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VARIATION FORMULAE FOR TIME VARYING SINGULAR FRACTIONAL DELAY DIFFERENTIAL SYSTEMS

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Abstract. In this paper, the Caputo time varying singular fractional differential systems with delay and the Riemann-Liouville time varying singular fractional differential systems with delay are considered . By the $D-$ inverse matrix and $\overline{\alpha}-\delta$ function, two fundamental solutions are given. The variation formulae for time varying singular fractional differential systems with delay are obtained.

1. Introduction

In recent years, fractional differential systems have gained many scholar's attention [1]–[10]. We notice also that in practical systems, such as economic systems, biological systems, space-light industry systems and so on, due to the transmission of the signal or the mechanical transmission, we must study time delay [4], [5], [11]–[15]. In [14]–[18], we see that singular differential systems have obtained considerable importance due to their application in various sciences.

The study for the systems, which have three characters (fractional differential, delay and singular), is obviously very important.

In this article, we will consider singular Caputo time varying fractional differential systems with delay:

$$\begin{cases} E({}^cD^{\alpha}x(t)) = A(t)x(t) + B(t)x(t-1) + f(t), \, t \geqslant 0, \\ x(t) = \varphi(t), & -1 \leqslant t \leqslant 0. \end{cases} \tag{1}$$

and singular Riemann-Liouville time varying fractional differential systems with delay

$$\begin{cases} E(D^{\alpha}x(t)) = A(t)x(t) + B(t)x(t-1) + f(t), \, t \geqslant 0, \\ x(t) = \varphi(t), & -1 \leqslant t \leqslant 0. \end{cases} \tag{2}$$

where $x(t) \in R^n$ is a state vector; $E \in R^{n\times n}$ is a singular matrix; $A(t), B(t) \in R^{n\times n}$ are time varying matrix , $f(t) \in R^n$ is a sufficiently smooth vector function and $\varphi(t)$ is the

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initial state vector function; $0<\alpha<1$; ${}^cD^\alpha x(t)$ denotes α order-Caputo fractional derivative; $D^\alpha x(t)$ denotes α order-Riemann-Liouville fractional derivative.

Variation formulae are very useful to further study differential systems. In paper [19], for the constant variation formula and the general solution of degenerate neutral differential systems we gave some results. In [20], we had the variation formula of time varying singular delay differential systems. In this paper, the Caputo time varying singular fractional differential systems with delay and the Riemann-Liouville time varying singular fractional differential systems with delay are considered . By the $D-$inverse matrix and $\alpha-\delta$ function, two fundamental solutions are given. The variation formulae for time varying singular fractional differential systems with delay are obtained.

2. Definitions and lemmas

Let us start with some definitions and lemmas.

DEFINITION 1. Riemann-Liouville's fractional integral of $\alpha>0$ order for a function $f:R^+\to R$ is defined as

$$ {}_aD_t^{-\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-\theta)^{\alpha-1}f(\theta)d\theta. $$

Simply, we note ${}_0D_t^{-\alpha}f(t)$ as $D^{-\alpha}f(t)$.

DEFINITION 2. Caputo's fractional derivative of α order $(0<\alpha<1)$ for a function $f:R^+\to R$ is defined as

$$ {}_a^cD_t^{\alpha}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t\frac{f'(\theta)}{(t-\theta)^\alpha}d\theta. $$

Simply, we note ${}_0^cD_t^{\alpha}f(t)$ as ${}^cD^\alpha f(t)$.

DEFINITION 3. Riemann-Liouville's fractional derivative of α order $(0<\alpha<1)$ for a function $f:R^+\to R$ is defined as

$$ {}_aD_t^{\alpha}f(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t\frac{f(\theta)}{(t-\theta)^\alpha}d\theta. $$

Simply, we note ${}_0D_t^{\alpha}f(t)$ as $D^\alpha f(t)$.

REMARK. Caputo's fractional partial derivative of α order $(0<\alpha<1)$ for a function $f(t,s)$ can be defined as

$$ {}_a^cD_t^{\alpha}f(t,s)=\frac{\partial^\alpha f(t,s)}{\partial^\alpha t}=\frac{1}{\Gamma(1-\alpha)}\int_a^t\frac{\frac{\partial f(\theta,s)}{\partial\theta}}{(t-\theta)^\alpha}d\theta. $$

From [2], [3] we have

LEMMA 1. *When* $0<\alpha<1$,

$$ D^\alpha f(t)={}^cD^\alpha f(t)+\frac{f(0)}{\Gamma(1-\alpha)}t^{-\alpha}. \tag{3} $$

LEMMA 2. *When* $0<\alpha<1$,

$$ {}^{c}D^{\alpha}\int_{0}^{t}K(t,s)ds=\int_{0}^{t}({}_{s}^{c}D_{t}^{\alpha}K(t,s))ds+\frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}(\lim_{s\to\theta-0}K(\theta,s))\frac{1}{(t-\theta)^{\alpha}}d\theta. $$

Proof. From Definition 2, we have

$$ \begin{aligned} &{}^{c}D^{\alpha}\textstyle\int_{0}^{t}K(t,s)ds\\ &\quad=\frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}\frac{\left(\int_{0}^{\theta}K(\theta,s)ds\right)'}{(t-\theta)^{\alpha}}d\theta\\ &\quad=\frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}\frac{\int_{0}^{\theta}\frac{\partial K(\theta,s)}{\partial\theta}ds+\lim_{s\to\theta-0}K(\theta,s)}{(t-\theta)^{\alpha}}d\theta\\ &\quad=\frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}\frac{\int_{0}^{\theta}\frac{\partial K(\theta,s)}{\partial\theta}ds}{(t-\theta)^{\alpha}}d\theta+\frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}(\lim_{s\to\theta-0}K(\theta,s))\frac{1}{(t-\theta)^{\alpha}}d\theta\\ &\quad=\frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}ds\int_{s}^{t}\frac{\frac{\partial K(\theta,s)}{\partial\theta}}{(t-\theta)^{\alpha}}d\theta+\frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}(\lim_{s\to\theta-0}K(\theta,s))\frac{1}{(t-\theta)^{\alpha}}d\theta. \quad\square \end{aligned} $$

LEMMA 3. *When* $0<\alpha<1$,

$$ {}_{a}^{c}D_{t}^{\alpha}({}_{a}D_{t}^{-\alpha}[x(t)])=x(t). $$

Proof. From [2], for $0<\beta\leqslant\alpha<1$, we have

$$ \begin{aligned} {}_{a}^{c}D_{t}^{\alpha}({}_{a}^{c}D_{t}^{-\beta}x(t)) &={}_{a}^{c}D_{t}^{\alpha-\beta}x(t)+\frac{1}{\Gamma(1-\alpha+\beta)}x(a)(t-a)^{(\alpha-\beta)}\\ &=\frac{1}{\Gamma(1-\alpha+\beta)}\int_{a}^{t}\frac{f'(\theta)}{(t-\theta)^{\alpha-\beta}}d\theta+\frac{1}{\Gamma(1-\alpha+\beta)}x(a)(t-a)^{(\alpha-\beta)}.\\ {}_{a}^{c}D_{t}^{\alpha}({}_{a}D_{t}^{-\alpha}[x(t)]) &=\frac{1}{\Gamma(1-\alpha+\alpha)}\int_{a}^{t}\frac{x'(\theta)}{(t-\theta)^{\alpha-\alpha}}d\theta+\frac{1}{\Gamma(1-\alpha+\alpha)}x(a)(t-a)^{(\alpha-\alpha)}\\ &=\int_{a}^{t}x'(\theta)d\theta+x(a)=x(t). \quad\square \end{aligned} $$

From [12], we have

LEMMA 4. *For any square matrix* E, *the Drazin inverse matrix* E^{d} *is exist and unique, and if the Jordan normalized form is*

$$ E=T\begin{pmatrix}J_{1}&0\\0&J_{0}\end{pmatrix}T^{-1}. $$

Here J_{0} *is a nilpotent matrix,* J_{1} *and* T *is an invertible matrix. Then*

$$ E^{d}=T\begin{pmatrix}J_{1}^{-1}&0\\0&0\end{pmatrix}T^{-1}. $$

3. Constant Variation Formula of the System (1)

We consider three systems

$$\begin{cases} E(^cD^\alpha x(t)) = A(t)x(t) + B(t)x(t-1) + EE^d f(t), & t \geqslant 0 \\ x(t) \equiv 0, & -1 \leqslant t \leqslant 0, \end{cases} \tag{4}$$

$$\begin{cases} E(^cD^\alpha x(t)) = A(t)x(t) + B(t)x(t-1) + (I - EE^d) f(t), & t \geqslant 0 \\ x(t) \equiv 0, & -1 \leqslant t \leqslant 0, \end{cases} \tag{5}$$

and

$$\begin{cases} E(^cD^\alpha x(t)) = A(t)x(t) + B(t)x(t-1), & t \geqslant 0 \\ x(t) = \varphi(t), & -1 \leqslant t \leqslant 0. \end{cases} \tag{6}$$

It's easy to see that

LEMMA 5. *Assume that* $x_1(t), x_2(t), x_3(t)$ *is the solution of the system* (4), (5), (6) *respectively, then* $x(t) = x_1(t) + x_2(t) + x_3(t)$ *is the solution of the system (1).*

DEFINITION 4. Let $X(t,s) \in R^{n\times n}$, and satisfy

$$\begin{cases} E(^cD^\alpha X(t,s)) = A(t)X(t,s) + B(t)X(t-1,s),\ t \geqslant s, \\ X(t,s) = \begin{cases} EE^d,\ t = s, \\ 0, \quad t < s, \end{cases} \end{cases} \tag{7}$$

we call $X(t,s)$ the fundamental solution matrix of the system (4). We also call $X(t,s)$ as the first fundamental solution of the system (1), the first fundamental solution for short.

To give the corresponding fundamental solution of the system (5), we define a new function:

DEFINITION 5. Let α $(0 < \alpha < 1)$, the function

$$\bar{\delta}^\alpha(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{\delta(\theta)}{(t-\theta)^\alpha} d\theta$$

is called $\alpha - \bar{\delta}$ function, where $\delta(t)$ is the δ-function.

DEFINITION 6. Let $Y(t,s) \in R^{n\times n}$, and satisfy

$$\begin{cases} E(^cD^\alpha Y(t,s)) = A(t)Y(t,s) + B(t)Y(t-1,s) + (I - EE^d)\bar{\delta}^\alpha(t-s), \\ Y(t,s) = \begin{cases} I - EE^d, & t = s, \\ 0, & -1 \leqslant t < s. \end{cases} \end{cases} \tag{8}$$

then $Y(t,s)$ is called the corresponding fundamental solution of the system (5). We also call $Y(t,s)$ as the second fundamental solution of the system (1), the second fundamental solution for short. There $\delta^{\alpha}(t)$ is the $\alpha-\delta$ function.

THEOREM 1. *Assume that $X(t.s)$ is the first fundamental solution, then the solution of the system (4)*

$$x(t)=\int_0^t (X(t,s))E^dD^{-\alpha}[f(s)]ds.$$

Proof. Let $K(t,s)=X(t,s)E^dD^{-\alpha}[f(s)]$. From Lemma 2, we have

$$\begin{aligned} E({}^cD^{\alpha}x(t)) &= E({}^cD^{\alpha}\textstyle\int_0^t(X(t,s))E^dD^{-\alpha}[f(s)]ds) \\ &= E({}^cD^{\alpha}\textstyle\int_0^t K(t,s)ds) \\ &= E\textstyle\int_0^t({}_s^cD_t^{\alpha}K(t,s))ds+E\frac{1}{\Gamma(1-\alpha)}\int_0^t(\lim\limits_{s\to\theta-0}K(\theta,s))\frac{1}{(t-\theta)^{\alpha}}d\theta. \end{aligned} \tag{9}$$

From (7), we have

$$\begin{aligned} E\textstyle\int_0^t({}_s^cD_t^{\alpha}K(t,s))ds &= E\textstyle\int_0^t({}_s^cD_t^{\alpha}X(t,s)E^dD^{-\alpha}[f(s)]ds \\ &= \textstyle\int_0^t(E({}_s^cD_t^{\alpha}X(t,s))E^dD^{-\alpha}[f(s)]ds \\ &= \textstyle\int_0^t(E({}^cD^{\alpha}X(t,s))E^dD^{-\alpha}[f(s)]ds \\ &= \textstyle\int_0^t((A(t)X(t,s)+B(t)X(t-1,s))E^dD^{-\alpha}[f(s)]ds \\ &= \textstyle\int_0^t((A(t)X(t,s))E^dD^{-\alpha}[f(s)])ds+\int_0^t((B(t)X(t-1,s))E^dD^{-\alpha}[f(s)])ds \\ &= A(t)\textstyle\int_0^t(X(t,s)E^dD^{-\alpha}[f(s)])ds+B(t)\int_0^t(X(t-1,s)E^dD^{-\alpha}[f(s)])ds \\ &= A(t)\textstyle\int_0^t(X(t,s)E^dD^{-\alpha}[f(s)])ds+B(t)\int_0^{t-1}(X(t-1,s)E^dD^{-\alpha}[f(s)]ds \\ &\quad +B(t)\textstyle\int_{t-1}^t(X(t-1,s)E^dD^{-\alpha}[f(s)])ds \\ &= A(t)\textstyle\int_0^t(X(t,s)E^dD^{-\alpha}[f(s)])ds+B(t)\int_0^{t-1}(X(t-1,s)E^dD^{-\alpha}[f(s)])ds \\ &= A(t)x(t)+B(t)x(t-1). \end{aligned} \tag{10}$$

From Lemma 3, we have

$$\begin{aligned} &E\frac{1}{\Gamma(1-\alpha)}\int_0^t(\lim_{s\to\theta-0}K(\theta,s))\frac{1}{(t-\theta)^{\alpha}}d\theta \\ &\quad= E\frac{1}{\Gamma(1-\alpha)}\int_0^t(\lim_{s\to\theta-0}X(\theta,s)E^dD^{-\alpha}[f(s)])\frac{1}{(t-\theta)^{\alpha}}d\theta \\ &\quad= E\frac{1}{\Gamma(1-\alpha)}\int_0^t(X(\theta,\theta-0)E^dD^{-\alpha}[f(\theta)])\frac{1}{(t-\theta)^{\alpha}}d\theta \\ &\quad= E\frac{1}{\Gamma(1-\alpha)}\int_0^t(EE^dE^dD^{-\alpha}[f(\theta)])\frac{1}{(t-\theta)^{\alpha}}d\theta \\ &\quad= EEE^dE^d\frac{1}{\Gamma(1-\alpha)}\int_0^t(D^{-\alpha}[f(\theta)])\frac{1}{(t-\theta)^{\alpha}}d\theta \\ &\quad= EE^d({}^cD^{\alpha}D^{-\alpha}[f(t)])=EE^df(t). \end{aligned} \tag{11}$$

From (9), (10) and (11), we have that

$$x(t)=\int_0^t (X(t,s))E^dD^{-\alpha}[f(s)]ds.$$

is the solution of the system (4). $\square$

LEMMA 6. *For any square matrix E, we have:*

$$(E+I)(I-EE^d)(I+E(I-EE^d))^{-1}=I-EE^d. \tag{10}$$

Proof. From Lemma 4, if $E=T^{-1}\begin{pmatrix} J_1 & 0 \\ 0 & J_0 \end{pmatrix}T$,

$$I-EE^d=T^{-1}\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}T,$$

$$E+I=T^{-1}\begin{pmatrix} J_1+I & 0 \\ 0 & J_0+I \end{pmatrix}T,$$

$$(E+I)(I-EE^d)=T^{-1}\begin{pmatrix} 0 & 0 \\ 0 & J_0+I \end{pmatrix}T.$$

$$\begin{aligned} I-EE^d &= T^{-1}\begin{pmatrix} 0 & 0 \\ 0 & J_0+I \end{pmatrix}TT^{-1}\begin{pmatrix} I & 0 \\ 0 & J_0+I \end{pmatrix}^{-1}T \\ &= (E+I)(I-EE^d)(I+E(I-EE^d))^{-1}. \quad \square \end{aligned}$$

From [2], the following expression holds: for $0\leqslant\alpha<1$ the Laplace transformation of $D^{-\alpha}f(t)$,

$$L(D^{-\alpha}f(t))=\lambda^{-\alpha}L[f(t)]. \tag{13}$$

LEMMA 7. *The Laplace transformation of $\alpha-\bar{\delta}$ function is*

$$L(\bar{\delta}^{\alpha}(t))=\lambda^{\alpha}.$$

Proof.

$$\begin{aligned} L(\bar{\delta}^{\alpha}(t)) &= \tfrac{1}{\Gamma(1-\alpha)} L\left(\tfrac{d}{dt}\int_0^t \tfrac{\delta(\theta)}{(t-\theta)^{\alpha}}d\theta\right) \\ &= \tfrac{\lambda}{\Gamma(1-\alpha)} L\left(\int_0^t \tfrac{\delta(\theta)}{(t-\theta)^{\alpha}}d\theta\right) \\ &= \tfrac{\lambda}{\Gamma(1-\alpha)} L(t^{\alpha})L(\delta(t)) \\ &= \tfrac{\lambda}{\Gamma(1-\alpha)} L(t^{\alpha}) \\ &= \tfrac{\lambda}{\Gamma(1-\alpha)} \int_0^{+\infty} t^{-\alpha}e^{-\lambda t}dt. \end{aligned}$$

Let $\xi = \lambda t$, we have

$$\begin{aligned} L(\bar{\delta}^{\alpha}(t)) &= \tfrac{\lambda}{\Gamma(1-\alpha)} \int_0^{+\infty} \left(\tfrac{\xi}{\lambda}\right)^{-\alpha} e^{-\xi}\tfrac{1}{\lambda}d\xi \\ &= \tfrac{\lambda}{\Gamma(1-\alpha)} \tfrac{1}{\lambda^{1-\alpha}} \int_0^{+\infty} \xi^{-\alpha}e^{-\xi}d\xi \\ &= \lambda^{\alpha}. \quad \square \end{aligned}$$

LEMMA 8. *For* $\bar{\delta}^{\alpha}(t)$,

$$\int_0^t \bar{\delta}^{\alpha}(t-s)D^{-\alpha}[f(s)]ds = f(t). \tag{14}$$

Proof. From (13) and Lemma 7, we have

$$\begin{aligned} L\left(\int_0^t \bar{\delta}^{\alpha}(t-s)D^{-\alpha}[f(s)]ds\right) &= L(\bar{\delta}^{\alpha}(t))L(D^{-\alpha}f(t)) \\ &= \lambda^{\alpha}\lambda^{-\alpha}L[f(t)] \\ &= L[f(t)]. \end{aligned}$$

That is

$$\int_0^t \bar{\delta}^{\alpha}(t-s)D^{-\alpha}[f(s)]ds = f(t). \quad \square$$

THEOREM 2. *Assume that $Y(t)$ is the second fundamental solution, then the solution of the system (5) can be expressed as*

$$x(t) = \int_0^t Y(t,s)(I+E(I-EE^d))^{-1}D^{-\alpha}[f(s)]ds.$$

Proof. Let $\bar{K}(t,s) = Y(t,s)(I+E(I-EE^d))^{-1}D^{-\alpha}[f(s)]$. From Lemma 2, we have

$$\begin{aligned} E({}^cD^{\alpha}x(t)) &= E({}^cD^{\alpha}\int_0^t Y(t,s)(I+E(I-EE^d))^{-1}D^{-\alpha}[f(s)]ds) \\ &= E({}^cD^{\alpha}\int_0^t \bar{K}(t,s)ds) \\ &= E\int_0^t ({}_s^cD_t^{\alpha}\bar{K}(t,s))ds + E\tfrac{1}{\Gamma(1-\alpha)}\int_0^t (\lim_{s\to\theta-0}\bar{K}(\theta,s))\tfrac{1}{(t-\theta)^{\alpha}}d\theta. \end{aligned} \tag{15}$$

From Lemma 8, we have

$$
\begin{aligned}
E\int_0^t ({}_s^cD_t^\alpha \bar{K}(t,s))ds &= E\int_0^t ({}_s^cD_t^\alpha Y(t,s))(I+E(I-EE^d))^{-1}D^{-\alpha}[f(s)]ds \\
&= \int_0^t E({}_s^cD_t^\alpha Y(t,s))(I+E(I-EE^d))^{-1}D^{-\alpha}[f(s)]ds \\
&= \int_0^t E({}^cD^\alpha Y(t,s))(I+E(I-EE^d))^{-1}D^{-\alpha}[f(s)]ds \\
&= \int_0^t (A(t)Y(t,s)+B(t)Y(t-1,s) \\
&\quad +(I-EE^d)\bar{\delta}^\alpha(t-s))(I+E(I-EE^d))^{-1}D^{-\alpha}[f(s)]ds \\
&= \int_0^t (A(t)Y(t,s))(I+E(I-EE^d))^{-1}D^{-\alpha}[f(s)]ds \\
&\quad +\int_0^t (B(t)Y(t-1,s)(I+E(I-EE^d))^{-1}D^{-\alpha}[f(s)]ds \\
&\quad +\int_0^t ((I-EE^d)\bar{\delta}^\alpha(t-s))(I+E(I-EE^d))^{-1}D^{-\alpha}[f(s)]ds \qquad (16)\\
&= \int_0^t (A(t)Y(t,s))(I+E(I-EE^d))^{-1}D^{-\alpha}[f(s)]ds \\
&\quad +\int_0^{t-1} (B(t)Y(t-1,s)(I+E(I-EE^d))^{-1}D^{-\alpha}[f(s)]ds \\
&\quad +(I-EE^d)(I+E(I-EE^d))^{-1}\int_0^t (\bar{\delta}^\alpha(t-s))D^{-\alpha}[f(s)]ds \\
&= A(t)\int_0^t (Y(t,s))(I+E(I-EE^d))^{-1}D^{-\alpha}[f(s)]ds \\
&\quad +B(t)\int_0^{t-1} (Y(t-1,s)(I+E(I-EE^d))^{-1}D^{-\alpha}[f(s)]ds \\
&\quad +(I-EE^d)(I+E(I-EE^d))^{-1}[f(t)] \\
&= A(t)x(t)+B(t)x(t-1)+(I-EE^d)(I+E(I-EE^d))^{-1}[f(t)].
\end{aligned}
$$

$$
\begin{aligned}
&E\frac{1}{\Gamma(1-\alpha)}\int_0^t (\lim_{s\to\theta-0}\bar{K}(\theta,s))\frac{1}{(t-\theta)^\alpha}d\theta \\
&= E\frac{1}{\Gamma(1-\alpha)}\int_0^t (\lim_{s\to\theta-0}Y(t,s)(I+E(I-EE^d))^{-1}D^{-\alpha}[f(s)]\frac{1}{(t-\theta)^\alpha}d\theta \\
&= E\frac{1}{\Gamma(1-\alpha)}\int_0^t (Y(\theta,\theta-0)(I+E(I-EE^d))^{-1}D^{-\alpha}[f(\theta)])\frac{1}{(t-\theta)^\alpha}d\theta \\
&= E\frac{1}{\Gamma(1-\alpha)}\int_0^t ((I-EE^d)(I+E(I-EE^d))^{-1}D^{-\alpha}[f(\theta)])\frac{1}{(t-\theta)^\alpha}d\theta \qquad (17)\\
&= E(I-EE^d)(I+E(I-EE^d))^{-1}\frac{1}{\Gamma(1-\alpha)}\int_0^t (D^{-\alpha}[f(\theta)])\frac{1}{(t-\theta)^\alpha}d\theta \\
&= E(I-EE^d)(I+E(I-EE^d))^{-1}({}^cD^\alpha D^{-\alpha}[f(t)]) \\
&= E(I-EE^d)(I+E(I-EE^d))^{-1}f(t).
\end{aligned}
$$

From (15), (16), (17) and Lemma 5, we have

$$
\begin{aligned}
E({}^cD^\alpha x(t)) &= A(t)x(t)+B(t)x(t-1)+(I-EE^d)(I+E(I-EE^d))^{-1}f(t) \\
&\quad +E(I-EE^d)(I+E(I-EE^d))^{-1}f(t) \\
&= A(t)x(t)+B(t)x(t-1)+(I+E)(I-EE^d)(I+E(I-EE^d))^{-1}f(t) \\
&= A(t)x(t)+B(t)x(t-1)+(I-EE^d)f(t).
\end{aligned}
$$

So $x(t)=\int_0^t Y(t,s)(I+E(I-EE^d))^{-1}D^{-\alpha}[f(s)]ds$ is the solution of the system (5). □

From Lemma 5, Lemma 6, Theorem 1 and Theorem 2, we have the constant variation formula for singular fractional differential the systems with delay (1).

THEOREM 3. *Assume that $x(t,\varphi(t),0)$ is the solution of the system (6), then that the solution of the system (1) $x(t,\varphi(t),f(t))$ can be written as*

$$\begin{aligned} x(t) = & \int_0^t (X(t,s))E^d + Y(t,s)(I+E(I-EE^d))^{-1})D^{-\alpha}[f(s)]ds \\ & + x(t,\varphi(t),0), \end{aligned} \tag{18}$$

where $X(t,s)$ is the first fundamental solution, $Y(t,s)$ is the second fundamental solution.

4. Constant Variation Formula of the System (2)

For the singular Riemann-Liouville time varying fractional differential the systems with delay (2), we can give the constant variation formula.

Take (3) to the system (2), the system (2) will become

$$\begin{cases} E({}^cD^{\alpha}x(t)) = A(t)x(t) + B(t)x(t-1) + f(t) - \frac{\varphi(0)}{\Gamma(1-\alpha)}t^{-\alpha}, & t \geqslant 0, \\ x(t) = \varphi(t), & -1 \leqslant t \leqslant 0. \end{cases} \tag{19}$$

THEOREM 4. *Assume that $x(t,\varphi(t),0)$ is the solution of the system (6), then that the solution $x(t,\varphi(t),f(t))$ of the singular Riemann-Liouville time varying fractional differential the systems with delay (2) can be written as*

$$\begin{aligned} x(t) = & \int_0^t (X(t,s))E^d + Y(t,s)(I+E(I-EE^d))^{-1})D^{-\alpha}[f(s)]ds \\ & -\varphi(0)\int_0^t (X(t,s))E^d + Y(t,s)(I+E(I-EE^d))^{-1})ds + x(t,\varphi(t),0), \end{aligned} \tag{20}$$

there $X(t,s)$ is the first fundamental solution, $Y(t,s)$ is the second fundamental solution.

Proof. From (19) and Theorem 3, we have

$$\begin{aligned} x(t) = & \int_0^t (X(t,s))E^d + Y(t,s)(I+E(I-EE^d))^{-1})D^{-\alpha}[f(s) - \tfrac{\varphi(0)}{\Gamma(1-\alpha)}s^{-\alpha}]ds \\ & + x(t,\varphi(t),0). \end{aligned}$$

For

$$D^{-\alpha}[s^{-\alpha}] = \frac{1}{\Gamma(\alpha)}\int_0^s (s-\theta)^{\alpha-1}\theta^{-\alpha}d\theta,$$

let $\xi = \frac{\theta}{s}$, we have

$$\begin{aligned} D^{-\alpha}[s^{-\alpha}] &= \tfrac{1}{\Gamma(\alpha)}\int_0^1 (1-\xi)^{\alpha-1}\xi^{-\alpha}d\xi = \tfrac{1}{\Gamma(\alpha)}B(\alpha,1-\alpha) \\ &= \tfrac{1}{\Gamma(\alpha)}\tfrac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(\alpha+(1-\alpha))} = \Gamma(1-\alpha). \end{aligned}$$

There $B(a,b)$ is the $\beta-$function [2]. So, we have

$$x(t) = \int_0^t (X(t,s))E^d + Y(t,s)(I+E(I-EE^d))^{-1})D^{-\alpha}[f(s) - \frac{\varphi(0)}{\Gamma(1-\alpha)}s^{-\alpha}]ds$$
$$+x(t,\varphi(t),0)$$
$$= \int_0^t (X(t,s))E^d + Y(t,s)(I+E(I-EE^d))^{-1})D^{-\alpha}[f(s)]ds$$
$$-\int_0^t (X(t,s))E^d + Y(t,s)(I+E(I-EE^d))^{-1})D^{-\alpha}[\frac{\varphi(0)}{\Gamma(1-\alpha)}s^{-\alpha}]ds + x(t,\varphi(t),0)$$
$$= \int_0^t (X(t,s))E^d + Y(t,s)(I+E(I-EE^d))^{-1})D^{-\alpha}[f(s)]ds$$
$$-\varphi(0)\int_0^t (X(t,s))E^d + Y(t,s)(I+E(I-EE^d))^{-1})ds + x(t,\varphi(t),0). \quad \square$$

THEOREM 5. *Assume that $x(t,\varphi(t),f(t))$ is the solution of the singular Caputo time varying fractional differential the systems with delay (1), $\bar{x}(t,\varphi(t),f(t))$ is the solution of the singular Riemann-Liouville time varying fractional differential the systems with delay (2), then*

$$\bar{x}(t,\varphi(t),f(t)) = x(t,\varphi(t),f(t))$$
$$-\varphi(0)\int_0^t (X(t,s))E^d + Y(t,s)(I+E(I-EE^d))^{-1})ds,$$

where $X(t,s)$ is the first fundamental solution, $Y(t,s)$ is the second fundamental solution.

Proof. It's easy to be proven by Theorem 3 and Theorem 4.

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