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# COMPARISON RESULTS FOR PERIODIC BOUNDARY VALUE PROBLEM OF FRACTIONAL DIFFERENTIAL EQUATIONS

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*Abstract.* We study a linear fractional differential equation with a periodic boundary condition and give the explicit form of the solution and the corresponding Green's function. Using some properties of the Green's function we present some new comparison results.

## 1. Introduction

Initial value problems for fractional differential equations have been considered by some authors recently [9, 10, 20]. However the study of boundary value problems need further study, and in particular the periodic boundary value problem for fractional differential equations. In that direction we present in this paper some new results on differential inequalities related to the periodic boundary value problem for fractional differential equations.

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. The subject is as old as the differential calculus, and goes back to time when Leibniz and Newton invented differential calculus. The idea of fractional calculus has been a subject of interest not only among mathematicians, but also among physicists and engineers, appearing in rheology, viscoelasticity, electrochemistry, electromagnetism, etc. For details, see the monographs of Kilbas *et al.* [7], Kiryakova [8], Lakshmikantham *et al.* [11], Miller and Ross [12], Oldham *et al.* [15], Podlubny [16], and Samko *et al.* [18] and the references therein. Some recent contributions to the theory of fractional differential equations can be seen in [1, 2, 3, 4, 5, 6, 20].

Comparison principles are a crucial tool to study nonlinear differential equations and obtain approximate solutions.

In section 2 we recall some basic definitions and facts on fractional calculus, on linear fractional differential equations, and on Mittag-Leffler functions. Then we prove in section 3 some comparison principles for the periodic boundary problem. Our results improve previous results and include the integer case.

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## 2. Preliminaries

Let $\alpha \in (0,1]$, $T > 0$, $\lambda \in \mathbb{R}$, and $\sigma : (0,T] \to \mathbb{R}$.

We study the linear fractional differential equation

$$D^{\alpha} u(t) - \lambda u(t) = \sigma(t), \ \ t \in J := (0,T], \tag{1}$$

where

$$D^{\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} u(s) ds$$

is the usual Riemann-liouville fractional derivative of order $\alpha$ of a function $u(0,T] \to \mathbb{R}$. We note that

$$D^{\alpha} u(t) = \frac{d}{dt} I^{1-\alpha} u(t),$$

where $I^{1-\alpha}$ is the Riemann-Liouville fractional primitive of order $1-\alpha$ of $u$. For details on fractional calculus, we refer to the monographs [7, 11, 16, 18].

Denote the set of functions $u : (0,T] \to \mathbb{R}$ such that $u(t)$ is continuous on $(0,T]$, $t^{1-\alpha} u(t)$ is continuous on $[0,T]$, and $I^{1-\alpha} u(t)$ is continuously differentiable on $[0,T]$, by $\mathscr{E}$. By $\mathscr{F}$ we denote the set of functions $v : (0,T] \to \mathbb{R}$ such that $v(t)$ is continuous on $(0,T]$, and $t^{1-\alpha} v(t)$ is continuous on $[0,T]$,

Let $\sigma \in \mathscr{F}$. By a solution of (1) we mean a function $u \in \mathscr{E}$ satisfying (1).

We are interested in solutions of (1) satisfying the periodic boundary condition

$$\lim_{t \to 0^+} t^{1-\alpha} u(t) = T^{1-\alpha} u(T). \tag{2}$$

Note that for $\alpha = 1$ the solution $u \in C^1[0,T]$ and the boundary condition (2) is the usual periodic boundary condition $u(0) = u(T)$.

Consider the linear fractional differential equation (1) with the periodic boundary condition (2). To find the possible solutions of this linear problem, we write the general solution of (1) [7]:

$$u(t) = u_0 \Gamma(\alpha) t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^{\alpha}) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-s)^{\alpha}) \sigma(s) ds, \tag{3}$$

where $\lim_{t \to 0^+} t^{1-\alpha} u(t) = u_0 \in \mathbb{R}$ is the initial condition, and $E_{\alpha,\alpha}$ is the Mittag-Leffler function

$$E_{\alpha,\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha(k+1))}, z \in \mathbb{C}.$$

We note that $E_{\alpha,\alpha}(x) > 0$ for every $x \in \mathbb{R}$ and $E_{\alpha,\alpha}(x)$ is strictly increasing in $x$ [13, 17, 19]. Then for $x > 0$ we have that $E_{\alpha,\alpha}(-x) < E_{\alpha,\alpha}(0) = \frac{1}{\Gamma(\alpha)} < E_{\alpha,\alpha}(x)$.

For $\alpha = 1$ this representation is still valid since $\Gamma(1) = 1$ and $E_{1,1}(z) = e^z$. We thus obtain the solution

$$u(t) = u_0 e^{\lambda t} + \int_0^t e^{\lambda(t-s)} \sigma(s) ds,$$

of the initial problem for a linear first order ordinary differential equation

$$u'(t) - \lambda u(t) = \sigma(t),\, u(0) = u_0.$$

Now, for $t = T$ in (3), we get

$$T^{1-\alpha}u(T) = u_0\Gamma(\alpha)E_{\alpha,\alpha}(\lambda T^{\alpha}) + T^{1-\alpha}\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(T-s)^{\alpha})\sigma(s)ds. \quad (4)$$

Using the periodic condition (2), it is possible to determine a unique value of $u_0$ if and only if $1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda T^{\alpha}) \neq 0$.

Hence for $\lambda \neq 0$ we have that $E_{\alpha,\alpha}(\lambda T^{\alpha}) \neq \frac{1}{\Gamma(\alpha)}$ and

$$u_0 = \frac{T^{1-\alpha}}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda T^{\alpha})}\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(T-s)^{\alpha})\sigma(s)ds.$$

Substituting into (3) we have

$$u(t) = \frac{T^{1-\alpha}\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda T^{\alpha})}\cdot\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(T-s)^{\alpha})\sigma(s)ds$$
$$+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})\sigma(s)ds. \quad (5)$$

Let

$$c_{\alpha} = \frac{T^{1-\alpha}\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda T^{\alpha})}.$$

Hence for $\lambda \neq 0$ we have that the linear fractional periodic boundary value problem (1)–(2) has a unique solution given by

$$u(t) = \int_0^T G_{\lambda,\alpha}(t,s)\sigma(s)ds, \quad (6)$$

where $G_{\lambda,\alpha}(t,s) = c_{\alpha}E_{\alpha,\alpha}(\lambda t^{\alpha})E_{\alpha,\alpha}(\lambda(T-s)^{\alpha})t^{\alpha-1}(T-s)^{\alpha-1} + (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})$, for $0 \leqslant s \leqslant t \leqslant 1$; and $G_{\lambda,\alpha}(t,s) = c_{\alpha}E_{\alpha,\alpha}(\lambda t^{\alpha})E_{\alpha,\alpha}(\lambda(T-s)^{\alpha})t^{\alpha-1}(T-s)^{\alpha-1}$, for $0 \leqslant t < s \leqslant 1$.

For $\alpha = 1$ the problem is $u'(t) - \lambda u(t) = \sigma(t)$, $t \in [0,1]$, $u(0) = u(1)$ and it has a unique solution [14] given by (6). It coincides with the usual representation of the solution [14].

Now, if $\lambda < 0$, then $1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda T^{\alpha}) > 0$ since $E_{\alpha,\alpha}(\lambda T^{\alpha}) < \frac{1}{\Gamma(\alpha)} = E_{\alpha,\alpha}(0)$.

Analogously, if $\lambda > 0$, then $1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda T^{\alpha}) < 0$ since $E_{\alpha,\alpha}(\lambda T^{\alpha}) > \frac{1}{\Gamma(\alpha)} = E_{\alpha,\alpha}(0)$.

Now, for $\mu \in \mathbb{R}$, consider the nonhomogeneous boundary condition

$$\lim_{t\to 0^+} t^{1-\alpha}u(t) = T^{1-\alpha}u(T) + \mu. \quad (7)$$

The solution of the problem

$$D^{\alpha}u(t) - \lambda u(t) = 0,\; t \in (0,T] \quad (8)$$

together with the boundary conditions (7) is obtained as follows. The general solution of the fractional equation (8) is

$$u(t) = u_0\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})$$

with $u_0 = \lim_{t\to 0^+} t^{1-\alpha}u(t)$. Hence (7) leads to the equation

$$u_0 = T^{1-\alpha}u(T) + \mu = u_0\Gamma(\alpha)E_{\alpha,\alpha}(\lambda T^{\alpha}) + \mu.$$

This equation has a unique solution if and only if $\lambda \neq 0$. Thus,

$$u_0 = \frac{\mu}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda T^{\alpha})}.$$

Consequently, the unique solution of (8)–(7) is

$$\theta_\mu(t) = \frac{\mu\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda T^{\alpha})}t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha}). \tag{9}$$

LEMMA 2.1. *For $\lambda \neq 0$, the solution of (1)–(7) is given by*

$$u(t) = \int_0^T G_{\lambda,\alpha}(t,s)\sigma(s)ds + \theta_\mu(t).$$

*Proof.* The function $\int_0^T G_{\lambda,\alpha}(t,s)\sigma(s)ds$ is the solution of the problem (1)–(2), and $\theta_\mu(t)$ is the solution of (8)–(7). In consequence, the sum of these two functions is indeed a solution of (1)–(7).

To prove that it is the unique solution, let $u_1$ and $u_2$ be solutions of the problem (1)–(7), then $u = u_1 - u_2$ is solution of the problem $D^{\alpha}u - \lambda u = 0$ and the boundary condition (2). Using the representation (5) we see that $u(t) = 0,\ t \in [0,T]$, that is $u_1 = u_2$. □

## 3. Comparison Results

We now present our main results.

THEOREM 3.1. *Let $\lambda < 0$. Suppose that $u \in \mathscr{E}$ with*

$$D^{\alpha}u(t) - \lambda u(t) \geqslant 0,\ t \in (0,T]$$

*and*

$$\lim_{t\to 0^+} t^{1-\alpha}u(t) \geqslant T^{1-\alpha}u(T).$$

*Then $u(t) \geqslant 0,\ t \in [0,T]$.*

*Proof.* The function $u$ satisfies (1) with $\sigma(t) \geqslant 0,\ t \in (0,T]$ and (7) with $\mu \geqslant 0$. Using the representation of the solution given by Lemma 2.1, we see that $u(t) \geqslant 0$ for every $t \in [0,T]$. □

Analogously, we have

THEOREM 3.2. *Let $\lambda > 0$. Suppose that $u \in \mathscr{E}$ with*

$$D^{\alpha}u(t) - \lambda u(t) \geqslant 0,\ t \in (0,T]$$

*and*

$$\lim_{t \to 0^+} t^{1-\alpha}u(t) \leqslant T^{1-\alpha}u(T).$$

*Then $u(t) \leqslant 0$, $t \in [0,T]$.*

Using these results it is possible to obtain approximate solutions for nonlinear fractional problems with periodic boundary conditions via the monotone iterative technique. It will be useful for practitioners working on dynamical systems of fractional order and will be published elsewhere.

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