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ON THE STABILITY ANALYSIS OF WEIGHTED AVERAGE FINITE DIFFERENCE METHODS FOR FRACTIONAL WAVE EQUATION

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Abstract. In this article, a numerical study for the fractional wave equations is introduced by using a class of finite difference methods. These methods are extension of the weighted average methods for ordinary (non-fractional) wave equations. The stability analysis of the proposed methods is given by a recently proposed procedure similar to the standard John von Neumann stability analysis. Simple and accurate stability criterion valid for different discretization schemes of the fractional derivative, arbitrary weight factor, and arbitrary order of the fractional derivative, is given and checked numerically. Numerical test example and comparisons have been presented for clarity.

1. Introduction

In the last few years, there are many studies for the fractional differential equations, because of their important applications in many areas like physics, medicine and engineering, and this field, fractional calculus allows us to study fractal phenomena which can not be studied by the ordinary case. There are many applications of the fractional differential equations see ([4]-[8], [10], [14], [16]-[22]), the studied models have received a great deal of attention like in the fields of viscoelastic materials [1], electrochemical processes [7], control theory [17], advection and dispersion of solutes in natural porous or fractured media [2], anomalous diffusion, signal processing and image filtering [12].
In this section, the definitions of Riemann-Liouville and the Grünwald-Letnikov fractional derivatives are given as follows:

DEFINITION 1. The Riemann-Liouville derivative of order α of the function $y(x)$ is defined by

$$D_x^{\alpha} y(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x \frac{y(\tau)}{(x-\tau)^{\alpha-n+1}} d\tau, \quad x > 0,$$

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where n is the smallest integer exceeding α and $\Gamma(.)$ is the Gamma function. If $\alpha = m \in \mathbb{N}$, then the above definition coincides with the classical m^{th} derivative $y^{(m)}(x)$.

DEFINITION 2.. The Grünwald-Letnikov definition for the fractional derivatives of order $\alpha > 0$ of the function $y(x)$ is defined by

$$D^{\alpha} y(x) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{[\frac{x}{h}]} w_k^{(\alpha)} y(x - hk), \quad x \geqslant 0,$$

where $[\frac{x}{h}]$ means the integer part of $\frac{x}{h}$ and $w_k^{(\alpha)}$ are the normalized Grünwald weights which are defined by $w_k^{(\alpha)} = (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix}$.

The Grünwald-Letnikov definition is simply a generalization of the ordinary discretization formula for integer order derivatives. The Riemann-Liouville and the Grünwald-Letnikov approaches coincide under relatively weak conditions; if $y(x)$ is continuous and $y'(x)$ is integrable in the interval $[0,x]$, then for every order $0 < \alpha < 1$ both the Riemann-Liouville and the Grünwald-Letnikov derivatives exist and coincide for any value inside the interval $[0,x]$. This fact of fractional calculus ensures the consistency of both definitions for most physical applications, where the functions are expected to be sufficiently smooth ([7], [10], [17]).

In this paper, we will study the numerical solution using the fractional WAM of the following fractional wave differential equation:

$$u_{tt}(x,t) = D_t^{2-\gamma} u_{xx}(x,t) + f(x,t), \tag{1}$$

on a finite domain $a < x < b$, $0 \leqslant t \leqslant T$, where $f(x,t)$ is the source term and $D_t^{2-\gamma}$ is the fractional derivative defined by the Riemann-Liouville operator, with $\gamma \in (1,2]$. Under the zero boundary conditions $u(a,t) = u(b,t) = 0$, and the following initial conditions

$$u(x,0) = g_1(x) \quad \text{and} \quad u_t(x,0) = g_2(x).$$

In the last few years appeared many papers to study this model (1) ([3], [9], [11], [13], [23], [24]).

The plan of the paper is as follows; In section 2 some approximate formulae of the fractional derivatives and numerical finite difference scheme are given. In section 3, we study the stability and the accuracy of the presented method. In section 4, numerical solutions and exact solutions of fractional wave problem are compared. The paper ends with some conclusions in section 5.

2. Finite difference scheme of the fractional wave differential equation

In this section, we will use the finite weighted average method to obtain the discretization finite difference formula of the wave equation (1). We use the following notations Δt and Δx, at time-step length and space-step length, respectively. The coordinates of the mesh points are $x_j = a + j\Delta x$ and $t_m = m\Delta t$, and the values of the solution $u(x,t)$ on these grid points are $u(x_j,t_m) \equiv u_j^m \simeq U_j^m$.

For more details about discretization in fractional calculus see [11].

In the first step, the ordinary differential operators are discretized as follows ([15], [18])

$$\frac{\partial^2 u}{\partial t^2}\mid_{x_j,t_m} = \delta_{tt} u_j^m + O((\Delta t)^2) \equiv \frac{u_j^{m-1} - 2u_j^m + u_j^{m+1}}{(\Delta t)^2} + O((\Delta t)^2), \tag{2}$$

and

$$\frac{\partial^2 u}{\partial x^2}\mid_{x_j,t_m} = \delta_{xx} u_j^m + O((\Delta x)^2) \equiv \frac{u_{j-1}^m - 2u_j^m + u_{j+1}^m}{(\Delta x)^2} + O((\Delta x)^2). \tag{3}$$

In the second step, the Riemann-Liouville operator is discretized as follows

$$D_t^{2-\gamma} u(x,t)\mid_{x_j,t_m} = \delta_{tt}^{2-\gamma} u_j^m + O((\Delta t)^p), \tag{4}$$

where

$$\delta_{tt}^{2-\gamma} u_j^m \equiv \frac{1}{(\Delta t)^{2-\gamma}} \sum_{k=0}^{[\frac{t_m}{\Delta t}]} w_k^{(2-\gamma)} u(x_j, t_m - k\Delta t) = \frac{1}{(\Delta t)^{2-\gamma}} \sum_{k=0}^{m} w_k^{(2-\gamma)} u_j^{m-k},$$

where $[\frac{t_m}{\Delta t}]$ means the integer part of $\frac{t_m}{\Delta t}$ and p is the order of the approximation which depends on the choice of $w_k^{(2-\gamma)}$. There are many choices of the weights $w_k^{(\alpha)}$ ([11], [17]) , so the above formula is not unique. Let us denote the generating function of the weights $w_k^{(\alpha)}$ by $w(z,\alpha)$, i.e.,

$$w(z,\alpha) = \sum_{k=0}^{\infty} w_k^{(\alpha)} z^k.$$

If

$$w(z,\alpha) = (1-z)^{\alpha}, \tag{5}$$

then (4) gives the backward difference formula of the first order, which is called the Grünwald-Letnikov formula. The coefficients $w_k^{(\alpha)}$ can be evaluated from (5) by the following formula

$$w_k^{(\alpha)} = (1 - \frac{\alpha+1}{k}) w_{k-1}^{(\alpha)}, \quad w_0^{(\alpha)} = 1. \tag{6}$$

For $\gamma = 2$, the operator $D_t^{2-\gamma}$ becomes the identity operator so that, the consistency of Eq.(4) requires $w_0^{(0)} = 1$, and $w_k^{(0)} = 0$ for $k \geqslant 1$, which in turn means that $w(z,0) = 1$.

Now, we are going to obtain the finite difference scheme of the fractional wave equation (1). To achieve this aim we evaluate this equation at the points of the grid (x_j, t_m)

$$[u_{tt}(x,t) - D_t^{2-\gamma} u_{xx}(x,t)]_{x_j, t_m} = f(x_j, t_m).$$

Then, we replace the second order time-derivative by the central difference formula (2) and replace the second order space-derivative by the three-point centered formula (3) with respect to the weighed average formula (4) at the times t_m and t_{m+1}

$$\delta_{tt} u_j^m - \{\lambda \delta_{tt}^{2-\gamma} \delta_{xx} u_j^m + (1-\lambda) \delta_{tt}^{2-\gamma} \delta_{xx} u_j^{m+1}\} - f(x_j, t_m) = T_j^m, \tag{7}$$

with λ being the weight factor and T_j^m is the resulting truncation error. The standard difference formula is given by

$$\delta_{tt} U_j^m - \{\lambda \delta_{tt}^{2-\gamma} \delta_{xx} U_j^m + (1-\lambda) \delta_{tt}^{2-\gamma} \delta_{xx} U_j^{m+1}\} - f(x_j, t_m) = 0. \tag{8}$$

Now, by substituting from the difference operators given by (2), (3) and (4) in (8), we can obtain the following scheme

$$-\phi U_{j-1}^{m+1} + (1+2\phi) U_j^{m+1} - \phi U_{j+1}^{m+1} = R, \tag{9}$$

where

$$\phi = (1-\lambda)\beta, \quad \beta = \frac{(\Delta t)^\gamma}{(\Delta x)^2}, \tag{10}$$

and

$$R = 2U_j^m - U_j^{(m-1)} + \beta \sum_{r=0}^{m} \left[\lambda w_r^{(2-\gamma)} + (1-\lambda) w_{r+1}^{(2-\gamma)}\right] \times \\ \times \left[U_{j-1}^{m-r} - 2U_j^{m-r} + U_{j+1}^{m-r}\right] + (\Delta t)^2 f(x_j, t_m). \tag{11}$$

Eq.(9) is the fractional weighted average difference scheme considered in this paper. Fortunately, Eq.(9) is a tridiagonal system that can be solved using the Thomas algorithm ([15], [18]). In the case of $\lambda = 1$ and $\lambda = \frac{1}{2}$ we have the backward Euler fractional quadrature method and the Crank-Nicholson fractional quadrature method, respectively, which have been studied, e.g., in [23], but at $\lambda = 0$ the scheme is called fully implicit.

3. Stability analysis and truncation error

3.1. Stability analysis

In this section, we use the John von Neumann method to study the stability analysis of the weighted average scheme (9) for the force free case.

THEOREM 1. *The fractional weighted average methods derived in (9) are* ***conditionally*** *stable under the following stability criterion*

$$\beta\beta_{\times} \leqslant 1, \tag{12}$$

where $\beta_{\times} = (2\lambda-1)w(-1,2-\gamma)$.

Proof. By using (11), we can write (9) with free source term in the following form

$$-\phi U_{j-1}^{m+1} + (1+2\phi)U_j^{m+1} - \phi U_{j+1}^{m+1} - 2U_j^m + U_j^{m-1} \\ -\beta\sum_{r=0}^{m}[\lambda w_r^{(2-\gamma)} + (1-\lambda)w_{r+1}^{(2-\gamma)}][U_{j-1}^{m-r} - 2U_j^{m-r} + U_{j+1}^{m-r}] = 0. \tag{13}$$

In the fractional John von Neumann stability procedure, the stability of the fractional weighted average methods is decided by putting $U_j^m = \xi_m e^{\mathrm{i}qj\Delta x}$. Inserting this expression into the weighted average difference scheme (13), we obtain

$$(1+2\phi)\xi_{m+1}e^{\mathrm{i}qj\Delta x} + \xi_{m-1}e^{\mathrm{i}qj\Delta x} - 2\xi_m e^{\mathrm{i}qj\Delta x} - \phi\xi_{m+1}e^{\mathrm{i}q(j-1)\Delta x} - \phi\xi_{m+1}e^{\mathrm{i}q(j+1)\Delta x} \\ -\beta\sum_{r=0}^{m}[\lambda w_r^{(2-\gamma)} + (1-\lambda)w_{r+1}^{(2-\gamma)}][e^{\mathrm{i}q(j-1)\Delta x} - 2e^{\mathrm{i}qj\Delta x} + e^{\mathrm{i}q(j+1)\Delta x}]\xi_{m-r} = 0, \tag{14}$$

substituting by $\phi = (1-\lambda)\beta$ and dividing (14) by $e^{\mathrm{i}qj\Delta x}$, we get

$$(1+2(1-\lambda)\beta)\xi_{m+1} + \xi_{m-1} - 2\xi_m - (1-\lambda)\beta\xi_{m+1}e^{-\mathrm{i}q\Delta x} - (1-\lambda)\beta. \\ \xi_{m+1}e^{\mathrm{i}q\Delta x} - \beta\sum_{r=0}^{m}[\lambda w_r^{(2-\gamma)} + (1-\lambda)w_{r+1}^{(2-\gamma)}][e^{-\mathrm{i}q\Delta x} - 2 + e^{\mathrm{i}q\Delta x}]\xi_{m-r} = 0.$$

Using the known Euler's formula $e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$, and under some simplifications we have

$$[1 + 4(1-\lambda)\beta\sin^2(\frac{q\Delta x}{2})]\xi_{m+1} + \xi_{m-1} - 2\xi_m \\ + 4\beta\sin^2(\frac{q\Delta x}{2})\sum_{r=0}^{m}[\lambda w_r^{(2-\gamma)} + (1-\lambda)w_{r+1}^{(2-\gamma)}]\xi_{m-r} = 0. \tag{15}$$

In the John von Neumann method, the stability analysis is carried out using the amplification factor η defined by

$$\xi_{m+1} = \eta\xi_m.$$

Sure, η depends on m. But, let us assume that, as in [23], η is independent of time. Then, inserting this expression into Eq.(15) one gets

$$[1+4(1-\lambda)\beta\sin^2(\frac{q\Delta x}{2})]\eta\xi_m+\eta^{-1}\xi_m-2\xi_m \\ +4\beta\sin^2(\frac{q\Delta x}{2})\sum_{r=0}^{m}[\lambda w_r^{(2-\gamma)}+(1-\lambda)w_{r+1}^{(2-\gamma)}]\eta^{-r}\xi_m=0,$$

divided by ξ_m to obtain the following formula of η

$$\eta=\frac{2-\eta^{-1}-4\beta\sin^2(\frac{q\Delta x}{2})\sum_{r=0}^{m}[\lambda w_r^{(2-\gamma)}+(1-\lambda)w_{r+1}^{(2-\gamma)}]\eta^{-r}}{1+4(1-\lambda)\beta\sin^2(\frac{q\Delta x}{2})}.$$

The mode will be stable as long as $|\,\eta\,|\leqslant 1$, i.e.,

$$-1\leqslant\frac{2-\eta^{-1}-4\beta\sin^2(\frac{q\Delta x}{2})\sum_{r=0}^{m}[\lambda w_r^{(2-\gamma)}+(1-\lambda)w_{r+1}^{(2-\gamma)}]\eta^{-r}}{1+4(1-\lambda)\beta\sin^2(\frac{q\Delta x}{2})}\leqslant 1,$$

considering the time-independent limit value $\eta=-1$ and since

$$1+4(1-\lambda)\beta\sin^2(\frac{q\Delta x}{2})>0,$$

then

$$-4-4(1-\lambda)\beta\sin^2(\frac{q\Delta x}{2})+4\beta\sin^2(\frac{q\Delta x}{2})\sum_{r=0}^{m}[\lambda w_r^{(2-\gamma)}+(1-\lambda)w_{r+1}^{(2-\gamma)}](-1)^{-r}\leqslant 0.$$

From the above equation we can obtain

$$(2\lambda-1)\sum_{r=0}^{m}(-1)^r w_r^{(2-\gamma)}+(-1)^m(\lambda-1)w_{m+1}^{(2-\gamma)}\leqslant\frac{1}{\beta\sin^2(\frac{q\Delta x}{2})},$$

put

$$\beta_m^{\times}=(2\lambda-1)\sum_{r=0}^{m}(-1)^r w_r^{(2-\gamma)}+(-1)^m(\lambda-1)w_{m+1}^{(2-\gamma)}, \tag{16}$$

one finds that the mode is stable when

$$\beta_m^{\times}\leqslant\frac{1}{\beta\sin^2(\frac{q\Delta x}{2})},$$

$\beta_m^{\times}$ depends on m, it turns out that $\beta_m^{\times}$ tends towards its limit value

$$\beta_{\times}=\lim_{m\to\infty}\beta_m^{\times}.$$

In this limit, the stability condition is

$$\frac{1}{\beta}\geqslant\beta_{\times}\sin^2(\frac{q\Delta x}{2}). \tag{17}$$

When $\lambda \neq \frac{1}{2}$, one can write $\beta_\times$ in terms of the generating function $w(z, 2-\gamma)$ of the coefficients $w_m^{(2-\gamma)}$

$$\beta_\times = (2\lambda - 1)w(-1, 2-\gamma). \tag{18}$$

Note that $w(-1,2-\gamma)$ is always positive (see section 2). Because (18) is negative when $\lambda < \frac{1}{2}$, then (17) holds for all β (β is always positive, see (10)). Therefore, any WA method with $\lambda < \frac{1}{2}$ is stable. However, $\beta_\times$ is positive and finite when $\lambda > \frac{1}{2}$, so that Eq.(17) tell us that, for any WA method with $\lambda > \frac{1}{2}$, there always exist values of β for which this WA method is unstable. Finally, from Eq.(16), if $\lambda = \frac{1}{2}$ (Crank-Nicholson method) then

$$\beta_m^\times = (-1)^m(\lambda - 1)w_{m+1}^{(2-\gamma)}.$$

But $w_{m+1}^{(2-\gamma)} \to 0$ for $m \to \infty$ so that $\beta_\times = 0$, and one concludes from Eq.(17) that the fractional method is stable for all β.

Proceeding as usual in the John von Neumann method, one can write a simpler and more conservative stability criterion than that given by Eq.(17) replacing $\sin^2(\frac{q\Delta x}{2})$ by its highest value, i.e., making $\sin^2(\frac{q\Delta x}{2}) \to 1$. Then the stability conditions for the fractional WA difference scheme (9) can be summarized in the following way: A weighted average method with weight factor $0 \leqslant \lambda \leqslant \frac{1}{2}$ is always stable; when $\frac{1}{2} < \lambda \leqslant 1$, the method is stable if $\frac{1}{\beta} \geqslant \beta_\times$, with $\beta_\times$ given by Eq.(18). Because β is always positive and $(2\lambda - 1)$ is negative for $0 \leqslant \lambda \leqslant \frac{1}{2}$, therefore we find that the sufficient condition for the presented method is stable and this completes the proof of the theorem.

3.2. Truncation error

THEOREM 2. *Assuming that u is sufficiently smooth at the origin $t = 0$ and the initial boundary data for u are consistent, then the truncation error in Eq.(7) for the free source term (i.e., $f(x,t) = 0$) is defined by*

$$T_j^m = O((\Delta t)^p) + (\frac{1}{2} - \lambda)O(\Delta t) + O((\Delta x)^2) + \frac{1}{(\Delta t)^{2-\gamma}} w_{m+1}^{(2-\gamma)} \delta_{xx} u_j^{(0)}.$$

Proof. From the definition of the truncation error which given in Eq.(7), one gets

$$T_j^m = \delta_{tt} u_j^m - \{\lambda \delta_{tt}^{2-\gamma} \delta_{xx} u_j^m + (1-\lambda)\delta_{tt}^{2-\gamma}\delta_{xx} u_j^{m+1}\},$$

i.e.,

$$T_j^m = \delta_{tt} u_j^m - \frac{1}{(\Delta t)^{2-\gamma}} \sum_{r=0}^{m} w_r^{(2-\gamma)}[(1-\lambda)\delta_{xx} u_j^{m+1-r} + \lambda \delta_{xx} u_j^{m-r}]$$
$$- \frac{1}{(\Delta t)^{2-\gamma}}(1-\lambda) w_{m+1}^{(2-\gamma)} \delta_{xx} u_j^{(0)}. \tag{19}$$

But

$$\delta_{xx} u_j^{m+1-r} = u_{xx} + \frac{(\Delta x)^2}{12} u_{xxxx} + \frac{\Delta t}{2}[u_{xxt} + \frac{(\Delta x)^2}{12} u_{xxxxt} + ...] + \frac{(\Delta t)^2}{8} u_{xxtt} + ...,$$

and

$$\delta_{xx} u_j^{m-r} = u_{xx} + \frac{(\Delta x)^2}{12} u_{xxxx} - \frac{(\Delta t)}{2}[u_{xxt} + \frac{(\Delta x)^2}{12} u_{xxxxt} + ...] + \frac{(\Delta t)^2}{8} u_{xxtt} + ...,$$

where the partial derivatives are evaluated at the point (x_j, t_{m-k}). Inserting these expressions into Eq.(19) and taking into account Eq.(1) with free force, (4), one gets

$$\begin{aligned} T_j^m = O((\Delta t)^p) - (\frac{1}{2} - \lambda)\Delta t D_\tau^{2-\gamma} u_{xxt} - \frac{(\Delta x)^2}{12} D_\tau^{2-\gamma} u_{xxxx} \\ - \frac{(\Delta t)^2}{8} D_\tau^{2-\gamma} u_{xxtt} - \frac{1}{(\Delta t)^{2-\gamma}}(1-\lambda) w_{m+1}^{(2-\gamma)} \delta_{xx} u_j^{(0)} + ..., \end{aligned}$$

with $\tau = t_m + \frac{\Delta t}{2}$, i.e.,

$$T_j^m = O((\Delta t)^p) + (\tfrac{1}{2} - \lambda) O(\Delta t) + O((\Delta x)^2) + \tfrac{1}{(\Delta t)^{2-\gamma}} w_{m+1}^{(2-\gamma)} \delta_{xx} u_j^{(0)},$$

where the terms of order $O[(\Delta t)^{a+p}(\Delta x)^b]$ with $a+b+p>2$ have not been included. This completes the proof of the theorem.

From this result, we can conclude that the truncation error is of first order in the time step if $\gamma = 2$ and $p \geqslant 1$. For $\gamma = 2$, $\lambda = 0.5$, and $p \geqslant 2$ the method is of second order.

4. Numerical results

In this section, we will test the proposed method by considering a numerical test example. Consider the fractional wave equation (1) with the following source term

$$f(x,t) = 2\sin(\pi x) + \frac{\pi^2}{\Gamma(\gamma+1)}[2t^\gamma - \gamma t^{\gamma-1}]\sin(\pi x),$$

and the initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = -\sin(\pi x).$$

The exact analytical solution of Eq.(1) in this case is

$$u(x,t) = \sin(\pi x)(t^2 - t).$$

Tables 1 and 2 show the numerical solution using the fractional WAM, the exact solution and the relative error.

x	u_{exact}	u_{FDM}	relative error
0.1	-0.0772542485	-0.0768635765	0.0051
0.2	-0.1469463130	-0.1462032106	0.0051
0.3	-0.2022542485	-0.2012314559	0.0051
0.4	-0.2377641290	-0.2365617641	0.0051
0.5	-0.2500000000	-0.2487357587	0.0051
0.6	-0.2377641290	-0.2365617641	0.0051
0.7	-0.2022542485	-0.2012314559	0.0051
0.8	-0.1469463130	-0.1462032106	0.0051
0.9	-0.0772542485	-0.0768635765	0.0051

Table 1: Comparison of the numerical solution, the exact solution, and the relative error of the fractional wave problem (1) by means of the fractional WAM for $\lambda = \frac{1}{2}$, $\gamma = 1.75$, $\Delta x = \frac{1}{20}$, $\Delta t = \frac{1}{200}$, $\beta = 0.0376$, and the final time $T = 0.5$.

x	u_{exact}	u_{FDM}	relative error
0.1	-0.0579406864	-0.0593982540	0.0252
0.2	-0.1102097348	-0.1129821931	0.0252
0.3	-0.1516906864	-0.1555066480	0.0252
0.4	-0.1783230968	-0.1828090286	0.0252
0.5	-0.1875000000	-0.1922167879	0.0252
0.6	-0.1783230968	-0.1828090286	0.0252
0.7	-0.1516906864	-0.1555066480	0.0252
0.8	-0.1102097348	-0.1129821931	0.0252
0.9	-0.0579406864	-0.0593982540	0.0252

Table 2: Comparison of the numerical solution, the exact solution, and the relative error of the fractional wave problem (1) by means of the fractional WAM for $\lambda = 0$, $\gamma = 1.8$, $\Delta x = \frac{1}{50}$, $\Delta t = \frac{1}{400}$, $\beta = 0.05178$, and the final time $T = 0.25$.

From these tables, we can conclude that the proposed method gives accurate results and stable solution, which satisfies the theoretical results, where for the values in table 1, we found that the stability bound (12), $\beta\beta_{\times} = 0 \leqslant 1$, and in table 2, we found that the stability bound $\beta\beta_{\times} = -0.05949 \leqslant 1$.

Tables 3 and 4 show the magnitude of the maximum error between the numerical solution and the exact solution obtained by using the fractional WAM discussed above with different values of Δx and Δt and also show the stability bound.

Δx	Δt	maximum error	stability bound
$\frac{1}{5}$	$\frac{1}{50}$	0.011494	0
$\frac{1}{10}$	$\frac{1}{50}$	0.010766	0
$\frac{1}{10}$	$\frac{1}{100}$	0.003611	0
$\frac{1}{20}$	$\frac{1}{100}$	0.003308	0
$\frac{1}{20}$	$\frac{1}{150}$	0.001205	0
$\frac{1}{30}$	$\frac{1}{150}$	0.001151	0
$\frac{1}{30}$	$\frac{1}{200}$	0.000212	0
$\frac{1}{40}$	$\frac{1}{200}$	0.000194	0
$\frac{1}{40}$	$\frac{1}{210}$	0.000069	0
$\frac{1}{45}$	$\frac{1}{220}$	0.000046	0

Table 3: This table shows the maximum error and the stability bound with different values of Δx and Δt for $\lambda = 0.5$, $\gamma = 1.5$, and the final time $T = 0.2$.

Δx	Δt	maximum error	stability bound
$\frac{1}{10}$	$\frac{1}{50}$	0.013961	-0.159242
$\frac{1}{20}$	$\frac{1}{100}$	0.010640	-0.196050
$\frac{1}{20}$	$\frac{1}{150}$	0.008519	-0.098404
$\frac{1}{30}$	$\frac{1}{200}$	0.007364	-0.135768
$\frac{1}{50}$	$\frac{1}{250}$	0.006538	-0.258078
$\frac{1}{50}$	$\frac{1}{300}$	0.005869	-0.189296
$\frac{1}{60}$	$\frac{1}{400}$	0.004946	-0.167151
$\frac{1}{60}$	$\frac{1}{450}$	0.004602	-0.136820
$\frac{1}{65}$	$\frac{1}{470}$	0.004483	-0.149131
$\frac{1}{70}$	$\frac{1}{480}$	0.004430	-0.166876

Table 4: This table shows the maximum error and the stability bound with different values of Δx, Δt for $\lambda = 0$, $\gamma = 1.7$, and the final time $T = 0.4$.

From the Figure 1, we can see that the numerical solution is unstable, since the stability condition $\beta\beta_{\times} \leqslant 1$ (i.e., $\beta\beta_{\times} = 33.27 > 1$) is not satisfy.

All the numerical calculations in this paper were carried out using BDF1 formula for the coefficients $w_k^{(\alpha)}$. There are three reasons for this: first, in contrast with other formulae, BDF1 coefficients can be easily computed using the recursive relation (6); second, although this formula is only of order 1, this is not relevant because the truncation error has (except for $\lambda = \frac{1}{2}$) a term of order $(\Delta t)^2$ (see Eq.(19)); and, third, because higher-order BDF formulae involve practical problems that in some cases may lead to completely useless results [6].

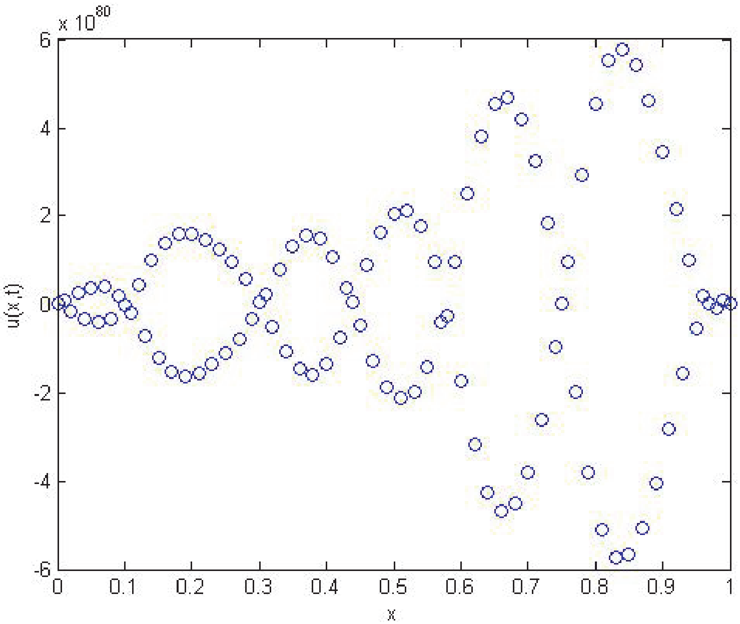

Figure 1. The behavior of the numerical solution of the proposed problem (1) by means of the fractional WAM for $\lambda = 1$, $\gamma = 1.54$, $\Delta x = \frac{1}{100}$, $\Delta t = \frac{1}{50}$, $\beta = 24.1873$, and the final time $T = 2$.

The stability limit $\beta_\times$ adopts an especiall simple form when BDF1 coefficients are used $\beta_\times = (2\lambda - 1)2^{2-\gamma}$, because from Eq.(5), $w(-1, 2-\gamma) = 2^{2-\gamma}$.

5. Conclusion

This paper presented a class of numerical methods for solving the fractional wave differential equations. This class of methods is very close to the weighted average finite difference method. Special attention is given to study the stability of proposed methods. To execute this aim we have resorted to the kind of fractional John von Neumann stability analysis. From the theoretical study we can conclude that, this procedure is suitable for the fractional weighted averages methods and lead to very good predictions for the stability bounds. The presented stability criterion of the fractional WAM depends strongly on the value of the weighting parameter λ; they are unconditionally stable for $0 < \lambda \leqslant \frac{1}{2}$ and conditionally stable for $\frac{1}{2} < \lambda \leqslant 1$.

Numerical solutions and exact solutions of the proposed problem are compared and the derived stability condition is checked numerically. From this comparison, we can see that, the numerical solutions are in excellent agreement with the exact solutions.

This work can be extended to solve numerically the multi dimensional fractional wave equation using the same procedure with the non-uniform meshes or use other weighted coefficient $w_k^{(\alpha)}$. Although we expect to accure some difficults.

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