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HOMOTOPY PERTURBATION METHOD TO FRACTIONAL BIOLOGICAL POPULATION EQUATION

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Abstract. In this paper, the Homotopy perturbation method is successfully extended to solve fractional biological population model and numerical results are obtained. The fractional derivatives are described in the Caputo sense, some examples are provided. And the solutions of the equation are continuous with the parameter α.

1. Introduction

Fractional derivative have been extensively investigated due to their broad applications in mathematics, physics and engineering [12, 2, 9], such as anomalous transport in disordered systems, some percolations in porous media, and the diffusion of biological populations. But these nonlinear fractional differential equations are difficult to get their exact solutions [14, 7]. An effective method for solving such equations is needed. In this paper, the Homotopy Perturbation Method (HPM)is considered. This method which provides analytical approximate solution was first presented by He [3, 4] and applied to various nonlinear problems [5, 6, 8]. Odibat and Momani [10, 11], Wang [15] applied the HPM to nonlinear fractional equations which have nonlinear terms in the equations. We extend the HPM to time-fractional biological population model [13], a representative biological population diffusion equation is $u_t = u_{xx}^2 + u_{xx}^2 + \sigma(u)$, where $u(x,y,t)$ denotes the population density and $\sigma(u)$ represents the population supply due to births and deaths. In this paper, we propose a generalized time-fractional nonlinear biological population diffusion equation as follows

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + hu^a(1-ru^b), \quad t>0, \quad x,y \in R \tag{1}$$

with given initial condition $u(x,y,0) = f_0(x,y)$, and according to *Malthusian* law and *Verhulst* law, we consider a more general form of $\sigma(u) = hu^a(1-ru^b)$, where h,a,r,b are real numbers. When choose special value, they change to *Malthusian* law and *Verhulst* law.

The derivatives in Eq. (1) is the Caputo derivative. Linear and nonlinear population systems were solved in [13] and [1] by using Viriational Iteration Method (VIM) and Adomian Decomposition Method (ADM). However, one of the disadvantages of ADM

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is the inherent difficulty in calculating the Adomian polynomials. In this letter, we are interested in extending the applicability of HPM to population systems of fractional differential equation (1). To demonstrate the effectiveness of the HPM algorithm, several numerical examples of fractional biological population systems shall be presented.

2. Preliminaries

DEFINITION 1. A real function $f(t)$, $t>0$, is said to be in the space C_μ, $\mu \in R$, if there exists a real number $p>\mu$, such that $f(t)=t^p f_1(t)$, where $f_1(t)\in C(0,\infty)$, and it is said to be in the space C_μ^n if and only if $f^{(n)}\in C_\mu$, $n\in N$.

DEFINITION 2. The Riemann-Liouville fractional integral operator $J^\alpha(\alpha \geqslant 0)$ of a function $f\in C_\mu$, $\mu \geqslant -1$, is defined as

$$J^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau, \quad (\alpha \geqslant 0) \tag{2}$$

where $\Gamma(\cdot)$ is the well-known gamma function, and some properties of the operator J^α are as follows

$$J^\alpha J^\beta f(t)=J^{\alpha+\beta}f(t), \quad (\alpha \geqslant 0,\ \beta \geqslant 0) \tag{3}$$

$$J^\alpha t^\gamma=\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)}t^{\alpha+\gamma}, \quad (\gamma \geqslant -1) \tag{4}$$

DEFINITION 3. The Caputo fractional derivative D^α of a function $f(t)$ is defined as

$${}_0D_t^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(t)d\tau}{(t-\tau)^{\alpha+1-n}}, \quad (n-1<Re(\alpha)\leqslant n,\ n\in N) \tag{5}$$

the following are two basic properties of the Caputo fractional derivative

$${}_0D_t^\alpha t^\beta=\frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}t^{\beta-\alpha}, \tag{6}$$

$$(J^\alpha D^\alpha)f(t)=f(t)-\sum_{k=0}^{n-1}f^{(k)}(0^+)\frac{t^k}{k!}, \tag{7}$$

we have chosen the Caputo fractional derivative because it allows traditional initial and boundary conditions to be included in the formulation of the problem. And some other properties of fractional derivative can be found in [12, 2].

The organization of this paper is as follows. In Section 2, we review the procedure of HPM and apply this technique to Eq. (1). To show the efficiency of this method, we present some examples in section 3, and some numerical results are obtained. The last section is a short summary and discussion.

3. Analysis of the HPM

The Homotopy analysis method which provides an analytical approximate solution is applied to various nonlinear problems [3, 5, 8, 10, 15]. In this section, we extend HPM to Eq. (1), according to this method, we construct the following simple homotopy

$$(1-p)\left(\frac{\partial^\alpha u}{\partial t^\alpha}-\frac{\partial^\alpha u_0}{\partial t^\alpha}\right)+p\left[\frac{\partial^\alpha u}{\partial t^\alpha}-\frac{\partial^2 u^2}{\partial x^2}-\frac{\partial^2 u^2}{\partial y^2}-hu^a(1-ru^b)\right]=0, \quad (8)$$

or

$$\frac{\partial^\alpha u}{\partial t^\alpha}-\frac{\partial^\alpha u_0}{\partial t^\alpha}+p\left[\frac{\partial^\alpha u_0}{\partial t^\alpha}-\frac{\partial^2 u^2}{\partial x^2}-\frac{\partial^2 u^2}{\partial y^2}-hu^a(1-ru^b)\right]=0, \quad (9)$$

where u_0 is an initial approximation of Eq. (1), and $p \in [0,1]$ is an embedding parameter. In case $p=0$, (9) is a fractional differential equation, $\frac{\partial^\alpha u}{\partial t^\alpha}-\frac{\partial^\alpha u_0}{\partial t^\alpha}=0$, which is easy to solve; and when $p=1$, (9) turns out to be the original one (1). The basic assumption is that the solutions can be written as a power series in p

$$u=u_0+pu_1+p^2u_2+p^3u_3+\ldots, \quad (10)$$

the approximate solutions of the original equations can be obtained by setting $p=1$, i.e.

$$u=\lim_{p\to 1}\sum_{n=0}^{\infty}p^n u_n=u_0+u_1+u_2+u_3+\cdots, \quad (11)$$

institute (10) into (9) and comparing coefficients of terms with identical powers of p, lead to

$$p^0: \quad \frac{\partial^\alpha u_0}{\partial t^\alpha}=0, \quad (12)$$

$$p^1: \quad \frac{\partial^\alpha u_1}{\partial t^\alpha}+\frac{\partial^\alpha u_0}{\partial t^\alpha}-\frac{\partial^2 u_0^2}{\partial x^2}-\frac{\partial^2 u_0^2}{\partial y^2}-hu_0^a(1-ru_0^b)=0, \quad (13)$$

$$p^2: \quad \frac{\partial^\alpha u_2}{\partial t^\alpha}-\frac{\partial^2 2u_0u_1}{\partial x^2}-\frac{\partial^2 2u_0u_1}{\partial y^2}-hau_0^{a-1}u_1+hr(a+b)u_0^{a+b-1}u_1=0, \quad (14)$$

$$\vdots$$

Because of the knowledge of various perturbation methods that low order approximate solution leads to high accuracy, there requires no infinite series. Then after a series of recurrent calculation by using Mathematica software we will get approximate solutions of fractional biological population model. In section 4, we show some examples that the Homotopy perturbation method gives a very good approximation of the exact solution.

4. Numerical results

In order to assess the advantages and the accuracy of the Homotopy perturbation method presented in this paper for nonlinear fractional differential equation, we have applied it to the following three problems.

EXAMPLE 1. Consider Eq. (1) with $a=1$, $r=0$, corresponding to *Malthusian* law, we have the following fractional biological population equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^2 u^2}{\partial x^2}+\frac{\partial^2 u^2}{\partial y^2}+hu, \tag{15}$$

subject to the initial condition

$$u_o=\sqrt{xy}. \tag{16}$$

Substituting (10) into (9), and comparing coefficients of terms with identical powers of p, leads to

$$p^0: \qquad \frac{\partial^\alpha u_0}{\partial t^\alpha}=0, \tag{17}$$

$$p^1: \qquad \frac{\partial^\alpha u_1}{\partial t^\alpha}-\frac{\partial^2 u_0^2}{\partial x^2}-\frac{\partial^2 u_0^2}{\partial y^2}-hu_0=0, \tag{18}$$

$$p^2: \qquad \frac{\partial^\alpha u_2}{\partial t^\alpha}-\frac{\partial^2 2u_0u_1}{\partial x^2}-\frac{\partial^2 2u_0u_1}{\partial y^2}-hu_1=0, \tag{19}$$

$$\vdots$$

operating with Riemann-Liouville fractional operator J^α, which is the inverse operator of Caputo derivative D^α, in both sides of (17-19) the solution reads

$$u_o=\sqrt{xy}, \tag{20}$$

$$u_1=\frac{ht^\alpha}{\Gamma(1+\alpha)}\sqrt{xy}, \tag{21}$$

$$u_2=\frac{h^2t^{2\alpha}}{\Gamma(1+2\alpha)}\sqrt{xy}, \tag{22}$$

$$\vdots$$

$$u_n=\frac{h^nt^{n\alpha}}{\Gamma(1+n\alpha)}\sqrt{xy}, \tag{23}$$

Then the approximate solution in a series form is

$$u(x,y,t)=\sum_{n=0}^{\infty}u_n=\sqrt{xy}\sum_{n=0}^{\infty}\frac{(ht^\alpha)^n}{\Gamma(1+n\alpha)}=\sqrt{xy}E_\alpha(ht^\alpha), \tag{24}$$

where $E_\alpha(ht^\alpha)$ is Mittag-leffler function, defined as $E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(1+n\alpha)}$. As $\alpha\to 1$ we have

$$u(x,y,t)=\sqrt{xy}\sum_{n=0}^{\infty}\frac{(ht)^n}{\Gamma(1+n)}=\sqrt{xy}e^{ht}, \tag{25}$$

which is an exact solution to the standard form biological population equation. The evolution results for the exact solution (25) and the approximate solution (24), for the case $\alpha = 1$, are shown in Fig. 1. It can be seen from Fig. 1 that the solution obtained by the HPM is nearly identical with the exact solution. Fig. 2 show the approximate solutions when $\alpha = 0.9$ and $\alpha = 0.5$ respectively. From Fig. 2, it is easy to conclude that the approximate solution of fractional biological population model is continuous with the parameter α.

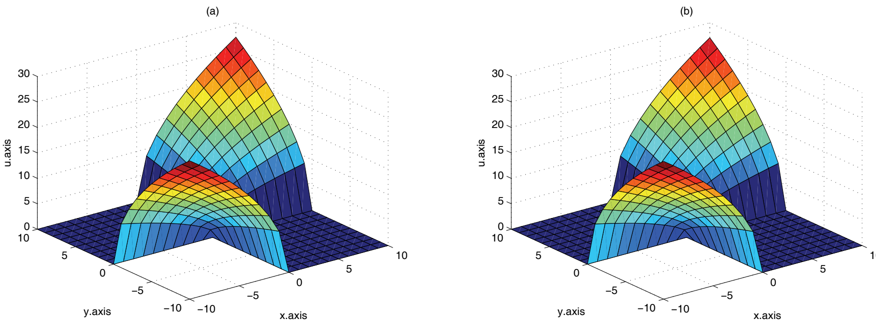

Fig. 1. The surface shows the solution $u(x,y,t)$ for(15): (a) exact solution (25); (b) numerical solution (24), when $h = 0.1,\ t = 10$.

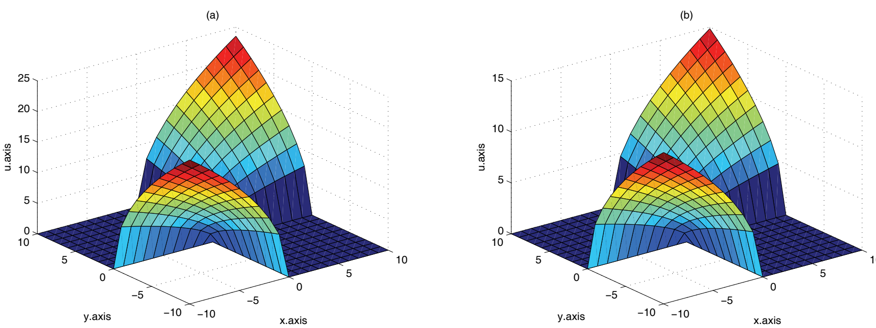

Fig. 2. The surface shows the approximate solution $u(x,y,t)$ for (15): $(a)\alpha = 0.9; (b)\alpha = 0.5$, when $h = 0.1,\ t = 10$.

EXAMPLE 2. Consider Eq. (1) with $a = 1$, $b = 1$, this leads to *Verhulst* law, and we have the following fractional biological population equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + hu(1 - ru), \tag{26}$$

subject to the initial condition $u_o = e^{\sqrt{\frac{hr}{8}}(x+y)}$, by using (10) and (9), we now successively obtain

$$u_o = e^{\sqrt{\frac{hr}{8}}(x+y)}, \tag{27}$$

$$u_1 = \frac{ht^{\alpha}}{\Gamma(1+\alpha)} e^{\sqrt{\frac{hr}{8}}(x+y)}, \tag{28}$$

$$u_2 = \frac{h^2 t^{2\alpha}}{\Gamma(1+2\alpha)} e^{\sqrt{\frac{hr}{8}}(x+y)}, \tag{29}$$

$$\vdots$$

Then the approximate solution in a series form is

$$u(x,y,t) = e^{\sqrt{\frac{hr}{8}}(x+y)} \sum_{n=0}^{\infty} \frac{(ht^{\alpha})^n}{\Gamma(1+n\alpha)} = e^{\sqrt{\frac{hr}{8}}(x+y)} E_{\alpha}(ht^{\alpha}), \tag{30}$$

as $\alpha \to 1$ we have

$$u(x,y,t) = e^{\sqrt{\frac{hr}{8}}(x+y)+ht}, \tag{31}$$

which is an exact solution of the integer order biological population.

EXAMPLE 3. Consider Eq. (1) with $a=-1$, $b=1$, we have the following fractional biological population equation

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + hu^{-1} - hr, \tag{32}$$

and the initial condition $u_o = \sqrt{\frac{hr}{4}x^2 + \frac{hr}{4}y^2 + y + 5}$, by using (10) and (9), we now obtain

$$u_o = \sqrt{\frac{hr}{4}x^2 + \frac{hr}{4}y^2 + y + 5}, \tag{33}$$

$$u_1 = \frac{ht^{\alpha}}{\Gamma(\alpha+1)\sqrt{\frac{hr}{4}x^2 + \frac{hr}{4}y^2 + y + 5}}, \tag{34}$$

$$u_2 = \frac{-2h^2 t^{2\alpha}}{\Gamma(1+2\alpha)(\sqrt{\frac{hr}{4}x^2 + \frac{hr}{4}y^2 + y + 5})^3}, \tag{35}$$

$$\vdots$$

Then the approximate solution in a series form is

$$u(x,y,t) = u_0 + \frac{ht^{\alpha}}{u_0} \sum_{n=0}^{\infty} \frac{n+1}{\Gamma(1+(n+1)\alpha)} \left(\frac{-ht^{\alpha}}{u_0^2}\right)^n, \tag{36}$$

as $\alpha \to 1$ we have

$$u(x,y,t) = u_0 + \frac{ht}{u_0} exp\left(\frac{-ht}{u_0^2}\right), \tag{37}$$

which is an exact solution of the integer order biological population. The evolution results for the exact solution (37) and the numerical solution (36), for the special case

Table 1: *Comparison of the exact and numerical values by HPM for* $t = 10$

(x, y)	exact value of Eq. (32)	numerical value by HPM	absolute error by HPM
(-20, -20)	20.8806	20.8806	0
(-10, -10)	10.4162	10.4162	9.3969e-013
(0, 0)	2.4384	2.4384	9.1337e-007
(10, 10)	11.3357	11.3357	4.3698e-013
(20, 20)	21.8174	21.8174	0

Table 2: *Comparison of the exact and numerical values by HPM for* $t = 20$

(x, y)	exact value of Eq. (32)	numerical value by HPM	absolute error by HPM
(-20, -20)	20.9045	20.9045	5.6843e-014
(-10, -10)	10.4638	10.4638	3.0036e-011
(0, 0)	2.6022	2.6022	2.8660e-005
(10, 10)	11.3795	11.3795	1.3941e-011
(20, 20)	21.8403	21.8403	3.5527e-014

$\alpha = 1$, $h = 0.05$, $r = 45$, are summarized in Tables 1-2 for $t = 10$, $t = 20$, and absolute errors are also calculated. Those results shows that even for small $n(n = 3)$, the nth approximation has high accuracy. And from above procedure, we can easily conclude that the HPM is an efficient and simple tool to solve approximate solution of nonlinear fractional differential equations.

5. Conclusion

In this work, the HPM was applied to derive approximate analytical solutions of time fractional degenerate parabolic equations arising in the spatial diffusion of biological populations subject to some initial conditions. The reliability of HPM and reduction in computations give this method a wider applicability. The corresponding solutions are obtained according to the recurrence relation using Mathematica.

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