time is not $dt$ but $(dt)^\alpha$. In the parlance of mechanics, the action integral turns to be an integral with respect to $(dt)^\alpha$. In our framework herein, $w(t)(dt)^\alpha$ will be substituted for $w(t)dt$.

With this modeling purpose in mind, we have the following alternative to define stochastic systems in the presence of coarse-grained time [21].

*First model*

The most direct way to generalize (5.1) is to write

$$dx = f(x,t)dt + g(x,t)w(t)(dt)^{\alpha}, \quad 0 < \alpha \leqslant 1, \tag{6.1}$$

which pictures the fact that the non-random dynamics $\dot{x} = f(x,t)$ is disturbed by the coarse-grained white noise $g(x,t)w(t)(dt)^{\alpha}$. On multiplying both sides of (6.1) by $\alpha!$ and on taking account of the equality $d^{\alpha}x = \alpha! dx$ (which holds only when $0 < \alpha < 1$), we obtain

$$d^{\alpha}x = f(x,t)d^{\alpha}t + \alpha! g(x,t)w(t)(dt)^{\alpha}. \tag{6.2}$$

This being the case, the $\alpha th$ derivative of $t$ yields

$$d^{\alpha}t = \Gamma^{-1}(2-\alpha)t^{1-\alpha}(dt)^{\alpha}, \tag{6.3}$$

and on substituting into (6.1), we obtain

$$d^{\alpha}x = \Gamma^{-1}(2-\alpha)f(x,t)t^{1-\alpha}(dt)^{\alpha} + \Gamma(1+\alpha)g(x,t)w(t)(dt)^{\alpha},$$

or again, on dividing both sides by $(dt)^{\alpha}$,

$$x^{(\alpha)} = \Gamma^{-1}(2-\alpha)f(x,t)t^{1-\alpha} + \Gamma(1+\alpha)g(x,t)w(t). \tag{6.4}$$

Next, on making the substitution $\Gamma^{-1}(2-\alpha)f \leftarrow f$ and $\Gamma(1+\alpha)g \leftarrow g$, we eventually have the general model

$$x^{(\alpha)} = f(x,t)t^{1-\alpha} + g(x,t)w(t). \tag{6.5}$$

or, in terms of fractional Brownian motion,

$$d^{\alpha}x = f(x,t)t^{1-\alpha}(dt)^{\alpha} + g(x,t)db(t,\alpha). \tag{6.6}$$

with the identification $db(t,\alpha) := w(t)(dt)^{\alpha}$.

*Second model*

Another approach is to generalize the equation (5.1) directly in the form

$$x^{(\alpha)}(t) = f(x,t) + g(x,t)w(t), \quad 0 < \alpha \leqslant 1. \tag{6.7}$$

or

$$d^{\alpha}x = f(x,t)(dt)^{\alpha} + g(x,t)db(t,\alpha). \tag{6.8}$$

*Third model*

The third model consists in identifying the fractional stochastic differential equation with the first two terms of the fractional Taylor's series, to yield

$$dx(t) = f(x,t)w^2(t)(dt)^{\alpha} + g(x,t)w(t)(dt)^{\alpha/2}. \tag{6.9}$$

In the Itô's mean square calculus, one can formally write the identity $w^2(t) \equiv \sigma^2(t)$, in such a manner that (6.9) turns to be

$$dx(t) = f(x,t)\sigma^2(t)(dt)^{\alpha} + g(x,t)w(t)(dt)^{\alpha/2},$$

where $\sigma^2(t)$ is the variance of $w(t)$, or more generally

$$dx(t) = f(x,t)(dt)^{\alpha} + g(x,t)w(t)(dt)^{\alpha/2}. \tag{6.10}$$

Here, when $\alpha = 1$, we have the Maruyama's equation

$$dx(t) = f(x,t)\sigma^2(t)dt + g(x,t)w(t)\sqrt{dt}, \tag{6.11}$$

*Further remarks and comments*

(i) The second model (6.7) can be thought of as an extension of the so-called Langevin equation in physics and which reads

$$\dot{x}(t) = f(x,t) + g(x,t)w(t).$$

(ii) The equation (6.10) of the third model is fully consistent with the fractional Taylor's series

$$dx = ((\alpha/2)!)^{-1}x^{(\alpha/2)}(t)(dt)^{\alpha/2} + (\alpha!)^{-1}x^{(\alpha)}(t)(dt)^{\alpha},$$

and formally, we have the correspondence

$$((\alpha/2)!)^{-1}x^{(\alpha/2)}(t) \equiv g(x,t)w(t)$$

and

$$(\alpha!)^{-1}x^{(\alpha)}(t) \equiv f(x,t).$$

We believe we can claim that this remark is a strong support to this model.

(iii) In all the derivation above, we have made the identification

$$db(t,\alpha) \equiv w(t)(dt)^{\alpha}$$

but this is a formal writing only. To the best of our knowledge, nowhere in the literature, $w(t)(dt)^{\alpha}$ has been used to define fractional Brownian motion, and this is the reason why we refer to white noise defined in coarse-grained time.

(iv) There are various models of fractional Brownian motion which are presently in use in the literature, and the most commonly accepted is the proposal of Mandelbrot and van Ness [29]

$$b(t,\alpha) - b(0,\alpha) = \frac{1}{\Gamma(1+\alpha/2)}\left\{\int_{-\infty}^{0}\left[(t-\tau)^{\alpha-1/2} - (-\tau)^{\alpha-1/2}\right]db(\tau) + \int_{0}^{t}(t-\tau)^{\alpha-1/2}db(\tau)\right\}$$

and, at first glance, this model is not quite identical to our one. So, with two different models, it is likely that we shall have different results.

### 6.2. Itô's lemma of fractional order

*Fractional Itô's lemma for differentiable functions*

The Itô's lemma reads as follows. If $h(x,t)$ is a function with first and second derivative with respect to $x$, and if $db(t) = w(t)\sqrt{dt}$, then one has the following equality, that is

$$h(x+db) = h(x,t) + h'_x(x,t)w\sqrt{dt} + (1/2)h''_{xx}(x,t)\sigma^2 dt + o(dt^{3/2}), \tag{6.12}$$

where $\sigma^2$ is the variance of $w(t)$. This formula is exact in the mean square stochastic calculus in the sense that $(db)^2$ and $\sigma^2 dt$ are equal in this framework.

We shall generalize this lemma in the form

$$h\left(x+w(dt)^\alpha\right) = h(x,t) + h'_x(x,t)w(dt)^\alpha + (1/2)h'_{xx}(x,t)\sigma^2(dt)^{2\alpha} + o(dt^{3\alpha}). \tag{6.13}$$

The easier way to prove this result is to show that it applies when $h(x,t)$ is a polynomial with respect to time, and then to use the Weierstrass theorem, of polynomial approximation.

In the equation (6.13), the identification with the usual Itô's lemma is achieved when $\alpha = 1/2$.

*Fractional Itô's lemma for non- differentiable functions*

Assume now that $h(x)$ is not differentiable, and has only derivatives of fractional order $k\beta$, where $k$ is any positive integer and $\beta$ is such that $0 < \beta < 1$. In such a case, $h(x)$ has a fractional Taylor's series of order $\beta$ expressed by (3.1), namely

$$h(x+db) = \sum_{k=0}^{\infty} \frac{(db)^{k\beta}}{(k\beta)!} f^{(k\beta)}(x), \quad 0 < \beta \leqslant 1, \tag{6.14}$$

and on taking account of the equality $db = w(t)(dt)^\alpha$, we have

$$f\left(x+w(dt)^\alpha\right) = f(x) + \sum_{k}^{\frac{f^{(k\beta)}(x)}{(k\beta)!}} w^{k\beta}(t)(dt)^{k\alpha\beta}, \quad k\alpha\beta \leqslant 1. \tag{6.15}$$

## 7. Path Probability Density of Coarse-grained Time Processes

### 7.1. Coarse-grained time processes with short-range memory

We refer to the fractional stochastic process $x(t)$ with short-range dependence (i.e. $0 < \alpha < 1/2$) defined by the equation

$$dx = f(x,t)dt + g(x,t)w(t)(dt)^\alpha, \quad 0 < \alpha < 1/2. \tag{7.1}$$

To handle this equation, we shall firstly re-write it in another form, and to this end, we shall proceed as follows. We multiply its both sides by $(2\alpha)!$, and on taking account of (3.10) which applies here since $0 < 2\alpha < 1$, we obtain

$$d^{2\alpha}x = f(x,t)d^{2\alpha}t + (2\alpha)!g(x,t)w(t)(dt)^\alpha, \tag{7.2}$$

This being the case, the fractional derivative of the function $t$ provides the equality

$$d^{2\alpha}t = \Gamma^{-1}(2-2\alpha)t^{1-2\alpha}(dt)^{2\alpha} \tag{7.3}$$

by the means of which we can re-write (7.2) in the sought form

$$\begin{aligned} d^{2\alpha}x &= \frac{f(x,t)}{\Gamma(2-2\alpha)}t^{1-2\alpha}(dt)^{2\alpha} + (2\alpha)!g(x,t)w(t)(dt)^{\alpha}, \\ &=: \tilde{f}(x,t)(dt)^{2\alpha} + \tilde{g}(x,t)w(t)(dt)^{\alpha}. \end{aligned} \tag{7.4}$$

with

$$\tilde{f}(x,t) := \Gamma^{-1}(2-2\alpha)f(x,t)t^{1-2\alpha}$$

and

$$\tilde{g}(x,t) := (2\alpha)!g(x,t).$$

Given this equation, we can duplicate step by step the derivation in the preceding Section 6. Dividing both sides of (7.4) by $(dt)^{2\alpha}$ yields the equation

$$\frac{d^{2\alpha}x}{dt^{2\alpha}} = \tilde{f}(x,t) + \tilde{g}(x,t)\frac{w}{(dt)^{\alpha}},$$

which provides the discrete approximation scheme

$$\frac{\Delta^{2\alpha}x}{\tau^{2\alpha}} = \tilde{f}(x_k,t_k) + \tilde{g}(x_k,t_k)\frac{w_{k+1}}{\tau^{\alpha}},$$

or in a like manner

$$\frac{w_{k+1}}{\tau^{\alpha}} = \frac{1}{\tilde{g}(x_k,t_k)}\left(\frac{\Delta^{2\alpha}x_k}{\tau^{2\alpha}} - \tilde{f}(x_k,t_k)\right).$$

This equation provides the probability density of $x_{k+\alpha} \equiv x(k\tau + \alpha\tau)$ which is

$$p(x_{k+\alpha}) = \frac{1}{\sigma\sqrt{2\pi\tau^{\alpha}}}\exp\left(-\frac{1}{2\sigma^2\tilde{g}^2(x_k,t_k)}\left(\frac{\Delta^{2\alpha}x_k}{\tau^{2\alpha}} - \tilde{f}(x_k,t_k)\right)^2\tau^{\alpha}\right). \tag{7.5}$$

On taking the limit as $\tau \downarrow 0$ and $N \uparrow \infty$, we find that

$$p_r(\{x(t)\}, t \in [0,T]) = K\exp\left(-\frac{1}{2\sigma^2}\int_0^T \tilde{g}^{-2}(x,t)\left(x^{(2\alpha)}(t) - \tilde{f}(x,t)\right)^2(dt)^{\alpha}\right). \tag{7.6}$$

The fact that the exact value of the constant $K$ is unknown should not be troublesome. Indeed, on a practical standpoint, we shall compare the likelihood of stochastic trajectories, in other words we shall work with the quotient of these probability in such a manner that the effect of $K$ will vanish.

EXAMPLE 7.1. We consider the fractional version of (6.9) which reads

$$dx = rxdt + xw(t)(dt)^{\alpha}, \quad 0 < \alpha < 1/2. \tag{7.7}$$

We have successively

$$\tilde{f}(x,t) = \frac{rx}{\Gamma(2-2\alpha)} t^{1-2\alpha},$$

$$\tilde{g}(x,t) = (2\alpha)!x,$$

therefore the expression

$$p_r(\{x(t)\}, t \in [0,T]) = K \exp\left( -\frac{1}{2\sigma^2} \int_0^T ((2\alpha)!x(t))^{-2} \times \right. \\ \left. \times \left( x^{(2\alpha)}(t) - \frac{rx}{\Gamma(2-2\alpha)} t^{1-2\alpha} \right)^2 (dt)^{\alpha} \right) \tag{7.8}$$

EXAMPLE 7.2. For the stochastic process defined by the equation

$$dx = -x^{2r+1}dt + w(t)(dt)^{\alpha}, \quad 0 < \alpha < 1/2, \quad r \in N^+, \tag{7.9}$$

one has successively

$$\tilde{f}(x,t) = -\frac{x^{2r+1}}{(1-2\alpha)!} t^{1-2\alpha},$$

$$\tilde{g}(x,t) = (2\alpha)!,$$

therefore

$$p_r(\{x(t)\}, t \in [0,T]) = K \exp\left( -\frac{1}{2((2\alpha)!\sigma)^2} \int_0^T \left( x^{(2\alpha)}(t) + \frac{x^{2r+1}(t)}{(1-2\alpha)!} t^{1-2\alpha} \right)^2 (dt)^{\alpha} \right). \tag{7.10}$$

in which the integral w.r.t. $(dt)^{2\alpha}$ is defined by (4.2).

When $r = 0$, one so recovers the Ornstein-Uhlenback fractional process.

EXAMPLE 7.3. We consider the coarse-grained Brownian motion $b(t,\alpha)$ defined by the equation

$$db(t,\alpha) = w(t)(dt)^{\alpha}, \quad 0 < \alpha < 1/2. \tag{7.11}$$

One has

$$\tilde{f}(x,t) = 0 \quad \text{and} \quad \tilde{g}(x,t) = (2\alpha)!$$

therefore

$$p_r(\{b(t,\alpha)\}, t \in [0,T]) = K \exp\left( -\frac{1}{2\sigma^2} \int_0^T \left( \frac{b^{(2\alpha)}(t,\alpha)}{(2\alpha)!} \right)^2 (dt)^{\alpha} \right). \tag{7.12}$$

### 7.2. Coarse-grained time processes with long-range memory

The process under consideration is still defined by the equation (7.1), but here we assume that $1/2 < \alpha < 1$,

$$dx = f(x,t)dt + g(x,t)w(t)(dt)^{\alpha}, \quad 1/2 < \alpha < 1. \tag{7.13}$$

Such a process is referred to as a stochastic process with long-range memory. Here the equation (7.2) dooes not hold because, as one has $2\alpha > 1$, the equation (3.30), which relates $d^{2\alpha}x$ to $dx$ no longer applies. We have to find another approach

Multiplying both sides of (7.13) by $\alpha!$ on the other hand, and taking account of the relation (7.3) in which $\alpha$ is substituted for $2\alpha$ on the other hand, we have the fractional differential

$$d^{\alpha}x = \left(\frac{f(x,t)}{\Gamma(2-\alpha)}t^{1-\alpha} + \alpha! g(x,t)w(t)\right)(dt)^{\alpha}, \tag{7.14}$$

or in a like manner

$$x^{(\alpha)}(t) = \breve{f}(x,t) + \breve{g}(x,t)w(t). \tag{7.15}$$

with

$$\breve{f}(x,t) := \Gamma^{-1}(2-\alpha)f(x,t)t^{1-\alpha}$$

and

$$\breve{g}(x,t) := \alpha! g(x,t)$$

We then have

$$w(t) = \left(\breve{g}(x,t)\right)^{-1}\left(x^{(\alpha)}(t) - \breve{f}(x,t)\right),$$

and the rationale of the preceding subsection directly yields

$$p_r(\{x(t)\}, t \in [0,T]) = K \exp\left(-\frac{1}{2\sigma^2}\int_0^T \breve{g}^{-2}(x,t)\left(x^{(\alpha)}(t) - \breve{f}(x,t)\right)^2 (dt)^{\alpha}\right). \tag{7.16}$$

EXAMPLE 7.4. For the parallel of (7.7) which reads

$$dx = rxdt + xw(t)(dt)^{\alpha}, \quad 1/2 < \alpha < 1, \tag{7.17}$$

one has

$$\breve{f}(x,t) = \frac{rx}{(1-\alpha)!}t^{1-\alpha}, \quad \breve{g}(x,t) = \alpha! x$$

therefore

$$p_r(\{x(t)\}, t \in [0,T]) = K \exp\left(-\frac{1}{2\sigma^2}\int_0^T (\alpha! x(t))^{-2}\left(x^{(\alpha)}(t) - \frac{rx}{(1-\alpha)!}t^{1-\alpha}\right)^2 (dt)^{\alpha}\right) \tag{7.18}$$

EXAMPLE 7.5. We once more consider the equation (7.11) in which now we assume that the order $\alpha$ satisfies the condition $1/2 < \alpha < 1$.

Here one has

$$\breve{f}(x,t) = -\frac{x^{2r+1}}{(1-\alpha)!}t^{1-\alpha}, \quad \breve{g}(x,t) = \alpha!$$

and the equation (7.16) direct yields

$$p_r(\{x(t)\}, t \in [0,T]) = K\exp\left(-\frac{1}{2(\sigma\alpha!)^2}\int_0^T \left(x^{(\alpha)}(t) + \frac{x^{2r+1}(t)}{(1-\alpha)!}t^{1-\alpha}\right)^2 (dt)^\alpha\right). \tag{7.19}$$

EXAMPLE 7.6. We consider the coarse-grained time Brownian motion of the example (7.11), but now we assume that $1/2 < \alpha < 1$,

$$db(t,\alpha) = w(t)(dt)^\alpha, \quad 1/2 < \alpha < 1., \tag{7.20}$$

We then have

$$\breve{f}(x,t) = 0 \quad and \quad \breve{g}(x,t) = \alpha!,$$

therefore

$$p_r(\{b(t,\alpha)\}, t \in [0,T]) = K\exp\left(-\frac{1}{2\sigma^2}\int_0^T \left(\frac{b^{(\alpha)}(t,\alpha)}{\alpha!}\right)^2 (dt)^\alpha\right). \tag{7.21}$$

When $\alpha = 1$, one so recovers (5.12)

## 8. Characteristic Function of Random Coarse-grained Processes

### 8.1. Coarse-grained time processes with short-range memory

We refer again to the stochastic process $x(t)$ defined by the stochastic differential equation

$$dx = f(x,t)dt + g(x,t)w(t)(dt)^\alpha, \quad 0 < \alpha < 1/2, \tag{8.1}$$

and to the function $\varphi(u,x) = \exp(iux)$; and since it is differentiable, we can use the corresponding Itô's lemma (6.13) to obtain

$$d\varphi = \varphi\left(iufdt + iugw(dt)^\alpha - \frac{u^2}{2}g^2\sigma^2(dt)^{2\alpha}\right). \tag{8.2}$$

On taking the mathematical expectation of both sides, we come across the approximate equation

$$d\left(\ln\hat{\Phi}(u,t)\right) = iu\langle f\rangle dt - \frac{u^2}{2}\langle g^2\rangle\sigma^2(dt)^{2\alpha}. \tag{8.3}$$

Looking for a solution in the form

$$\hat{\Phi}(u,t) = \hat{\Phi}_1(u,t)\hat{\Phi}_2(u,t),$$

we finds that it is determined by the two equations

$$d\hat{\Phi}_1(u,t) = iu\langle f\rangle\,\hat{\Phi}_1(u,t)dt, \tag{8.4}$$

and

$$d\hat{\Phi}_2(u,t) = -\frac{u^2}{2}\left\langle g^2\right\rangle\sigma^2\hat{\Phi}_2(u,t)(dt)^{2\alpha}. \tag{8.5}$$

Equation (8.4) yields

$$\hat{\Phi}_1(u,t) = \Phi_1(u,0)\exp\left(iu\int_0^t\langle f(x,s)\rangle\,ds\right). \tag{8.6}$$

The solution to (8.5) can be obtained as follows. Multiplying its both sides by $(2\alpha)!$ and taking account of (3.10), we obtain the equation

$$d^{2\alpha}\hat{\Phi}_2(u,t) = -(2\alpha)!\frac{u^2}{2}\left\langle g^2\right\rangle\sigma^2\hat{\Phi}_2(u,t)(dt)^{2\alpha}$$

of which the solution is

$$\hat{\Phi}_2(u,t) = \hat{\Phi}_2(u,0)E_{2\alpha}\left(-\frac{(2\alpha)!}{2}u^2\sigma^2\int_0^T\left\langle g^2(x,s)\right\rangle(ds)^{2\alpha}\right). \tag{8.7}$$

We then have the sought result

$$\hat{\Phi}(u,t) = \Phi(u,0)\exp\left(iu\int_0^t\langle f(x,s)\rangle\,ds\right)\,E_{2\alpha}\left(-\frac{(2\alpha)!}{2}u^2\sigma^2\int_0^t\left\langle g^2(x,s)\right\rangle(ds)^{2\alpha}\right). \tag{8.8}$$

EXAMPLE 8.1. For the equation (7.7) of the example 7.1, the solution (8.8) yields

$$\hat{\Phi}(u,t) = \Phi(u,0)\exp\left(iru\int_0^t\langle x\rangle\,ds\right)\,E_{2\alpha}\left(-\frac{(2\alpha)!}{2}u^2\sigma^2\int_0^t\left\langle x^2\right\rangle(ds)^{2\alpha}\right). \tag{8.9}$$

EXAMPLE 8.2. For the stochastic process defined by (7.15), one has

$$\begin{aligned}\hat{\Phi}(u,t) &= \Phi(u,0)\exp\left(-iu\int_0^t\left\langle x^{2r+1}\right\rangle ds\right)\ \mathrm{E}_{2\alpha}\left(-\frac{(2\alpha)!}{2}u^2\sigma^2\int_0^t(ds)^{2\alpha}\right).\\ &= \Phi(u,0)\exp\left(-iu\int_0^t\left\langle x^{2r+1}\right\rangle ds\right)\ \mathrm{E}_{2\alpha}\left(-\frac{(2\alpha)!}{2}u^2\sigma^2t^{2\alpha}\right).\end{aligned} \tag{8.10}$$

EXAMPLE 8.3. Refer to the Brownian motion with short-range memory defined by the equation (7.11). Then the expression (8.8) yields

$$\begin{aligned}\hat{\Phi}(u,t) &= \Phi(u,0)E_{2\alpha}\left(-\frac{(2\alpha)!}{2}u^2\sigma^2\int_0^t (ds)^{2\alpha}\right)\\ &= \Phi(u,0)E_{2\alpha}\left(-\frac{(2\alpha)!}{2}u^2\sigma^2 t^{2\alpha}\right), \quad 0<\alpha<1/2. \qquad (8.11)\end{aligned}$$

## 8.2. Coarse-grained time processes with long-range memory

We can duplicate step by step the calculus above related to fractional processes with short-range memory, up to the equations (8.4) and (8.5) which still hold here.

$\hat{\Phi}_1(u,t)$ is still given by the equation (8.6).

The solution of (8.5) with $1<2\alpha<2$ is now given by the expression (see the Appendix for the details)

$$\hat{\Phi}_2(u,t) = \hat{\Phi}_2(u,0)E^2_{1/2}\left((1/2)!\frac{iu\sigma}{\sqrt{2}}\int_0^t \left\langle g^2(x,s)\right\rangle^{1/2}(ds)^\alpha\right)$$

therefore the general solution

$$\hat{\Phi}(u,t) = \Phi(u,0)\exp\left(iu\int_0^t \langle f(x,s)ds\rangle\right)E^2_{1/2}\left((1/2)!\frac{iu\sigma}{\sqrt{2}}\int_0^t \left\langle g^2(x,s)\right\rangle^{1/2}(ds)^\alpha\right). \qquad (8.12)$$

EXAMPLE 8.4. For the equation (7.7) in the example (7.1), one has

$$\hat{\Phi}(u,t) = \Phi(u,0)\exp\left(iru\int_0^t \langle x\rangle\, ds\right)E^2_{1/2}\left((1/2)!\frac{iu\sigma}{\sqrt{2}}\int_0^t \left\langle x^2\right\rangle^{1/2}(ds)^\alpha\right). \qquad (8.13)$$

EXAMPLE 8.5. We now refer to the stochastic process defined by the equation (7.15) in the example (7.2). Here one has

$$\begin{aligned}\hat{\Phi}(u,t) &= \Phi(u,0)\exp\left(iru\int_0^t \langle x\rangle\, ds\right)E^2_{1/2}\left((1/2)!\frac{iu\sigma}{\sqrt{2}}\int_o^t (ds)^\alpha\right)\\ &= \Phi(u,0)\exp\left(iru\int_0^t \langle x\rangle\, ds\right)E^2_{1/2}\left((1/2)!\frac{iu\sigma}{\sqrt{2}}t^\alpha\right). \qquad (8.14)\end{aligned}$$

EXAMPLE 8.6. For the Brownian motion with long-range memory, one finds easily

$$\hat{\Phi}(u,t) = \Phi(u,0)E_{1/2}^{2}\left((1/2)!\frac{iu\sigma}{\sqrt{2}}\int_0^t (ds)^{\alpha}\right)$$
$$= \Phi(u,0)E_{1/2}^{2}\left((1/2)!\frac{iu\sigma}{\sqrt{2}}t^{\alpha}\right). \tag{8.15}$$

## 9. Application of the Maximum Entropy Principle

### 9.1. Fractional dynamical equations of the state moments

*Preliminary remarks*

In the following, we shall denote the state moment of the random variable $x(t)$ by

$$m_k(t) := \left\langle x^k(t) \right\rangle, \quad k \geqslant 1.$$

We refer to the stochastic differential equation *with short range dependence* (6.10) which we re-write below for convenience, that is

$$dx(t) = f(x,t)(dt)^{\alpha} + g(x,t)w(t)(dt)^{\alpha/2}, \tag{9.1}$$

with $\langle w(t) = 0 \rangle$ and $\left\langle w^2(t) \right\rangle = \sigma^2(t)$.

First of all, we notice that one has successively

$$\langle dx \rangle = \langle f(x,t) \rangle (dt)^{\alpha}$$
$$\left\langle (dx)^2 \right\rangle = \left\langle g^2(x,t)\,\sigma^2(x,t) \right\rangle (dt)^{\alpha}$$
$$\left\langle (dx)^k \right\rangle = 0, \quad k \geqslant 3$$

*Dynamical equation of the first order moment.*

We have successively

$$dm_1 = \langle x + dx \rangle - \langle x \rangle$$
$$dm_1 = \langle f(x,t) \rangle (dt)^{\alpha}$$
$$\alpha! dm_1 = \alpha! \langle f(x,t) \rangle (dt)^{\alpha}$$

therefore

$$m_1^{(\alpha)}(t) = \alpha! \langle f(x,t) \rangle . \tag{9.2}$$

*Dynamical equation of the second order moment.*

In a like manner one has

$$dm_2 = \left\langle (x+dx)^2 \right\rangle - \left\langle x^2 \right\rangle$$

$$= \left\langle 2xdx + (dx)^2 \right\rangle$$

$$= \langle 2xf(x,t) \rangle (dt)^\alpha + \left\langle g^2(x,t)\,\sigma^2(x,t) \right\rangle (dt)^\alpha .$$

Multiplying both-sides by $\alpha!$ and dividing by $(dt)^\alpha$ yields

$$m_2^{(\alpha)}(t) = \alpha! \langle 2xf(x,t) \rangle + \alpha! \left\langle g^2(x,t)\,\sigma^2(t) \right\rangle . \tag{9.3}$$

*Dynamical equation of the moment of order $k, k \geqslant 3$.*

In the general case one has

$$dm_k = \left\langle (x+dx)^k - x^k \right\rangle$$

$$= \left\langle kx^{k-1}dx + \frac{k(k-1)}{2}x^{k-2}(dx)^2 \right\rangle$$

$$= k\left\langle x^{k-1}f \right\rangle (dt)^\alpha + \frac{k(k-1)}{2}\sigma^2(t)\left\langle x^{k-2}g^2 \right\rangle (dt)^\alpha$$

therefore

$$m_k^{(\alpha)}(t) = \alpha! \left\langle kx^{k-1}f(x,t) + \frac{k(k-1)}{2}\sigma^2(t)x^{k-2}g^2(x,t) \right\rangle . \tag{9.4}$$

In the special case when $f(x,t)$ and $g(x,t)$ are polynomials with respect to $x$, these equations define a set of linear differential equations of fractional order to determine the moments.

EXAMPLE 9.1. Let us consider the popular equation (see (7.17))

$$dx = rx(dt)^\alpha + xw(t)(dt)^{\alpha/2}, \tag{9.5}$$

where $r$ is a constant with $\langle w \rangle = 0$ and $\left\langle w^2 \right\rangle = \sigma^2(t)$. We then have the identification

$$f(x,t) = rx \qquad \textit{and} \qquad g(x) = x$$

and according to (9.2) and (9.3) one has the equations

$$m_1^{(\alpha)}(t) = \alpha! r m_1(t)$$

$$m_2^{(\alpha)}(t) = \alpha!(2r + \sigma^2(t))\,m_2(t)$$

of which the solutions are (with the modified Riemann-Liouville derivative)

$$m_1(t) = m_1(0)E_\alpha(\alpha! r t^\alpha) \tag{9.6}$$

and

$$m_2(t) = m_2(0)E_\alpha\left( 2r\alpha! t^\alpha + \alpha! \int_0^t \sigma^2(\tau)(d\tau)^\alpha \right) . \tag{9.7}$$

## 9.2. Application of Jaynes maximum entropy principle

*Background on the MEP*. Jaynes maximum entropy principle [11] states that if, all you know about a random variable $X \in \Re$ is summarized in some constraints of statistic nature, such as some mathematical expectations, for instance; then, as the best estimate $\hat{p}(x,t)$ of its probability density $p(x,t)$, you should select that one which maximizes its informational entropy

$$H(X) := -\int_{\Re} p(x)\ln p(x)dx, \tag{9.8}$$

subject to these constraints.

The classical example is provided by the case when one has at hand the first two moments $m_1$ and $m_2$ of $X$, therefore one obtains, via a simple optimization with Lagrange parameters

$$\hat{p}(x,t) = \left(s\sqrt{2\pi}\right)^{-1} \exp\left(-\frac{(x-m_1)^2}{2s^2}\right). \tag{9.9}$$

with

$$s^2 := m_2 - (m_1)^2.$$

EXAMPLE 9.2. We come back to the equation (9.5) with the maximization of

$$H(X,t) := -\int_{\Re} p(x,t)\ln p(x,t)dx,$$

and the calculation yields the equation (9.9) in which $m_1$ and $m_2$ are defined by (9.6) and (9.7).

## 9.3. An approach to informational entropy of fractional order

Recently [22] we have proposed a new model of entropy of fractional order to take account of some defects in observation, and of course, formally it could be possible to use it to apply the maximum entropy principle. In the present subsection, we shall introduce the reader to new results on this fractional entropy, mainly to show why it is quite relevant in our problem of determining path probability density. We shall display its main properties, and by this way, we shall exhibit its practical meaning.

*Discrete entropy of fractional order*

DEFINITION 9.1. Given a random variable $X$ which takes on the values $x_1, x_2, ..., x_m$ with the probabilities $p_1, p_2, ..., p_m$, $p_1 + p_2 + ... + p_m = 1$, its fractional entropy of order $\alpha$ is defined by the expression

$$\tilde{H}_\alpha(X) := -\sum_{i=1}^{m} p_i \left(\mathrm{Ln}_\alpha\, p_i\right)^{1/\alpha}, \tag{9.10}$$

where $\mathrm{Ln}_\alpha\, p$ denotes the inverse of the Mittag-Leffler function, clearly $p = E_\alpha\left(\mathrm{Ln}_\alpha\, p\right)$, and with the convention $0(\mathrm{Ln}_\alpha 0)^{1/\alpha} = 1(\mathrm{Ln}_\alpha 1)^{1/\alpha} = 0$.

The motivation for this definition comes from the relation [17]

$$(\mathrm{Ln}_\alpha uv)^{1/\alpha} = (\mathrm{Ln}_\alpha u)^{1/\alpha} + (\mathrm{Ln}_\alpha v)^{1/\alpha}. \tag{9.11}$$

Indeed, according to (9.11), $(\mathrm{Ln}_\alpha p_i)^{1/\alpha}$ is considered as measuring the quantum of information associated to $p_i$, and its mathematical expectation yields (9.10). O

It follows that, as expected, when $X$ is a non-random variable which takes on one value only, then its fractional entropy is zero.

Remark that, according to the equality [17]

$$\left(\mathrm{Ln}_\alpha \frac{1}{p}\right)^{1/\alpha} = -(\mathrm{Ln}_\alpha p)^{1/\alpha}$$

one has the new expression

$$\tilde{H}_\alpha(X) := \sum_{i=1}^{m} p_i \left(\mathrm{Ln}_\alpha \frac{1}{p_i}\right)^{1/\alpha}. \tag{9.12}$$

*Continuous entropy of fractional order*

DEFINITION 9.2. Given a real valued random variable $X$ with the probability density $p(x)$, its fractional probability entropy of order $\alpha$, $0 < \alpha < 1$, is defined by the expression

$$\tilde{H}_\alpha(X) = -\int_{\Re} p(x)\,(\mathrm{Ln}_\alpha p(x))^{1/\alpha} dx. \tag{9.13}$$

*Motivation.* The derivation of this definition in order that it be fully consistent with the entropy of discrete variable can be achieved by the total entropy [13]

$$H(X_d, h) = \sum_{i=1}^{N} P_i \left[-(\mathrm{Ln}_\alpha P_i)^{1/\alpha} + (\mathrm{Ln}_\alpha h)^{1/\alpha}\right],$$

and by using the equality

$$\mathrm{Ln}_\alpha(x^y) = y^\alpha \,\mathrm{Ln}_\alpha(x),$$

we obtain the new expression

$$H(X_d, h) = -\sum_{i=1}^{N} P_i \left(\mathrm{Ln}_i \frac{P_i}{h_i}\right)^{1/\alpha},$$

of which the limit when $h \downarrow 0$ is (9.13). It can be shown that

LEMMA 9.1. *When $X$ and $Y$ are independent, then the following equation holds, which is*

$$\tilde{H}_\alpha(X, Y) = \tilde{H}_\alpha(X) + \tilde{H}_\alpha(Y).$$

*Proof.* A simple calculation yields

$$\begin{aligned}\tilde{H}_\alpha(X,Y) &= -\sum_{i=1}^{m}(\mathrm{Ln}_\alpha\, p_i)^{1/\alpha}\sum_{j=1}^{n} p_i q_{ji} - \sum_{i=1}^{m}\sum_{j=1}^{n} p_i q_{ji}\,(\mathrm{Ln}_\alpha\, q_{ji})^{1/\alpha}, \\ &= -\sum_{i=1}^{m}(\mathrm{Ln}_\alpha\, p_i)^{1/\alpha} p_i \sum_{j=1}^{n} q_{ji} - \sum_{i=1}^{m} p_i \sum_{j=1}^{n} q_{ji}\,(\mathrm{Ln}_\alpha\, q_{ji})^{1/\alpha} \end{aligned} \tag{9.14}$$

In a like manner one has the

LEMMA 9.2. *Define $\Phi_\alpha(p_1,p_2,..,p_m) := H_\alpha(X)$; then the Faddeev formula holds, which is*

$$\begin{aligned}\Phi_\alpha(p_1,p_2,..p_M,..,p_m) &:= \Phi_\alpha\left(\sum_{i=1}^{M} p_i, p_{M+1},....,p_m\right) \\ &\quad + \left(\sum_{i=1}^{M} p_i\right)\Phi_\alpha\left(\frac{p_1}{\sum_{i=1}^{M} p_i},.....,\frac{p_M}{\sum_{i=1}^{M} p_i}\right). \end{aligned} \tag{9.15}$$

*Proof.* The proof is based on the relation

$$\left(\mathrm{Ln}_\alpha\, x\frac{1}{x}\right)^{1/\alpha} = (\mathrm{Ln}_\alpha\, x)^{1/\alpha} + \left(\mathrm{Ln}_\alpha\, \frac{1}{x}\right)^{1/\alpha} = (\mathrm{Ln}_\alpha\, 1)^{1/\alpha} = 0$$

which provides

$$\left(\mathrm{Ln}_\alpha(x^{-1})\right)^{1/\alpha} = -(\mathrm{Ln}_\alpha\, x)^{1/\alpha}. \tag{9.16}$$

This being the case, for the sake of pedagogy, we consider the special case

$$\Phi_\alpha(p_1,p_2,p_3) := \Phi_\alpha(p_1+p_2,p_3) + (p_1+p_2)\,\Phi\left(\frac{p_1}{p_1+p_2},\frac{p_2}{p_1+p_2}\right),$$

It is easy to show that one has successively,

$$\Phi\left(\frac{p_1}{p_1+p_2},\frac{p_2}{p_1+p_2}\right) = \left(\frac{p_1}{p_1+p_2}\right)\mathrm{Ln}_\alpha^{1/\alpha}\frac{p_1}{p_1+p_2} + \left(\frac{p_2}{p_1+p_2}\right)\mathrm{Ln}_\alpha^{1/\alpha}\frac{p_2}{p_1+p_2}$$

with

$$\mathrm{Ln}_\alpha^{1/\alpha}\frac{p_1}{p_1+p_2} = \mathrm{Ln}_\alpha^{1/\alpha} p_1 - \mathrm{Ln}_\alpha^{1/\alpha}(p_1+p_2),$$

therefore

$$(p_1+p_2)\Phi\left(\frac{p_1}{p_1+p_2},\frac{p_2}{p_1+p_2}\right) = \Phi(p_1,p_2) - (p_1+p_2)\,\mathrm{Ln}_\alpha^{1/\alpha}(p_1+p_2).$$

Moreover, a straightforward calculation yields

$$\Phi(p_1+p_2,p_3)=(p_1+p_2)\,\mathrm{Ln}_\alpha^{1/\alpha}(p_1+p_2)+p_3\,\mathrm{Ln}_\alpha^{1/\alpha}\,p_3.$$

It follows that, as expected, when $X$ is a non-random variable which takes on one value only, then its fractional entropy is zero.

This entropy has properties similar to those of the entropies of Renyi [39] and Tsallis [44] in which the parameter $\alpha$ can be thought of as measuring an information loss, but in addition, and this is of paramount importance, it satisfies all the main basic composition rules of the classical information. We believe that this entropy could replace Renyi's and Tsallis entropies and provide new approaches to various problems in statistical mechanics.

*On the significance of fractional entropy*

*(First remark) Observation with information loss.* Let us consider the random variable $X$ which takes on the value $x_i$ with the probability $p_i$, $i=1,2,..,n$. In the absolute, the amount of information involved in $x_i$ is $\ln(1/p_i)$. Assume that the process is observed with some defect in observation, in such a manner that the amount of information which is so seized by the observer is lower than $\ln(1/p_i)$. An approach to describe this phenomenon is to select a model in the form

$$\textit{observed information} = \alpha\ln(1/p_i), \quad 0<\alpha<1,$$

therefore the entropy $\langle\alpha\ln(1/p)\rangle$, that is to say

$$\begin{aligned} H_{obs}(X) &= \sum_{i=1}^{n} p_i \ln(1/p_i^\alpha) \\ &= -\sum_{i=1}^{n} p_i \ln p_i^\alpha. \end{aligned}$$

Let us bear in mind that transinformation, or merely information, is a difference of entropies, whilst, strictly speaking, entropy measures the amount of uncertainty we have about a phenomenon prior to any experiment. Sometimes $H(X)$ is referred to as information (instead of uncertainty) mainly because it appears to be the maximum value of the amount of information which we need to identify $X$.

*(Second remark)* The equality

$$x=\exp(\ln x)=E_\alpha(\mathrm{Ln}_\alpha x)$$

provides the approximation

$$\mathrm{Ln}_\alpha x \cong \ln(x^{\alpha!}), \quad 0<\alpha!<1$$

which suggests that the entropy

$$K_\alpha(X) := -\sum_{i=1}^{n} p_i\,\mathrm{Ln}_\alpha\,p_i, \tag{9.17}$$

is also a valuable candidate to measure of information in the presence of defect in the observation process.

*(Third remark)* At first glance, the entropy (9.17) could be considered as the generalized entropy which we are looking for. This being the case, if in addition we require that a suitable generalization of entropy should satisfy all the basic functional properties of the classical Shannon entropy, then we have to go a step farther, and select the entropy $\tilde{H}_\alpha(X)$ instead of $K_\alpha(X)$.

Remark that, in term of physical dimensions, $K_\alpha$ compares with $(\tilde{H}_\alpha)^\alpha$: $K_\alpha \propto (\tilde{H}_\alpha)^\alpha$. In other words, in a practical problem of encoding, $(\tilde{H}_\alpha)^\alpha$ would be more relevant than $\tilde{H}_\alpha$.

For instance, if we want to encode $X$ by $Y$, which are characterized by the entropies $\tilde{H}_\alpha(X)$ and $\tilde{H}_\beta(Y)$ respectively, then one should have the necessary encoding condition $\left(\tilde{H}_\beta(Y)\right)^\beta \geqslant \left(\tilde{H}_\alpha(X)\right)^\alpha$. When $X$ has $M$ results which occur with the same probability and, likewise, $Y$ provides $N$ results with the same probability, then the balance condition turns to be simply $\mathrm{Ln}_\beta N \geqslant \mathrm{Ln}_\alpha M$.

*(Fourth remark) On the meaning of information.* Several authors have suggested to extend entropy of random variables in the form

$$H_{obs}(X) = -\int_{\Re} \mu(x)p(x)\ln p(x)dx$$

where $\mu(x)$ is a density function which would take account of the meaning of the information. If we assume that $\mu(x)$ is a constant, then we once more recover the uncertainty density $-\ln p^\mu$.

*(Fifth remark) Observed probability.* By using observation with informational invariance, we obtained a model of observed probability distribution in the form

$$p_{obs}(x) = ap^b(x),$$

where $p(x)$ is a probability density, and $a$ together with $b$ denotes two constants which characterize the observation process. Taking the mathematical expectation $\left\langle \ln(ap^b)\right\rangle$ yields the observed entropy

$$H_{obs}(X) = \ln a + \tilde{H}_b(X).$$

In the following we shall give more explanations on the heuristic derivation of the fractional entropy.

## 9.4. Heuristic derivation of entropy of fractional order

Let $X \in \Re$ denote the observed result of an experiment which may take on the value $x \in R$, to which we can ascribe a weight function $\varphi(x) \geqslant 0$. We assume that the uncertainty that an observer has about the value of X, prior to any experiment, can be defined by a function

$$H(X) = -\int_{\Re} \varphi(x)h(\varphi)dx, \tag{9.18}$$

where $-h(\varphi)$ is the density of uncertainty involved in the event $(X = x)$.

Assume now that, due to some coarse-graining effect in space, the increment of weighted uncertainty varies like $(dx)^\alpha$. Clearly it is not $-\varphi(x)h(\varphi)dx$, but rather it is $-\varphi(x)h(\varphi)(dx)^\alpha$ , $0 < \alpha < 1$, in such a manner that the amount of uncertainty involved in $X$ would be rather defined by the formal expression

$$H(X) = -\int_{\Re} \varphi(x)h(\varphi)(dx)^\alpha$$

of which the meaning would remain to be clarified.

So, in order to circumvent this difficulty and to be fully consistent with classical calculus, we shall rather assume that the amount of uncertainty involved in $X$ is defined by the integral

$$H(X) = -\int_{\Re} \varphi^{1/\alpha} h^{1/\alpha}(\varphi)dx. \tag{9.19}$$

in which the integrand is $(\varphi(x)h(\varphi)(dx)^\alpha)^{1/\alpha}$.

In the information theory, it is quite right to set the requirement that $h(.)$ should fulfill the condition (additive law of uncertainty)

$$h^{1/\alpha}(\varphi\psi) = h^{1/\alpha}(\varphi) + h^{1/\alpha}(\psi), \tag{9.20}$$

in which case we are led to set

$$h(\varphi) \equiv \mathrm{Ln}_\alpha \varphi$$

therefore the eventual expression

$$\tilde{H}_\alpha(X) = -\int_{\Re} \varphi^{1/\alpha}(x)(\mathrm{Ln}_\alpha \varphi)^{1/\alpha} dx. \tag{9.21}$$

The counterpart for discrete probability distributions reads

$$\tilde{H}_\alpha(X) = -\sum_{i=1}^{m} \varphi_i^{1/\alpha} (\mathrm{Ln}_\alpha \varphi_i)^{1/\alpha}. \tag{9.22}$$

This being the case, if we make the transformation

$$p = \varphi^{1/\alpha}, \tag{9.23}$$

then we obtain the entropies

$$\tilde{H}_\alpha(X) = -\alpha \int_{\Re} p(x) (\mathrm{Ln}_\alpha p(x))^{1/\alpha} dx, \tag{9.24}$$

and

$$\tilde{H}_\alpha(X) = -\alpha \sum_{i=1}^{m} p_i (\mathrm{Ln}_\alpha p_i)^{1/\alpha}. \tag{9.25}$$

It follows that, at first glance, this fractional entropy could be quite meaningful here to apply the maximum entropy principle in the presence of defect in observation.

When the solution defined by the maximum entropy principle cannot be obtained by the moment equations, we have to work via the detailed distributed optimization technique, and in the following we shall show how this can be done.

## 10. Application of the Maximum Fractional Entropy Principle

*MEP applied to the entropy $\tilde{H}_\alpha(X)$*

*Statement of the problem.* In the setting of fractional entropy, according to Jaynes [11] maximum entropy, our purpose will be to determine the probability distribution $p(x)$ which maximizes the entropy $\langle -(\mathrm{Ln}_\alpha p)^{1/\alpha} \rangle$ under the condition that the mathematical expectations $\langle 1 \rangle$, $\langle X^r \rangle$ and $\langle X^s \rangle$ , $r > 0, s > 0$, have given values.

Introducing the Lagrange parameters $\lambda$ , $\mu$ and $\upsilon$, this amounts to maximize the augmented Hamiltonian

$$H_{aug} := \int_{\Re} p \left( -(\mathrm{Ln}_\alpha p)^{1/\alpha} + (\lambda x^r + \mu x^s + \upsilon) \right) dx. \tag{10.1}$$

Due to the presence of $\mathrm{Ln}_\alpha p$, we shall work in the setting of fractional calculus and we shall equate to zero the $\alpha th$ fractional variation of the augmented Hamiltonian to obtain the optimality condition

$$\left( -D_p^\alpha (\mathrm{Ln}_\alpha)^{1/\alpha} - pD_p^\alpha (\mathrm{Ln}_\alpha p)^{1/\alpha} + (\lambda x^r + \mu x^s + \upsilon) D_p^\alpha p \right) \; \delta p(x) \, dx = 0$$

which provides the equation (on taking account of (3.24))

$$-(1-\alpha)!(\mathrm{Ln}_\alpha p)^{1/\alpha} - \alpha + (1-\alpha)!(\lambda x^r + \mu x^s + \upsilon) = 0,$$

therefore

$$\mathrm{Ln}_\alpha p = \left( \lambda x^r + \mu x^s + \upsilon - \frac{\alpha}{(1-\alpha)!} \right)^\alpha , \tag{10.2}$$

and

$$p = E_\alpha \left\{ \left( \lambda x^r + \mu x^s + \upsilon - \frac{\alpha}{(1-\alpha)!} \right)^\alpha \right\} \tag{10.3}$$

Remark that when $\alpha = 1$, $r = 2$ and $s = 1$, $E_\alpha(.)$ turns to be the exponential $\exp(.)$ and we obtain exactly the wel known result corresponding to the normal law.

*Practical meaning of the fractional MEP.* This result can be understood as follows. If all we know about a probability density $p(x)$ is summarized in the knowledge of $\langle X^r \rangle$ and $\langle X^s \rangle$, and if further these values have been obtained via an observation process involving an information loss described by the presence of $\alpha$, then we should select (9.8) as the likely value of $p(x)$.

*On the modeling standpoint, according to the maximum entropy principle, in the presence of defect in observation (or fuzzy observation) the probability density defined by the Mittag-Leffler function would be more relevant than the Gaussian law.*

*MEP applied to the entropy $K_\alpha(X)$*

Assume now that we use $K_\alpha$ instead of $\tilde{H}_\alpha$, to have the augmented entropy

$$K_{aug}(X) = \int_{\Re} p(-\mathrm{Ln}_\alpha p + \lambda x^r + \mu x^s + \upsilon) dx. \tag{10.4}$$

Here we shall use the equality

$$\frac{d^\alpha \mathrm{Ln}_\alpha p}{dp^\alpha} = \left(\frac{\alpha}{(1-\alpha)!}\right)^2 \frac{(\mathrm{Ln}_\alpha p)^{1-(1/\alpha)}}{p^\alpha}$$

which can be obtained by combining (3.24) with the chain fractional derivative rule (3.10); and we shall obtain the optimization condition

$$(1-\alpha)!\,\mathrm{Ln}_\alpha p + \alpha^2 (\mathrm{Ln}_\alpha p)^{1-(1/\alpha)} - (1-\alpha)!(\lambda x^r + \mu x^s + \upsilon) = 0. \tag{10.5}$$

As in evidence, the closed solution cannot be obtained in the general case, but when $\alpha = 1/2$ one has $(1-\alpha)! = (1/2)!\Gamma(1+1/2) = \sqrt{\pi}/2$, therefore the equation

$$\frac{\sqrt{\pi}}{2}\left(\mathrm{Ln}_{1/2} p\right)^2 - \frac{\sqrt{\pi}}{2}(\lambda x^r + \mu x^s + \upsilon)\,\mathrm{Ln}_{1/2} p + \left(\frac{1}{2}\right)^2 = 0 \tag{10.6}$$

which provides

$$2\,\mathrm{Ln}_{1/2} p = \lambda x^r + \mu x^s + \upsilon \pm \left(\frac{\sqrt{\pi}}{2}\right)^{-1} \sqrt{\frac{\pi}{4}(\lambda x^r + \mu x^s + \upsilon)^2 - \frac{\sqrt{\pi}}{2}}. \tag{10.7}$$

## 11. Maximum Entropy via Distributed Parameter Optimization

*Statement of the problem.* We consider the problem of maximizing

$$\max H_\alpha(X,t) := -\int_0^T \int_{\Re} p(x,t)\ln p(x,t)\,dx(dt)^\alpha \tag{11.1}$$

subject to the dynamics

$$\partial_t^\alpha p(x,t) = -\alpha!\partial_x(fp) + (\alpha!/2)\partial_{xx}^2(g^2\sigma^2 p). \tag{11.2}$$

*Further comments* The equation (11.2) has been obtained in the Ref [20]. Regarding the total entropy, at first glance it could be selected in the form of an integral with respect to $dt$, but we have rather selected an integral w.r.t. $(dt)^\alpha$ to save some consistency between the two equations (11.1) and (11.2).

*Optimization.* On introducing the Lagrange parameter function $\lambda(x,t)$ the problem turns to maximize the augmented entropy

$$\tilde{H}_\alpha(X,t) = \int_0^T \int_{\Re} \left\{-p\ln p + \lambda\left[p_t^{(\alpha)} + \alpha!\partial_x(fp) - (\alpha!/2)\partial_{xx}^2(g^2\sigma^2 p)\right]\right\} dx(dt)^\alpha \tag{11.3}$$

Detailed calculations yield successively

$$\int_0^T \int_{\Re} \lambda p_t^{(\alpha)} dx(dt)^\alpha = \int_{\Re} [\lambda p]_0^T dx - \int_0^T \int_{\Re} \lambda_t^{(\alpha)} p\,dx(dt)^\alpha, \tag{11.4}$$

$$\alpha!\int_0^T (dt)^\alpha \int_{\Re} \lambda\,\partial_x(fp)dx = \alpha!\int_0^T [\lambda fp]_{-\infty}^{+\infty}(dt)^\alpha - \alpha!\int_0^T \int_{\Re} \lambda_x fp\,dx(dt)^\alpha, \tag{11.5}$$

$$(\alpha!/2)\int_0^T (dt)^\alpha \int_{\Re} \lambda \partial_x^2(g^2\sigma^2 p)dx = (\alpha!/2)\int_0^T \left[\lambda \partial_x(g^2\sigma^2 p)\right]_{-\infty}^{+\infty}(dt)^\alpha$$

$$-(\alpha!/2)\int_0^T \left[\lambda_x(g^2\sigma^2 p\right]_{-\infty}^{+\infty}(dt)^\alpha + (\alpha!/2)\int_0^T \int_{\Re} \lambda_{xx}(g^2\sigma^2 p)dx(dt)^\alpha, \qquad (11.6)$$

and on substituting (11.4) to (11.6) into (11.3) yields the final form

$$\tilde{H}_\alpha = \int_0^T \int_{\Re} L(p,\lambda,x,t)dx(dt)^\alpha + \int_{\Re} [\lambda p]_0^T dx + \alpha! \int_0^T [\lambda f p]_{-\infty}^{+\infty}(dt)^\alpha$$

$$-(\alpha!/2)\int_0^T \left[\lambda \partial_x(g^2\sigma^2 p)\right]_{-\infty}^{+\infty}(dt)^\alpha + (\alpha!/2)\int_0^T \left[\lambda_x(g^2\sigma^2 p\right]_{-\infty}^{+\infty}(dt)^\alpha,$$

with

$$L(p,\lambda,x,t) := -p\ln p - \lambda_t^{(\alpha)} p - \alpha!\lambda_x f p - (\alpha!/2)\lambda_{xx}(g^2\sigma^2 p). \qquad (11.7)$$

The necessary optimality condition reads $\partial L/\partial p = 0$ which provides

$$p(x,t) = \exp\left(-1 - \lambda_t^{(\alpha)} - \alpha!\lambda_x f - (\alpha!/2)\lambda_{xx} g^2\sigma^2\right), \qquad (11.8)$$

and the optimal solution is then defined as the solution of the equations (11.2) and (11.8).

The boundary conditions are defined by the condition

$$\int_{\Re} [\lambda \delta p]_0^T dx = 0$$

therefore

$$\lambda(T) = 0.$$

It is clear that it is not a simple task to find the solution of these equation, and this remains all a programme for future research.

*Remark of importance.* If instead of the additive equation(9.20) we have the condition

$$h^{1/\alpha}(\varphi\psi) \leqslant h^{1/\alpha}(\varphi) + h^{1/\alpha}(\psi)$$

or

$$h^{1/\alpha}(\varphi\psi) \geqslant h^{1/\alpha}(\varphi) + h^{1/\alpha}(\psi),$$

then we shall talk in terms of sub entropy and super entropy, and we shall expand the theory in the same manner. But here the equations so obtained will have different practical significances and will apply to different frameworks.

## 12. Concluding Remarks

Stochastic differential equations of fractional order are by now pervasive in many areas of science, and presently, to the best of our knowledge, there is no sound theory for this kind of equations, and it is likely that, due to the high discontinuities involved in these processes, if theory there is, it will not be the only one, but rather one among several ones. For instance with (standard) stochastic differential equations driven by (standard) Brownian motion, we have the Itô's theory and the Stratonovich's theory. Up to now, there is only one model for fractional Brownian motion, the copyrighted one, but we believe that thare is room for further points of view.There is not only one model of fractional Brownian motion, but several ones. This being the case, the modeling of real problems in the real physical world involving coarse-grained phenomenon results in this family of equations, and we need at least some approximate approaches to their solutions. Despite that the system state $x(t)$ itself cannot be easily defined, we can nevertheless obtain some estimates of its probability densities, and the present paper shows how this can be done simply, on a practical standpoint.

In this way of thought, by using path integral, we have calculated the probability density of the trajectory generated by the fractional system, a tool which will allow us to determine which trajectory is more likely to occur. By the night, the computer can systematically simulate trajectories and then calculate their probabilities, what may be of help to the practitioner for making decisions.

As the second result, we have obtained approximate expressions for the characteristic function of the state probability of some fractional stochastic systems. These expressions involve the state moments of the system state of which the values can be easily obtained by preliminary experiments.

The last result is related to the potential application of the maximum entropy principle in this kind of problem. Two cases have been considered, depending upon the structural definition of the system. Sometimes, it is easy to obtain the fractional differential equation of the state moments of the system, and the optimization is then a classical one with Lagrange parameter. Otherwise, in the more general case, we shall use the entropy of fractional order which takes account of defects in observation..

In a first step we have considered one-dimensional systems only, but at first glance, it seems that the extension to multi-dimensional systems should not give rise to much more problems.

We shall finish with a remark on the basic stochastic process to be used in the modelling of stochastic systems. There are two schools. Most mathematicians define stochastic processes by means of the Brownian motion which is as such considered as the basic generating stochastic process, but there is also another school of applied mathematicians who focus on physics and who use white noises to generate stochastic process. The present paper could be considered as being in this trend, and consider that the solution of the equation

$$dx = w(t)\,(dt)^{\alpha}$$

is fully defined in the framework of fractional calculus with modified Riemann-Liouville derivative.

There are many traps in the manipulation and the fractional calculus of non-differentiable functions, and we beg the leniency of the reader who will probably find here topics to be improved.

## Appendix

### Solution of a special fractional differential equation

Our purpose in this appendix is to solve the fractional differential equation

$$dy = yf(t)(dt)^{2\alpha}, \quad 1 < 2\alpha < 2. \tag{A.1}$$

and to this end we proceed as follows. Firstly, one can write

$$(dy)^{1/2} = \sqrt{y}\sqrt{f}\,(dt)^{\alpha}. \tag{A.2}$$

On taking account of the equality (provided by fractional derivative)

$$(dy)^{1/2} = \frac{1}{(1/2)!} d^{1/2}\left(y^{1/2}\right), \tag{A.3}$$

We can re-write (A.2) in the form

$$\frac{1}{(1/2)!} d^{1/2}\left(y^{1/2}\right) = \sqrt{y}\sqrt{f}(dt)^{\alpha}. \tag{A.4}$$

This suggests to make the transformation (or change of function) $u = \sqrt{y}$, which yields

$$d^{1/2}u = (1/2)!u\sqrt{f}(dt)^{\alpha}.$$

We then have successively

$$\frac{d^{1/2}u}{u} = (1/2)!\sqrt{f}(dt)^{\alpha},$$

$$\int_0^t \frac{d^{1/2}}{u} = (1/2)!\int_0^t \sqrt{f}(ds)^{\alpha},$$

$$\mathrm{Ln}_{1/2}\,u = (1/2)!\int_0^t \sqrt{f}(ds)^{\alpha},$$

$$u = E_{1/2}\left((1/2)!\int_0^t \sqrt{f}(ds)^{\alpha}\right)$$

therefore

$$y = E_{1/2}^2\left((1/2)!\int_0^t \sqrt{f}(ds)^{\alpha}\right).$$

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