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OPIAL TYPE INTEGRAL INEQUALITIES FOR FRACTIONAL DERIVATIVES

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Abstract. We consider a certain class of convex functions in an integral inequality. Mean value theorems, Cauchy means, exponential convexity, and monotonicity are proved. Applications of Riemann-Liouville fractional integral, Caputo fractional derivative and integral representation of Riemann-Liouville fractional derivative are given.

1. Introduction and Preliminaries

Our object is to derive some results which reflect the importance of Opial type inequalities, in the field of analysis. Inequalities of this type are investigated for example in [7] and [8]. For the sake of good understanding see the following notions and Theorem 1 in [11, p. 236, 237, 238].

We say that a function $u:[a,b]\to\mathbb{R}$ belongs to the class $U(v,K)$ if it admits the representation

$$u(x)=\int_a^x K(x,t)v(t)\,dt$$

where v is a continuous function and K is an arbitrary non-negative kernal such that $v(x)>0$ implies $u(x)>0$ for every $x\in[a,b]$.

THEOREM 1. *Let $h:[0,\infty)\to\mathbb{R}$ be a differentiable function such that for $q>1$ the function $h(x^{\frac{1}{q}})$ is convex and $h(0)=0$. Let $u\in U(v,K)$ where*

$$\left(\int_a^x (K(x,t))^p\,dt\right)^{\frac{1}{p}}\leqslant N,\qquad p^{-1}+q^{-1}=1.$$

Then

$$\int_a^b |u(x)|^{1-q}h'(|u(x)|)|v(x)|^q dx\leqslant \frac{q}{N^q}h\Big(N\Big(\int_a^b |v(x)|^q dx\Big)^{\frac{1}{q}}\Big). \tag{1.1}$$

If the function $h(x^{\frac{1}{q}})$ is concave, then the reverse of the inequality in (1.1) holds.

There is given exponential-convexity of a class of certain functions so, we have the following definition and proposition in [4].

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DEFINITION 1. A function $h:(a,b)\to\mathbb{R}$ is exponentially convex if it is continuous and

$$\sum_{i,j=1}^{n} u_i u_j h(x_i+x_j) \geqslant 0,$$

for all $n\in\mathbb{N}$ and all choices $u_i\in\mathbb{R}$, $i=1,2,\dots,n$ and $x_i\in(a,b)$, such that $x_i+x_j\in(a,b)$, $1\leqslant i,j\leqslant n$.

PROPOSITION 1. *Let $h:(a,b)\to\mathbb{R}$. The following are equivalent.*
(i) h is exponentially convex.
(ii) h is continuous and

$$\sum_{i,j=1}^{n} u_i u_j h\Big(\frac{x_i+x_j}{2}\Big) \geqslant 0,$$

for every $u_i\in\mathbb{R}$ and every $x_i,x_j\in(a,b)$, $1\leqslant i,j\leqslant n$.
(iii) h is continuous and for every $x_i\in(a,b)$, $i=1,2,\dots,n$,

$$\det\Big[h\Big(\frac{x_i+x_j}{2}\Big)\Big]_{i,j=1}^{k} \geqslant 0,\ k=1,2,\dots,n.$$

In [4] we also have the following corollary.

COROLLARY 1. *If $h:(a,b)\to(0,\infty)$ is exponentially convex function then h is a log-convex function.*

The presentation of the paper is as follows. The Section 2 contains mean value theorems, exponential convexity and Cauchy's means for a certain class of linear functionals. In Section 3 theorems for Riemann-Liouville fractional integral, Caputo fractional derivative as well as for integral representation of Riemann-Liouville fractional derivative are given.

2. Preparatory Inequalities

DEFINITION 2. Let $h:[0,\infty)\to\mathbb{R}$ be a real valued continuously differentiable function and u,v be continuous functions, for $q>1$ we define the linear functional $\alpha_h(u,v)$ as

$$\alpha_h(u,v)=\frac{q}{N^q}h\bigg(N\Big(\int_a^b |v(x)|^q dx\Big)^{\frac{1}{q}}\bigg)-\int_a^b |u(x)|^{1-q}h'(|u(x)|)|v(x)|^q dx. \qquad (2.1)$$

The following definition is given in [11, p. 7].

DEFINITION 3. If g is strictly monotonic, then f is said to be (strictly) convex with respect to g if $f\circ g^{-1}$ is (strictly) convex.

LEMMA 1. *Let $h \in C^2(I)$, $I \subseteq (0,\infty)$, and $g(x) = x^q$, $q > 1$ with*

$$m \leqslant \frac{\xi h''(\xi) - (q-1)h'(\xi)}{q^2 \xi^{2q-1}} \leqslant M \quad \text{for all } \xi \in I.$$

Then the functions ϕ_1, ϕ_2 defined as:

$$\phi_1(x) = \frac{Mx^{2q}}{2} - h(x), \quad \phi_2(x) = h(x) - \frac{mx^{2q}}{2},$$

are convex functions with respect to $g(x) = x^q$, that is $\phi_i(x^{\frac{1}{q}})$, $i = 1,2$, are convex.

Proof. Let $F(x) = \phi_1(x^{\frac{1}{q}}) = \frac{Mx^2}{2} - h(x^{\frac{1}{q}})$. We have

$$F''(x) = M - \frac{yh''(y) - (q-1)h'(y)}{q^2 y^{2q-1}} \geqslant 0,$$

where $y = x^{\frac{1}{q}}$. By Definition 3, this shows that ϕ_1 is convex with respect to $g(x) = x^q$, so $\phi_1(x^{\frac{1}{q}})$ is convex.

Similarly, if we put $G(x) = \phi_2(x^{\frac{1}{q}})$ then $G''(x) > 0$, so by Definition 3, we have that ϕ_2 is convex with respect to $g(x) = x^q$, so $\phi_2(x^{\frac{1}{q}})$ is convex.

THEOREM 2. *Let $h : [0,\infty) \to \mathbb{R}$ be a function which satisfies the assumptions of Theorem 1. If $h \in C^2(I)$, where $I \subseteq (0,\infty)$ is compact interval, then there exists $\xi \in I$ such that the following equality holds*

$$\alpha_h(u,v) = \frac{\xi h''(\xi) - (q-1)h'(\xi)}{2q^2 \xi^{2q-1}} \left(N^q \left(\int_a^b |v(x)|^q dx \right)^2 - 2\int_a^b |u(x)|^q |v(x)|^q dx \right). \tag{2.2}$$

Proof. Suppose that $\min(\psi(y)) = m$ and $\max(\psi(y)) = M$ where

$$\psi(y) = \frac{yh''(y) - (q-1)h'(y)}{q^2 y^{2q-1}}.$$

Using ϕ_1 instead of h in (1.1) we get

$$\frac{q}{N^q} h\Big(N\Big(\int_a^b |v(x)|^q dx \Big)^{\frac{1}{q}} \Big) - \int_a^b |u(x)|^{1-q} h'(|u(x)|) |v(x)|^q dx \leqslant \frac{qM}{2} \left(N^q \left(\int_a^b |v(x)|^q dx \right)^2 - 2\int_a^b |u(x)|^q |v(x)|^q dx \right). \tag{2.3}$$

Similarly, using ϕ_2 instead of h in (1.1) we get

$$\frac{q}{N^q}h\Big(N\Big(\int_a^b |v(x)|^q dx\Big)^{\frac{1}{q}}\Big) - \int_a^b |u(x)|^{1-q}h'(|u(x)|)|v(x)|^q dx$$
$$\geqslant \frac{qm}{2}\left(N^q\left(\int_a^b |v(x)|^q dx\right)^2 - 2\int_a^b |u(x)|^q|v(x)|^q dx\right). \quad (2.4)$$

By combining the above two inequalities and using the fact that

$$m \leqslant \frac{yh''(y)-(q-1)h'(y)}{q^2y^{2q-1}} \leqslant M$$

there exists $\xi \in I$ such that we get (2.2).

THEOREM 3. *Let $h_1, h_2 : [0,\infty) \to \mathbb{R}$ be functions which satisfy the assumptions of Theorem 1. If $h_1, h_2 \in C^2(I)$, where $I \subseteq (0,\infty)$ is compact interval and*

$$N^q\left(\int_a^b |v(x)|^q dx\right)^2 - 2\int_a^b |u(x)|^q|v(x)|^q dx \neq 0,$$

then there exists an $\xi \in I$ such that we have

$$\frac{\alpha_{h_1}(u,v)}{\alpha_{h_2}(u,v)} = \frac{\xi h_1''(\xi)-(q-1)h_1'(\xi)}{\xi h_2''(\xi)-(q-1)h_2'(\xi)},$$

provided the denominators are not equal to zero.

Proof. The proof is similar to the proof of such Theorems for example see in [5]. □

Throughout the paper we frequently use the following family of convex functions with respect to $g(x) = x^q$ $(q > 1)$ on $(0,\infty)$.

$$\varphi_s(x) = \begin{cases} \frac{q^2}{s(s-q)}x^s, & s \neq 0,q; \\ -q\log x, & s = 0; \\ qx^q\log x, & s = q. \end{cases} \quad (2.5)$$

In the following we use $\Gamma_{\varphi_s}(u,v)$ in the place of $\alpha_{\varphi_s}(u,v)$, when we put $h = \varphi_s$ in (2.1), that is

$$\Gamma_{\varphi_s}(u,v) = \begin{cases} \frac{q^2}{N^q s(s-q)}\Big(qN^s(\int_a^b |v(x)|^q dx)^{\frac{s}{q}} - sN^q\int_a^b |u(x)|^{s-q}|v(x)|^q dx\Big), & s \neq 0,q; \\ \frac{q}{N^q}\Big(-q\log(N(\int_a^b |v(x)|^q dx)^{\frac{1}{q}}) + N^q\int_a^b |u(x)|^{-q}|v(x)|^q dx\Big), & s = 0; \\ \frac{q^2}{N^q}\Big(N^q\int_a^b |v(x)|^q \log(N(\int_a^b |v(x)|^q dx)^{\frac{1}{q}}) \\ \quad + \int_a^b (1+q\log|u(x)|)|v(x)|^q dx\Big), & s = q. \end{cases} \quad (2.6)$$

THEOREM 4. *For $\Gamma_{\varphi_s}(u,v)$ we have:*

a) for every $n \in N$ and $p_i \in \mathbb{R}$ the matrix $A = \left[\Gamma_{\varphi_{\frac{p_i+p_j}{2}}}(u,v)\right]_{i,j=1}^{n}$, is a positive semi-definite matrix.

b) the function $s \mapsto \Gamma_{\varphi_s}(u,v)$ is exponentially convex.

c) if $\Gamma_{\varphi_s}(u,v)$ is positive, then the function $s \mapsto \Gamma_{\varphi_s}(u,v)$ is log-convex.

Proof. a) Define the function $f(x) = \sum_{i,j=1}^{n} u_i u_j \varphi_{p_{ij}}(x)$, where $p_{ij} = \frac{p_i+p_j}{2}$. Set

$$F(x) = f(x^{\frac{1}{q}}) = \sum_{i,j=1}^{n} u_i u_j \varphi_{p_{ij}}(x^{\frac{1}{q}}).$$

Then

$$F''(x) = \left(\sum_{i=1}^{n} u_i x^{\frac{p_i-2q}{2q}}\right)^2 \geqslant 0.$$

This implies that f is convex with respect to $g(x) = x^q$, and also $f(0) = 0$. So using this f in the place of h in (1.1) we have

$$\sum_{i,j=1}^{n} u_i u_j \Gamma_{\varphi_{p_{ij}}}(u,v) \geqslant 0. \tag{2.7}$$

Hence the matrix, $A = \left[\Gamma_{\varphi_{\frac{p_i+p_j}{2}}}(u,v)\right]_{n\times n}$ is positive semi-definite.

b) After some computation we have $\lim_{s\to 0} \Gamma_{\varphi_s}(u,v) = \Gamma_{\varphi_0}(u,v)$ and $\lim_{s\to q} \Gamma_{\varphi_s}(u,v) = \Gamma_{\varphi_q}(u,v)$, so $\Gamma_{\varphi_s}(u,v)$ is continuous. Then by (2.7) and Proposition 1 we conclude that $s \mapsto \Gamma_{\varphi_s}(u,v)$ is exponentially convex.

c) As $\Gamma_{\varphi_s}(u,v)$ is positive and exponentially convex, so by Corollary 1, $\Gamma_{\varphi_s}(u,v)$ it is log-convex.

If we put $h_1 = \varphi_s$, $h_2 = \varphi_r$ in Theorem 3, then we have a mean defined as:

$$M_{s,r}^{[q]}(u,v) = \left(\frac{\alpha_{\varphi_s}(u,v)}{\alpha_{\varphi_r}(u,v)}\right)^{\frac{1}{s-r}}, \quad s \neq r, \tag{2.8}$$

that is

$$M_{s,r}^{[q]}(u,v) = \left(\frac{r(r-q)}{s(s-q)} \frac{qN^s(\int_a^b |v(x)|^q dx)^{\frac{s}{q}} - sN^q \int_a^b |u(x)|^{s-q}|v(x)|^q dx}{qN^r(\int_a^b |v(x)|^q dx)^{\frac{r}{q}} - rN^q \int_a^b |u(x)|^{r-q}|v(x)|^q dx}\right)^{\frac{1}{s-r}}, \tag{2.9}$$

where $s,r \neq q$, $s \neq r$.

In limiting cases we have: when s goes to r

$$M_{r,r}^{[q]}(u,v) = \exp\left(\frac{A}{B} - \frac{2r-q}{r(r-q)}\right), \quad r \neq q, \tag{2.10}$$

where

$$A = qN^r(\int_a^b |v(x)|^q dx)^{\frac{r}{q}} \log(N(\int_a^b |v(x)|^q dx)^{\frac{1}{q}}) \\ - N^q(\int_a^b |u(x)|^{r-q}|v(x)|^q dx + r\int_a^b |u(x)|^{r-q} \log(|u(x)|)|v(x)|^q dx,$$

and

$$B = qN^r(\int_a^b |v(x)|^q dx)^{\frac{r}{q}} - rN^q \int_a^b |u(x)|^{r-q}|v(x)|^q dx.$$

In (2.9) when r goes to q we get for $s \neq q$

$$M_{s,q}^{[q]}(u,v) = M_{q,s}^{[q]}(u,v) = \\ \left(\frac{q(s(s-q)N^q)^{-1}(qN^s(\int_a^b |v(x)|^q dx)^{\frac{s}{q}} - sN^q \int_a^b |u(x)|^{s-q}|v(x)|^q dx)}{(q\int_a^b |v(x)|^q dx \log(N(\int_a^b |v(x)|^q dx)^{\frac{1}{q}}) - (\int_a^b |v(x)|^q dx + q\int_a^b \log(|u(x)|)|v(x)|^q dx))}\right)^{\frac{1}{s-q}}. \tag{2.11}$$

When s goes to q we have

$$M_{q,q}^{[q]}(u,v) = \exp\left(\frac{1}{2}\left(\frac{P}{Q} - \frac{2}{q}\right)\right), \tag{2.12}$$

where

$$P = qN^q(q\int_a^b |v(x)|^q dx(\log(N(\int_a^b |v(x)|^q dx)^{\frac{1}{q}}))^2 \\ - (2\int_a^b \log|u(x)||v(x)|^q dx + q\int_a^b (\log|u(x)|)^2|v(x)|^q dx)),$$

and

$$Q = qN^q(q\int_a^b |v(x)|^q dx \log(N(\int_a^b |v(x)|^q dx)^{\frac{1}{q}}) \\ - (\int_a^b |v(x)|^q dx + q\int_a^b \log|u(x)||v(x)|^q dx)).$$

Now we prove monotonicity of means.

THEOREM 5. *Let $t,s,l,m \in \mathbb{R}_+$ such that $t \leqslant l$, $s \leqslant m$. Then*

$$M_{t,s}^{[q]}(u,v) \leqslant M_{l,m}^{[q]}(u,v).$$

Proof. The following inequality holds for convex function φ see in [11, p. 4].

$$\frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leqslant \frac{\varphi(y_2) - \varphi(y_1)}{y_2 - y_1}, \tag{2.13}$$

where $x_1 \leqslant y_1$, $x_2 \leqslant y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$.

Since by Theorem 4, $\Gamma_s(u,v)$ is log-convex, we can put in (2.13) $\varphi = \log\Gamma_{\varphi_s}(u,v)$, $x_1 = s$, $x_2 = t$, $y_1 = l$, $y_2 = m$. We get for $s \neq t$, $l \neq m$

$$\frac{\log\Gamma_{\varphi_t}(u,v) - \log\Gamma_{\varphi_s}(u,v)}{t-s} \leqslant \frac{\log\Gamma_{\varphi_m}(u,v) - \log\Gamma_{\varphi_l}(u,v)}{m-l},$$

therefore we have

$$\left(\frac{\Gamma_{\varphi_t}(u,v)}{\Gamma_{\varphi_s}(u,v)}\right)^{\frac{1}{t-s}} \leqslant \left(\frac{\Gamma_{\varphi_m}(u,v)}{\Gamma_{\varphi_l}(u,v)}\right)^{\frac{1}{m-l}}. \tag{2.14}$$

From (2.14) we get our result for $t \neq s$, $l \neq m$ and for $t = s$, $l = m$; $t \neq s$, $l = m$; $t = s$, $l \neq m$ we can consider limiting cases.

3. Inequalities for Fractional Derivatives

DEFINITION 4. Let $\alpha > 0$. For any $f \in L(a,b)$ the Riemann-Liouville fractional integral of f of order α is defined by

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x \in [a,b]. \tag{3.1}$$

THEOREM 6. *Let $h : [0,\infty) \to \mathbb{R}$ be a function which satisfies the assumptions of Theorem 1. If $h \in C^2(I)$, where $I \subseteq (0,\infty)$ is compact interval, let $v \in C[a,b]$ has Riemann-Liouville fractional integral of order $\alpha > \frac{1}{q}$. Then there exists $\xi \in I$ such that*

$$\alpha_h(I_a^\alpha v, v) = \frac{\xi h''(\xi) - (q-1)h'(\xi)}{2q^2\xi^{2q-1}}\Bigg(\frac{(b-a)^{q\alpha-1}}{\Gamma^q(\alpha)(p\alpha-p+1)^{\frac{q}{p}}}\left(\int_a^b |v(x)|^q dx\right)^2 - 2\int_a^b |I_a^\alpha v(x)|^q |v(x)|^q dx\Bigg). \tag{3.2}$$

Proof. From Theorem 2, we have

$$\alpha_h(u,v) = \frac{\xi h''(\xi) - (q-1)h'(\xi)}{2q^2\xi^{2q-1}}\left(N^q\left(\int_a^b |v(x)|^q dx\right)^2 - 2\int_a^b |u(x)|^q |v(x)|^q dx\right), \tag{3.3}$$

and v has Riemann-Liouville fractional integral of order α, so

$$u(x) = I_a^\alpha v(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} v(t)\,dt, x \in [a,b].$$

Here

$$K(x,t) = \begin{cases} \frac{1}{\Gamma(\alpha)}(x-t)^{\alpha-1}, & a \leqslant t \leqslant x; \\ 0, & x < t \leqslant b. \end{cases}$$

Let

$$P(x) = \left(\int_a^x (K(x,t))^p dt\right)^{\frac{1}{p}} = \frac{(x-a)^{\alpha-\frac{1}{q}}}{\Gamma(\alpha)(p\alpha-p+1)^{\frac{1}{p}}}.$$

Then

$$P'(x) = \frac{(\alpha - \frac{1}{q})(x-a)^{(\alpha - \frac{1}{q} - 1)}}{\Gamma(\alpha)(p\alpha - p + 1)^{\frac{1}{p}}} \geqslant 0,$$

for $\alpha > \frac{1}{q}$, $x \in [a,b]$. Therefore, $P(x)$ is increasing in $[a,b]$, and

$$\max_{x \in [a,b]} (P(x)) = \frac{(b-a)^{\alpha - \frac{1}{q}}}{\Gamma(\alpha)(p\alpha - p + 1)^{\frac{1}{p}}},$$

for $\alpha > \frac{1}{q}$. It follows that

$$\left(\int_a^x K(x,t)^p dt\right)^{\frac{1}{p}} \leqslant \frac{(b-a)^{\alpha - \frac{1}{q}}}{\Gamma(\alpha)(p\alpha - p + 1)^{\frac{1}{p}}},$$

so we can choose

$$N = \frac{(b-a)^{\alpha - \frac{1}{q}}}{\Gamma(\alpha)(p\alpha - p + 1)^{\frac{1}{p}}},$$

and by putting the values of $u(x)$ and N in (3.3) we get $\alpha_h(I_a^\alpha v, v)$ as required in (3.2).

THEOREM 7. *Let $h_1, h_2 : [0,\infty) \to \mathbb{R}$ be functions which satisfy the assumptions of Theorem 1. If $h_1, h_2 \in C^2(I)$, where $I \subseteq (0,\infty)$ is compact interval, let $v \in C[a,b]$ has Riemann-Liouville fractional integral of order $\alpha > \frac{1}{q}$. Then there exists $\xi \in I$ such that*

$$\frac{\alpha_{h_1}(I_a^\alpha v, v)}{\alpha_{h_2}(I_a^\alpha v, v)} = \frac{\xi h_1''(\xi) - (q-1)h_1'(\xi)}{\xi h_2''(\xi) - (q-1)h_2'(\xi)}, \tag{3.4}$$

provided that denominators are not equal to zero.

Proof. By Theorem 3 we have

$$\frac{\alpha_{h_1}(u,v)}{\alpha_{h_2}(u,v)} = \frac{\xi h_1''(\xi) - (q-1)h_1'(\xi)}{\xi h_2''(\xi) - (q-1)h_2'(\xi)},$$

and from the proof of Theorem 6, we can easily get (3.4) with required conditions.

If v has Riemann-Liouville fractional integral of order α, $\alpha > \frac{1}{q}$, then (2.6) becomes

$$\Gamma_{\varphi_s}(I_a^\alpha v, v) = \begin{cases} \frac{q^2}{s(s-q)}\left(q\frac{(b-a)^{(s-q)(\alpha - \frac{1}{q})} D^{\frac{s}{q}}}{\Gamma^{s-q}(\alpha)(p\alpha - p+1)^{\frac{s-q}{p}}} - s\int_a^b |I_a^\alpha v(x)|^{s-q}|v(x)|^q dx\right), & s \neq 0, q; \\ q\left(-q\frac{\Gamma^q(\alpha)(p\alpha-p+1)^{\frac{q}{p}}}{(b-a)^{q\alpha-1}}\log\left(\frac{(b-a)^{\alpha-\frac{1}{q}} D^{\frac{1}{q}}}{\Gamma(\alpha)(p\alpha-p+1)^{\frac{1}{p}}}\right) + \int_a^b |I_a^\alpha v(x)|^{-q}|v(x)|^q dx\right), & s = 0; \\ q^2\left(D\log\left(\frac{(b-a)^{\alpha-\frac{1}{q}} D^{\frac{1}{q}}}{\Gamma(\alpha)(p\alpha-p+1)^{\frac{1}{p}}}\right)\right. & \\ \left. \quad + \frac{\Gamma^q(\alpha)(p\alpha-p+1)^{\frac{q}{p}}}{(b-a)^{q\alpha-1}}\int_a^b (1+q\log|I_a^\alpha v(x)|)|v(x)|^q dx\right), & s = q, \end{cases} \tag{3.5}$$

where $D = \int_a^b |v(x)|^q dx$.

THEOREM 8. *For $\Gamma_{\varphi_s}(I_a^\alpha v, v)$ defined above we have:*

a) for every $n \in N$ and $p_i \in \mathbb{R}$ the matrix $A = \left[\Gamma_{\varphi_{\frac{p_i+p_j}{2}}}(I_a^\alpha v, v)\right]_{i,j=1}^n$, is a positive semi-definite matrix.

b) the function $s \mapsto \Gamma_{\varphi_s}(I_a^\alpha v, v)$ is exponentially convex.

c) if $\Gamma_{\varphi_s}(I_a^\alpha v, v)$ is positive, then the function $s \mapsto \Gamma_{\varphi_s}(I_a^\alpha v, v)$ is log-convex.

Proof. The proof is similar to the proof of Theorem 4.

If we put $h_1 = \varphi_s$, $h_2 = \varpi_r$ in Theorem 7, then we have a mean defined as

$$\Omega_{s,r}^{[q]}(I_a^\alpha v, v) = \left(\frac{\alpha_{\varphi_s}(I_a^\alpha v, v)}{\alpha_{\varphi_r}(I_a^\alpha v, v)}\right)^{\frac{1}{s-r}}, \quad s \neq r \tag{3.6}$$

That is

$$\Omega_{s,r}^{[q]}(I_a^\alpha v, v) =$$
$$\left(\frac{r(r-q)}{s(s-q)} \frac{q\Gamma^{q-s}(\alpha)(p\alpha-p+1)^{\frac{q-s}{p}}(b-a)^{(s-q)(\alpha-\frac{1}{q})}D^{\frac{s}{q}} - s\int_a^b |I_a^\alpha v(x)|^{s-q}|v(x)|^q dx}{q\Gamma^{q-r}(\alpha)(p\alpha-p+1)^{\frac{q-r}{p}}(b-a)^{(r-q)(\alpha-\frac{1}{q})}D^{\frac{r}{q}} - r\int_a^b |I_a^\alpha v(x)|^{r-q}|v(x)|^q dx}\right)^{\frac{1}{s-r}}. \tag{3.7}$$

In limiting cases we have: when s goes to r,

$$\Omega_{r,r}^{[q]}(I_a^\alpha v, v) = \exp\left(\frac{A_1}{B_1} - \frac{2r-q}{r(r-q)}\right), \quad r \neq q, \tag{3.8}$$

where

$$A_1 = \Gamma^{q-r}(\alpha)(p\alpha-p+1)^{\frac{q-r}{p}}(b-a)^{(r-q)(\alpha-\frac{1}{q})}D^{\frac{r}{q}}\log D$$
$$-q\log(\Gamma(\alpha)(p\alpha-p+1)^{\frac{1}{p}}(b-a)^{-(\alpha-\frac{1}{q})})\Gamma^{q-r}(\alpha)(p\alpha-p+1)^{\frac{q-r}{p}}(b-a)^{(r-q)(\alpha-\frac{1}{q})}D^{\frac{r}{q}}$$
$$-\int_a^b |I_a^\alpha v(x)|^{r-q}|v(x)|^q dx - r\int_a^b |I_a^\alpha v(x)|^{r-q}\log|I_a^\alpha v(x)||v(x)|^q dx$$

and

$$B_1 = q\Gamma^{q-r}(\alpha)(p\alpha-p+1)^{\frac{q-r}{p}}(b-a)^{(r-q)(\alpha-\frac{1}{q})}D^{\frac{r}{q}} - r\int_a^b |I_a^\alpha v(x)|^{r-q}|v(x)|^q dx.$$

In (3.7) when r goes to q we get for $s \neq q$

$$\Omega_{s,q}^{[q]}(I_a^\alpha v, v) = \Omega_{q,s}^{[q]}(I_a^\alpha v, v) =$$
$$\left(\frac{(s(s-q))^{-1}q\left(\Gamma^{q-s}(\alpha)(p\alpha-p+1)^{\frac{q-s}{p}}(b-a)^{(s-q)(\alpha+\frac{1}{p})}D^{\frac{s}{q}} - s\int_a^b |I_a^\alpha v(x)|^{s-q}|v(x)|^q dx\right)}{\left((\log D-1)D - q\log(\Gamma(\alpha)(p\alpha-p+1)^{\frac{1}{p}}(b-a)^{-(\alpha+\frac{1}{p})}) - q\int_a^b \log|I_a^\alpha v(x)||v(x)|^q dx\right)}\right)^{\frac{1}{s-q}}. \tag{3.9}$$

When s goes to q we have

$$\Omega^{[q]}_{q,q}(u,v) = \exp\left(\frac{1}{2}\left(\frac{P_1}{Q_1} - \frac{2}{q}\right)\right), \tag{3.10}$$

where

$$\begin{aligned} P_1 = \frac{D(\log D)^2}{q} &- D\log D\log(\Gamma(\alpha)(p\alpha - p + 1)^{\frac{1}{p}}(b-a)^{-(\alpha-\frac{1}{q})}) \\ &+ q(\log(\Gamma(\alpha)(p\alpha - p + 1)^{\frac{1}{p}}(b-a)^{-(\alpha-\frac{1}{q})}))^2 \\ &- 2\int_a^b \log|I^\alpha_a v(x)||v(x)|^q dx - q\int_a^b (\log|I^\alpha_a v(x)|)^2|v(x)|^q dx \end{aligned}$$

and

$$\begin{aligned} Q_1 = (\log D - 1)D - q\log\left(\Gamma(\alpha)(p\alpha - p + 1)^{\frac{1}{p}}(b-a)^{-(\alpha-\frac{1}{q})}\right) \\ - q\int_a^b \log|I^\alpha_a v(x)||v(x)|^q dx. \end{aligned}$$

Now we prove the monotonicity.

THEOREM 9. *Let $t,s,l,m \in \mathbb{R}_+$ such that $t \leqslant l$, $s \leqslant m$. Then*

$$\Omega^{[q]}_{t,s}(I^\alpha_a v, v) \leqslant \Omega^{[q]}_{l,m}(I^\alpha_a v, v).$$

Proof. The following inequality holds for convex function φ see in [11, p. 4].

$$\frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leqslant \frac{\varphi(y_2) - \varphi(y_1)}{y_2 - y_1}, \tag{3.11}$$

where $x_1 \leqslant y_1$, $x_2 \leqslant y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$. Since by Theorem 8, $\Gamma_s(I^\alpha_a v, v)$ is log-convex, we can put in (3.11): $\varphi = \log\Gamma_{\varphi_s}(I^\alpha_a v, v)$, $x_1 = s$, $x_2 = t$, $y_1 = l$, $y_2 = m$, we get for $s \neq t$, $l \neq m$

$$\frac{\log\Gamma_{\varphi_t}(I^\alpha_a v, v) - \log\Gamma_{\varphi_s}(I^\alpha_a v, v)}{t - s} \leqslant \frac{\log\Gamma_{\varphi_m}(I^\alpha_a v, v) - \log\Gamma_{\varphi_l}(I^\alpha_a v, v)}{m - l},$$

therefore we have

$$\left(\frac{\Gamma_{\varphi_t}(I^\alpha_a v, v)}{\Gamma_{\varphi_s}(I^\alpha_a v, v)}\right)^{\frac{1}{t-s}} \leqslant \left(\frac{\Gamma_{\varphi_m}(I^\alpha_a v, v)}{\Gamma_{\varphi_l}(I^\alpha_a v, v)}\right)^{\frac{1}{m-l}}. \tag{3.12}$$

From (3.12), we get our result for $t \neq s$, $l \neq m$ and for $t = s$, $l = m$; $t \neq s$, $l = m$; $t = s$, $l \neq m$ we can consider limiting cases. □

Next, we define Caputo fractional derivative, for details see [6, p. 449]. The Caputo fractional derivative is defined as: Let $\alpha \geqslant 0$, $n = \lceil \alpha \rceil$, $f \in AC^n([a,b])$. The Caputo fractional derivative is given by

$$D^{\alpha}_{*a} f(t) = \frac{1}{\Gamma(n-\alpha)} \int\limits_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}\, ds, \tag{3.13}$$

for all $x \in [a,b]$. The above function exists almost everywhere for $x \in [a,b]$.

THEOREM 10. *Let $h : [0,\infty) \to \mathbb{R}$ be a function with assumptions of Theorem 1. If $h \in C^2(I)$, where $I \subseteq (0,\infty)$ is compact interval and let $v \in AC^n[a,b]$, has Caputo fractional derivative of order α, $0 < \alpha - [\alpha] < \frac{1}{p}$. Then there exists $\xi \in I$ such that*

$$\begin{aligned}\alpha_h(D^{\alpha}_{*a} v, v^{(n)}) = {} & \frac{\xi h''(\xi) - (q-1)h'(\xi)}{2q^2 \xi^{2q-1}} \times \\ & \times \left(\frac{(b-a)^{q([\alpha]-\alpha+\frac{1}{p})}}{\Gamma^q([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{q}{p}}} \left(\int_a^b |v^{(n)}(x)|^q dx \right)^2 \right. \\ & \left. - 2\int_a^b D^{\alpha}_{*a} v(x)|^q |v^{(n)}(x)|^q dx \right). \end{aligned} \tag{3.14}$$

Proof. From Theorem 2 we have

$$\begin{aligned}\alpha_h(u,v) = {} & \frac{\xi h''(\xi) - (q-1)h'(\xi)}{2q^2 \xi^{2q-1}} \left(N^q \left(\int_a^b |v(x)|^q dx \right)^2 \right. \\ & \left. - 2\int_a^b |u(x)|^q |v(x)|^q dx \right), \quad \xi \in I, \end{aligned} \tag{3.15}$$

and $v \in AC^n[a,b]$, has Caputo fractional derivative of order α, so

$$u(x) = D^{\alpha}_{*a} v(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} v^{(n)}(t)\, dt.$$

Here

$$K(x,t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}(x-t)^{n-\alpha-1}, & a \leqslant t \leqslant x; \\ 0, & x < t \leqslant b. \end{cases}$$

Let

$$Q(x) = \left(\int_a^x (K(x,t))^p dt \right)^{\frac{1}{p}} = \frac{(x-a)^{n-\alpha-\frac{1}{q}}}{\Gamma(\alpha)(p(n-\alpha)-p+1)^{\frac{1}{p}}}, \quad n = [\alpha]+1.$$

Then

$$Q'(x) = \frac{(p([\alpha]-\alpha)+1)^{\frac{1}{q}}(x-a)^{(n-\alpha-\frac{1}{q}-1)}}{p\Gamma([\alpha]-\alpha+1)} \geqslant 0,$$

for $0<\alpha-[\alpha]<\frac{1}{p}$ and $x\in[a,b]$. $Q(x)$ is increasing in $[a,b]$. Therefore,

$$\max_{x\in[a,b]}(Q(x))=\frac{(b-a)^{[\alpha]-\alpha+\frac{1}{p}}}{\Gamma([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{1}{p}}}$$

for $0<\alpha-[\alpha]<\frac{1}{p}$. That is

$$\left(\int_a^x K(x,t)^p dt\right)^{\frac{1}{p}}\leqslant\frac{(b-a)^{[\alpha]-\alpha+\frac{1}{p}}}{\Gamma([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{1}{p}}},$$

for $0<\alpha-[\alpha]<\frac{1}{p}$. Therefore, here we can take

$$N=\frac{(b-a)^{[\alpha]-\alpha+\frac{1}{p}}}{\Gamma([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{1}{p}}},$$

for $0<\alpha-[\alpha]<\frac{1}{p}$. By putting $v=v^{(n)}$ and the values of $u(x)$ and N in (3.15) we get $\alpha_h(D^{\alpha}_{*a}v,v^{(n)})$ as required in (3.14).

THEOREM 11. *Let $h_1,h_2:[0,\infty)\to\mathbb{R}$ be the function with assumptions of Theorem 1. If $h_1,h_2\in C^2(I)$, where $I\subseteq(0,\infty)$ is compact interval, let $v\in AC^n[a,b]$ has Caputo fractional derivative of order α, $0<\alpha-[\alpha]<\frac{1}{p}$. Then there exists $\xi\in I$ such that*

$$\frac{\alpha_{h_1}(D^{\alpha}_{*a}v,v^{(n)})}{\alpha_{h_2}(D^{\alpha}_{*a}v,v^{(n)})}=\frac{\xi h_1''(\xi)-(q-1)h_1'(\xi)}{\xi h_2''(\xi)-(q-1)h_2'(\xi)},\tag{3.16}$$

provided that denominators are not equal to zero.

Proof. By Theorem 3 we have

$$\frac{\alpha_{h_1}(u,v)}{\alpha_{h_2}(u,v)}=\frac{\xi h_1''(\xi)-(q-1)h_1'(\xi)}{\xi h_2''(\xi)-(q-1)h_2'(\xi)},$$

and from the proof of Theorem 10, we get (3.16) with required condition.

If $v\in AC^n[a,b]$ has Caputo fractional derivative of order α, $\alpha-[\alpha]<\frac{1}{p}$, then (2.6) becomes

$$\Gamma_{\varphi_s}(D^{\alpha}_{*a}v,v^{(n)})=$$
$$\begin{cases}\frac{q^2}{s(s-q)}\left(\frac{q(b-a)^{(s-q)([\alpha]-\alpha+\frac{1}{p})}E^{\frac{s}{q}}}{\Gamma^{s-q}([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{s-q}{p}}}-s\int_a^b|D^{\alpha}_{*a}v(x)|^{s-q}|v^{(n)}(x)|^q dx\right), & s\neq 0,q;\\ q\left(\frac{-q\Gamma^q([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{q}{p}}}{(b-a)^{q([\alpha]-\alpha+\frac{1}{p})}}\log\left(\frac{(b-a)^{[\alpha]-\alpha+\frac{1}{p}}E^{\frac{1}{q}}}{\Gamma([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{1}{p}}}\right)\right.\\ \qquad\left.+\int_a^b|D^{\alpha}_{*a}v(x)|^{s-q}|v^{(n)}(x)|^q dx\right), & s=0;\\ q^2\left(E\log\left(\frac{(b-a)^{[\alpha]-\alpha+\frac{1}{p}}E^{\frac{1}{q}}}{\Gamma([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{1}{p}}}\right)\right.\\ \qquad\left.+\frac{(b-a)^{q([\alpha]-\alpha+\frac{1}{p})}}{\Gamma^q([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{q}{p}}}\int_a^b(1+q\log|D^{\alpha}_{*a}v(x)|)|v^{(n)}(x)|^q dx\right), & s=q,\end{cases}\tag{3.17}$$

where $E=\int_a^b|v^{(n)}(x)|^q dx$.

THEOREM 12. *For $\Gamma_{\varphi_s}(D^{\alpha}_{*a}v, v^{(n)})$ defined above we have:*

a) for every $n \in N$ and $p_i \in \mathbb{R}$ the matrix $A = \left[\Gamma_{\varphi_{\frac{p_i+p_j}{2}}}(D^{\alpha}_{*a}v, v^{(n)})\right]_{i,j=1}^{n}$, is a positive semi-definite matrix.

b) the function $s \mapsto \Gamma_{\varphi_s}(D^{\alpha}_{*a}v, v^{(n)})$ is exponentially convex.

c) if $\Gamma_{\varphi_s}(D^{\alpha}_{*a}v, v^{(n)})$ is positive, then the function $s \mapsto \Gamma_{\varphi_s}(D^{\alpha}_{*a}v, v^{(n)})$ is log-convex.

Proof. For proof see the proof of Theorem 4.

If we put $h_1 = \varphi_s$, $h_2 = \varpi_r$ in Theorem 11, then we have a mean defined as:

$$\Upsilon^{[q]}_{s,r}(D^{\alpha}_{*a}v, v^{(n)}) = \left(\frac{\alpha_{\varphi_s}(D^{\alpha}_{*a}v, v^{(n)})}{\alpha_{\varphi_r}(D^{\alpha}_{*a}v, v^{(n)})}\right)^{\frac{1}{s-r}}, \quad s \neq r \tag{3.18}$$

That is

$$\Upsilon^{[q]}_{s,r}(D^{\alpha}_{*a}v, v^{(n)}) =$$
$$\left(\frac{r(r-q)}{s(s-q)} \cdot \frac{q\Gamma^{q-s}([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{q-s}{p}}(b-a)^{(s-q)([\alpha]-\alpha+\frac{1}{p})}E^{\frac{s}{q}} - L_1}{q\Gamma^{q-r}([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{q-r}{p}}(b-a)^{(r-q)([\alpha]-\alpha+\frac{1}{p})}E^{\frac{r}{q}} - M_1}\right)^{\frac{1}{s-r}}, \tag{3.19}$$

for $s, r \neq q$, $s \neq r$. where

$$L_1 = s\int_a^b |D^{\alpha}_{*a}v(x)|^{s-q}|v^{(n)}(x)|^q dx, M_1 = r\int_a^b |D^{\alpha}_{*a}v(x)|^{r-q}|v^{(n)}(x)|^q dx.$$

In limiting cases we have, when s goes to r

$$\Upsilon^{[q]}_{r,r}(D^{\alpha}_{*a}v, v^{(n)}) = \exp\left(\frac{A_2}{B_2} - \frac{2r-q}{r(r-q)}\right), \quad r \neq q, \tag{3.20}$$

where

$$\begin{aligned} A_2 = {} & \Gamma^{q-r}([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{q-r}{p}}(b-a)^{(r-q)([\alpha]-\alpha+\frac{1}{p})}E^{\frac{r}{q}}\log E \\ & - q\log(\Gamma([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{1}{p}}(b-a)^{-([\alpha]-\alpha+\frac{1}{p})} \times \\ & \times \Gamma^{q-r}([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{q-r}{p}}(b-a)^{(r-q)([\alpha]-\alpha+\frac{1}{p})}E^{\frac{r}{q}} \\ & - \int_a^b |D^{\alpha}_{*a}v(x)|^{r-q}|v^{(n)}(x)|^q dx - r\int_a^b |D^{\alpha}_{*a}v(x)|^{r-q}\log|D^{\alpha}_{*a}v(x)||v^{(n)}(x)|^q dx \end{aligned}$$

and

$$\begin{aligned} B_2 = {} & q\Gamma^{q-r}([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{q-r}{p}}(b-a)^{(r-q)([\alpha]-\alpha+\frac{1}{p})}E^{\frac{r}{q}} \\ & - r\int_a^b |D^{\alpha}_{*a}v(x)|^{r-q}|v^{(n)}(x)|^q dx. \end{aligned}$$

In (3.19) when r goes to q we get

$$\Upsilon^{[q]}_{s,q}(D^{\alpha}_{*a}v,v^{(n)})=\Upsilon^{[q]}_{q,s}(D^{\alpha}_{*a}v,v^{(n)})=$$

$$\left(\frac{q\left(\Gamma^{q-s}([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{q-s}{p}}(b-a)^{(s-q)([\alpha]-\alpha+\frac{1}{p})}E^{\frac{s}{q}}-L_2\right)}{s(s-q)\left((\log E-1)E-q\log(\Gamma([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{1}{p}}(b-a)^{-([\alpha]-\alpha+\frac{1}{p})})-M_2\right)}\right)^{\frac{1}{s-q}}, \tag{3.21}$$

where

$$L_2=s\int_a^b|D^{\alpha}_{*a}v(x)|^{s-q}|v^n(x)|^q dx, M_2=q\int_a^b\log|D^{\alpha}_{*a}v(x)||v^{(n)}(x)|^q dx.$$

When s goes to q we have

$$\Upsilon^{[q]}_{q,q}(D^{\alpha}_{*a}v,v^{(n)})=\exp\left(\frac{1}{2}\left(\frac{P_2}{Q_2}-\frac{2}{q}\right)\right), \tag{3.22}$$

where

$$\begin{aligned}P_2=\frac{E(\log E)^2}{q}&-E\log E\log(\Gamma([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{1}{p}}(b-a)^{-([\alpha]-\alpha+\frac{1}{p})}\\&+q(\log(\Gamma([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{1}{p}}(b-a)^{-([\alpha]-\alpha+\frac{1}{p})}))^2\\&-2\int_a^b\log|D^{\alpha}_{*a}v(x)||v(x)|^q dx-q\int_a^b(\log|D^{\alpha}_{*a}v(x)|)^2|v(x)|^q dx,\end{aligned}$$

and

$$\begin{aligned}Q_2=(\log E-1)E-q\log(\Gamma([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{1}{p}}(b-a)^{-([\alpha]-\alpha+\frac{1}{p})})\\-q\int_a^b\log|D^{\alpha}_{*a}v(x)||v^{(n)}(x)|^q dx.\end{aligned}$$

Now we prove monotonicity.

THEOREM 13. *Let $t,s,l,m\in\mathbb{R}^+$ such that $t\leqslant l, s\leqslant m$. Then*

$$\Upsilon^{[q]}_{t,s}(D^{\alpha}_{*a}v,v^{(n)})\leqslant\Upsilon^{[q]}_{l,m}(D^{\alpha}_{*a}v,v^{(n)}).$$

Proof. The following inequality holds for convex function φ see in [11, p. 4],

$$\frac{\varphi(x_2)-\varphi(x_1)}{x_2-x_1}\leqslant\frac{\varphi(y_2)-\varphi(y_1)}{y_2-y_1}, \tag{3.23}$$

where $x_1\leqslant y_1$, $x_2\leqslant y_2$, $x_1\neq x_2$, $y_1\neq y_2$. Since by Theorem 12, $\Gamma_s(D^{\alpha}_a v,v^{(n)})$ is log-convex, we can put in (3.23): $\varphi=\log\Gamma_{\varphi_s}(D^{\alpha}_{*a}v,v^{(n)})$, $x_1=s$, $x_2=t$, $y_1=l$, $y_2=m$. We get for $s\neq t$, $l\neq m$

$$\frac{\log\Gamma_{\varphi_t}(D^{\alpha}_{*a}v,v^{(n)})-\log\Gamma_{\varphi_s}(D^{\alpha}_{*a}v,v^{(n)})}{t-s}\leqslant\frac{\log\Gamma_{\varphi_m}(D^{\alpha}_{*a}v,v^{(n)})-\log\Gamma_{\varphi_l}(D^{\alpha}_{*a}v,v^{(n)})}{m-l},$$

therefore we have

$$\left(\frac{\Gamma_{\varphi_t}(D^{\alpha}_{*a}v, v^{(n)})}{\Gamma_{\varphi_s}(D^{\alpha}_{*a}v, v^{(n)})}\right)^{\frac{1}{t-s}} \leqslant \left(\frac{\Gamma_{\varphi_m}(D^{\alpha}_{*a}v, v^{(n)})}{\Gamma_{\varphi_l}(D^{\alpha}_{*a}v, v^{(n)})}\right)^{\frac{1}{m-l}}. \tag{3.24}$$

From (3.24) we get our result for $t \neq s$, $l \neq m$ and for $t = s$, $l = m$; $t \neq s$, $l = m$; $t = s$, $l \neq m$ we can consider limiting cases.

We continue with the following lemma that is given in [2].

LEMMA 2. *Let* $\nu > \gamma \geqslant 0$, $n = [\nu] + 1$, $m = [\gamma] + 1$ *and* $f \in AC^n([a,b])$. *Suppose that one of the following conditions hold:*

(a) $\nu, \gamma \notin \mathbb{N}_0$ *and* $f^i(a) = 0$ *for* $i = m, m+1, ..., n-1$.

(b) $\nu \in \mathbb{N}, \gamma \notin \mathbb{N}_0$ *and* $f^i(a) = 0$ *for* $i = m, m+1, ..., n-2$.

(c) $\nu \notin \mathbb{N}, \gamma \in \mathbb{N}_0$ *and* $f^i(a) = 0$ *for* $i = m-1, ..., n-1$.

(d) $\nu \in \mathbb{N}, \gamma \in \mathbb{N}_0$ *and* $f^i(a) = 0$ *for* $i = m-1, ..., n-2$.

Then

$$D^{\gamma}_{*a}f(t) = \frac{1}{\Gamma(\nu-\gamma)} \int_a^t (t-s)^{\nu-\gamma-1} D^{\nu}_{*a}f(s)\,ds. \tag{3.25}$$

for all $a \leqslant t \leqslant b$.

By using Lemma 2 similar results to previous ones can be proved. They can be stated as follows:

THEOREM 14. *Let* h, q *and* p *be defined as in Theorem 1, and* $0 < \gamma < \nu - \frac{1}{q}$. *If one of the conditions in Lemma 2 is satisfied, then*

$$\int_a^b |D^{\gamma}_{*a}u(x)|^{1-q} h'(|D^{\gamma}_{*a}u(x)|)|D^{\nu}_{*a}v(x)|^q dx \leqslant \frac{q\Gamma^q(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{q}{p}}}{(b-a)^{q(\nu-\gamma-\frac{1}{q})}}$$
$$h\left(\frac{(b-a)^{\nu-\gamma-\frac{1}{q}}}{\Gamma(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{1}{p}}}\left(\int_a^b |D^{\nu}_{*a}v(x)|^q dx\right)^{\frac{1}{q}}\right), \tag{3.26}$$

If the function $h(x^{\frac{1}{q}})$ *is concave, then the reverse of the inequality* (3.26) *holds.*

THEOREM 15. *Let* $h : [0,\infty) \to \mathbb{R}$ *be the function with assumptions of Theorem 1. If* $h \in C^2(I)$, *where* $I \subseteq (0,\infty)$ *is compact interval, also let* $0 < \gamma < \nu - \frac{1}{q}$ *and one of the conditions in Lemma 2 is satisfied, then there exists* $\xi \in I$ *such that*

$$\alpha_h(D^{\gamma}_{*a}v, D^{\nu}_{*a}v) = \frac{\xi h''(\xi) - (q-1)h'(\xi)}{2q^2\xi^{2q-1}} \left(\frac{(b-a)^{q(\nu-\gamma-\frac{1}{q})}}{\Gamma^q(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{q}{p}}} \cdot \right.$$
$$\left. \left(\int_a^b |D^{\nu}_{*a}v(x)|^q dx\right)^2 - 2\int_a^b |D^{\gamma}_{*a}v(x)|^q |D^{\nu}_{*a}v(x)|^q dx\right).$$

THEOREM 16. *Let $h_1, h_2 : [0,\infty) \to \mathbb{R}$ be functions which satisfy the assumptions of Theorem 1. If $h_1, h_2 \in C^2(I)$, where $I \subseteq (0,\infty)$ is compact interval, let $0 < \gamma < \nu - \frac{1}{q}$ and if one of the conditions in Lemma 2 is satisfied, then there exists $\xi \in I$ such that*

$$\frac{\alpha_{h_1}(D^{\gamma}_{*a}v, D^{\nu}_{*a}v)}{\alpha_{h_2}(D^{\gamma}_{*a}v, D^{\nu}_{*a}v)} = \frac{\xi h_1''(\xi) - (q-1)h_1'(\xi)}{\xi h_2''(\xi) - (q-1)h_2'(\xi)}.$$

provided that denominators are not equal to zero.

In the next we give results using an improved composition rule for Canavati fractional derivatives (see, [3]). The subspace $C^{\nu}_{x_0}([a,b])$ of $C^n([a,b])$ defined as

$$C^{\nu}_{x_0}([a,b]) = \{f \in C^n([a,b]) : I^{1-\bar{\nu}}_{x_0} f^{(n)}\} \in C^1([x_0,b]),\ \bar{\nu} = \nu - n,\ n = [\nu].$$

For $f \in C^{\nu}_{x_0}([a,b])$ the generalized Canavati ν-fractional derivative of f over $[x_0,b]$ is given by

$$D^{\nu}_{x_0} f = D I^{1-\bar{\nu}}_{x_0} f^{(n)}.$$

An improved composition rule for Canavati fractional derivatives is given in the following result [3].

LEMMA 3. *Let $\nu > \gamma > 0, n = [\nu], m = [\gamma]$. Let $f \in C^{\nu}_{a}([a,b])$ be such that $f^{i}(a) = 0$ for $i = m, m+1, \ldots, n-1$. Then*

(i)

$$f \in C^{\gamma}_{a}([a,b])$$

(ii)

$$(D^{\gamma}_{a} f)(x) = \frac{1}{\Gamma(\nu-\gamma)} \int_a^x (x-t)^{\nu-\gamma-1} (D^{\nu}_{a} f)(t)\, dt \tag{3.27}$$

for every $x \in [a,b]$.

THEOREM 17. *Let h, q and p be defined as in Theorem 1, and $0 < \gamma < \nu - \frac{1}{q}$. Let $v \in C^{\nu}_{a}([a,b])$ be such that $f^{i}(a) = 0$ for $i = m, m+1, \ldots, n-1$. Then*

$$\int_a^b |D^{\gamma}_{a}u(x)|^{1-q} h'(|D^{\gamma}_{a}u(x)|) |D^{\nu}_{a}v(x)|^q dx \leqslant \frac{q\Gamma^q(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{q}{p}}}{(b-a)^{q(\nu-\gamma-\frac{1}{q})}}$$
$$h\left(\frac{(b-a)^{\nu-\gamma-\frac{1}{q}}}{\Gamma(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{1}{p}}} \left(\int_a^b |D^{\nu}_{a}v(x)|^q dx\right)^{\frac{1}{q}}\right), \tag{3.28}$$

If the function $h(x^{\frac{1}{q}})$ is concave, then the reverse of the inequality (3.28) holds.

Proof. By applying Lemma 3 we have

$$u(x) = (D_a^{\gamma} v)(x) = \frac{1}{\Gamma(\nu - \gamma)} \int_a^x (x-t)^{\nu-\gamma-1} (D_a^{\nu} v)(t) dt, \; x \in [a,b], \tag{3.29}$$

Here

$$K(x,t) = \begin{cases} \frac{1}{\Gamma(\nu-\gamma)} (x-t)^{\nu-\gamma-1}, & a \leqslant t \leqslant x; \\ 0, & x < t \leqslant b. \end{cases} \tag{3.30}$$

Let

$$R(x) = \left(\int_a^x (K(x,t))^p dt \right)^{\frac{1}{p}} = \frac{(x-a)^{\nu-\gamma-\frac{1}{q}}}{\Gamma(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{1}{p}}}.$$

Then

$$R'(x) = \frac{(\nu-\gamma-\frac{1}{q})(x-a)^{(\nu-\gamma-\frac{1}{q}-1)}}{\Gamma(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{1}{p}}} \geqslant 0,$$

for $0 < \gamma < \nu - \frac{1}{p}$, $x \in [a,b]$. $R(x)$ is hence increasing in $[a,b]$. Therefore,

$$\max_{x \in [a,b]} (R(x)) = \frac{(b-a)^{\nu-\gamma-\frac{1}{q}}}{\Gamma(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{1}{p}}},$$

for $0 < \gamma < \nu - \frac{1}{q}$. That is

$$\left(\int_a^x K(x,t)^p \right)^{\frac{1}{p}} \leqslant \frac{(b-a)^{\nu-\gamma-\frac{1}{q}}}{\Gamma(\nu-\gamma)(p(\beta-\alpha)-p+1)^{\frac{1}{p}}},$$

for $0 < \gamma < \nu - \frac{1}{q}$, so we can take here

$$N = \frac{(b-a)^{\nu-\gamma-\frac{1}{q}}}{\Gamma(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{1}{p}}},$$

for $0 < \gamma < \nu - \frac{1}{q}$. Therefore by putting $v = D_a^\nu v$ and the values of $u(x)$ and N in (1.1) we get $\alpha_h(D_a^\gamma v, D_a^\nu v)$ as required in (3.28).

THEOREM 18. *Let $h : [0,\infty) \to \mathbb{R}$ be the function with assumptions of Theorem 1. If $h \in C^2(I)$, where $I \subseteq (0,\infty)$ is compact interval, also let $0 < \gamma < \nu - \frac{1}{q}$, $v \in C_a^\nu([a,b])$ be such that $f^i(a) = 0$ for $i = m, m+1, ..., n-1$.*
Then there exists $\xi \in I$ such that

$$\alpha_h(D_a^\gamma v, D_a^\nu v) = \frac{\xi h''(\xi) - (q-1)h'(\xi)}{2q^2\xi^{2q-1}} \left(\frac{(b-a)^{q(\nu-\gamma-\frac{1}{q})}}{\Gamma^q(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{q}{p}}} \cdot \left(\int_a^b |D_a^\nu v(x)|^q dx \right)^2 - 2\int_a^b |D_a^\gamma v(x)|^q |D_a^\nu v(x)|^q dx \right). \quad (3.31)$$

Proof. From Theorem 2 we have

$$\alpha_h(u,v) = \frac{\xi h''(\xi) - (q-1)h'(\xi)}{2q^2\xi^{2q-1}} \left(N^q \left(\int_a^b |v(x)|^q dx \right)^2 - 2\int_a^b |u(x)|^q |v(x)|^q dx \right), \quad (3.32)$$

By using Lemma 3 we have

$$u(x) = D_a^\gamma v(x) = \frac{1}{\Gamma(\nu-\gamma)} \int_a^x (x-t)^{\nu-\gamma-1} D_a^\nu v(t) dt, \; x \in [a,b].$$

Here we have

$$K(x,t) = \begin{cases} \frac{1}{\Gamma(\nu-\gamma)}(x-t)^{\nu-\gamma-1}, & a \leqslant t \leqslant x; \\ 0, & x < t \leqslant b, \end{cases}$$

and from the proof of Theorem 17, we easily get (3.31).

THEOREM 19. *Let $h_1, h_2 : [0,\infty) \to \mathbb{R}$ be functions which satisfy the assumptions of Theorem 1. If $h_1, h_2 \in C^2(I)$, where $I \subseteq (0,\infty)$ is compact interval, let $0 < \gamma < \nu - \frac{1}{q}$, $v \in C_a^\nu([a,b])$ be such that $f^i(a) = 0$ for $i = m, m+1, ..., n-1$. Then there exists $\xi \in I$ such that*

$$\frac{\alpha_{h_1}(D_a^\gamma v, D_a^\nu v)}{\alpha_{h_2}(D_a^\gamma v, D_a^\nu v)} = \frac{\xi h_1''(\xi) - (q-1)h_1'(\xi)}{\xi h_2''(\xi) - (q-1)h_2'(\xi)}. \quad (3.33)$$

provided that denominators are not equal to zero.

Proof. By Theorem 3, we have

$$\frac{\alpha_{h_1}(u,v)}{\alpha_{h_2}(u,v)} = \frac{\xi h_1''(\xi)-(q-1)h_1'(\xi)}{\xi h_2''(\xi)-(q-1)h_2'(\xi)},$$

and by using Lemma 3 and the proof of Theorem 10, we can easily get (3.33) with required condition.

If for $0<\gamma<\nu-\frac{1}{q}$, $v\in C_a^{\nu}([a,b])$ be such that $f^i(a)=0$ for $i=m,m+1,\dots,n-1$. Then (2.6) becomes

$$\Gamma_{\varphi_s}(D_a^{\gamma}u, D_a^{\nu}v) = \begin{cases} \frac{q^2}{s(s-q)}\left(\frac{q(b-a)^{(s-q)(\nu-\gamma-\frac{1}{q})}F^{\frac{s}{q}}}{\Gamma^{s-q}(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{s-q}{p}}} - s\int_a^b |D_a^{\alpha}v(x)|^{s-q}|D_a^{\nu}v(x)|^q dx\right), & s\neq 0,q; \\ q\left(\frac{-q\Gamma^q(\nu-\alpha)(p(\nu-\gamma)-p+1)^{\frac{q}{p}}}{(b-a)^{q(\nu-\gamma-\frac{1}{q})}}\log\left(\frac{(b-a)^{\nu-\gamma-\frac{1}{q}}F^{\frac{1}{q}}}{\Gamma(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{1}{p}}}\right) \right. \\ \qquad \left. + \int_a^b |D_a^{\gamma}v(x)|^{s-q}|D_a^{\nu}v(x)|^q dx\right), & s=0; \\ q^2\left(F\log\left(\frac{(b-a)^{\nu-\gamma-\frac{1}{q}}F^{\frac{1}{q}}}{\Gamma(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{1}{p}}}\right)\right. \\ \qquad \left. + \frac{(b-a)^{q(\nu-\gamma-\frac{1}{q})}}{\Gamma^q(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{q}{p}}}\int_a^b (1+q\log|D_a^{\gamma}v(x)|)|D_a^{\nu}v(x)|^q dx\right), & s=q; \end{cases} \tag{3.34}$$

where $F=\int_a^b |D_a^{\nu}v(x)|^q dx$.

THEOREM 20. *For $\Gamma_{\varphi_s}(D_a^{\gamma}v, D_a^{\nu}v)$ defined above we have:*

a) for every $n\in N$ and $p_i\in\mathbb{R}$ the matrix $A=\left[\Gamma_{\varphi_{\frac{p_i+p_j}{2}}}(D_a^{\gamma}v, D_a^{\nu}v)\right]_{i,j=1}^{n}$, is a positive semi-definite matrix.

b) the function $s\mapsto \Gamma_{\varphi_s}(D_a^{\gamma}v, D_a^{\nu}v)$ is exponentially convex.

c) if $\Gamma_{\varphi_s}(D_a^{\gamma}v, D_a^{\nu}v)$ is positive, then the function $s\mapsto \Gamma_{\varphi_s}(D_a^{\gamma}v, D_a^{\nu}v)$ is log-convex.

Proof. For proof see the proof of Theorem 4.

If we put $h_1=\varphi_s$, $h_2=\varphi_r$ in Theorem 19, then we have a mean defined as:

$$\Sigma_{s,r}^{[q]}(D_a^{\gamma}v, D_a^{\nu}v) = \left(\frac{\alpha_{\varphi_s}(D_a^{\gamma}v, D_a^{\nu}v)}{\alpha_{\varphi_r}(D_a^{\gamma}v, D_a^{\nu}v)}\right)^{\frac{1}{s-r}}, \quad s\neq r, \tag{3.35}$$

that is

$$\Sigma_{s,r}^{[q]}(D_a^{\gamma}v, D_a^{\nu}v) = \left(\frac{r(r-q)}{s(s-q)}\,\frac{q\Gamma^{q-s}(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{q-s}{p}}(b-a)^{(s-q)(\nu-\gamma-\frac{1}{q})}F^{\frac{s}{q}}-L_3}{q\Gamma^{q-r}(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{q-r}{p}}(b-a)^{(r-q)(\nu-\gamma-\frac{1}{q})}F^{\frac{r}{q}}-M_3}\right)^{\frac{1}{s-r}}, \tag{3.36}$$

where

$$L_3 = s\int_a^b |D_a^\gamma v(x)|^{s-q}|D_a^\nu v(x)|^q dx,$$
$$M_3 = r\int_a^b |D_a^\gamma v(x)|^{r-q}|D_a^\nu v(x)|^q dx.$$

In limiting cases we have, for $r \neq q$:

$$\Sigma_{r,r}^{[q]}(D_a^\gamma v, D_a^\nu v) = \exp\left(\frac{A_3}{B_3} - \frac{2r-q}{r(r-q)}\right),$$

for $s \neq q$

$$\Sigma_{s,q}^{[q]}(D_a^\gamma v, D_a^\nu v) = \Sigma_{q,s}^{[q]}(D_a^\gamma v, D_a^\nu v) =$$
$$\left(\frac{q\left(\Gamma^{q-s}(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{q-s}{p}}(b-a)^{(s-q)(\nu-\gamma-\frac{1}{q})}F^{\frac{s}{q}} - L_4\right)}{s(s-q)\left((\log F - 1)F - q\log(\Gamma(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{1}{p}}(b-a)^{-(\nu-\gamma-\frac{1}{q})}) - M_4\right)}\right)^{\frac{1}{s-q}},$$
$$s \neq q. \quad (3.37)$$

and finally

$$\Sigma_{q,q}^{[q]}(D_a^\gamma v, D_a^\nu v) = \exp\left(\frac{1}{2}\left(\frac{P_3}{Q_3} - \frac{2}{q}\right)\right), \quad (3.38)$$

where

$$\begin{aligned} A_3 = {} & \Gamma^{q-r}(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{q-r}{p}}(b-a)^{(r-q)(\nu-\gamma-\frac{1}{q})}F^{\frac{r}{q}}\log F \\ & - q\log(\Gamma(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{1}{p}}(b-a)^{-(\nu-\gamma-\frac{1}{q})}\Gamma^{q-r}(\nu-\gamma)\times \\ & \times (p(\nu-\gamma)-p+1)^{\frac{q-r}{p}}(b-a)^{(r-q)(\nu-\gamma-\frac{1}{q})} \\ & - \int_a^b |D_a^\gamma v(x)|^{r-q}|D_a^\nu v(x)|^q dx - r\int_a^b |D_a^\gamma v(x)|^{r-q}\log|D_a^\gamma v(x)||D_a^\nu v(x)|^q dx, \end{aligned}$$

$$\begin{aligned} B_3 = {} & q\Gamma^{q-r}(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{q-r}{p}}(b-a)^{(r-q)(\nu-\gamma-\frac{1}{q})}F^{\frac{r}{q}} \\ & - r\int_a^b |D_a^\alpha v(x)|^{r-q}|D_a^\beta v(x)|^q dx. \end{aligned}$$

and

$$L_4 = s\int_a^b |D_a^\gamma v(x)|^{s-q}|D_a^\nu v(x)|^q dx,$$
$$M_4 = q\int_a^b \log|D_a^\gamma v(x)||D_a^\nu v(x)|^q dx.$$

and

$$P_3 = \frac{F(\log F)^2}{q} - F\log F \log(\Gamma(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{1}{p}}(b-a)^{-(\nu-\gamma-\frac{1}{q})}$$
$$+q(\log(\Gamma(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{1}{p}}(b-a)^{-(\nu-\gamma-\frac{1}{q})}))^2$$
$$-2\int_a^b \log|D_a^\gamma v(x)||D_a^\nu v(x)|^q dx - q\int_a^b (\log|D_a^\gamma v(x)|)^2|D_a^\nu v(x)|^q dx,$$

$$Q_3 = (\log F - 1)F - q\log(\Gamma(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{1}{p}}(b-a)^{-(\nu-\gamma-\frac{1}{q})})$$
$$-q\int_a^b \log|D_a^\gamma v(x)||D_a^\nu v(x)|^q dx.$$

Now we prove monotonicity.

THEOREM 21. *Let $t,s,l,m \in \mathbb{R}^+$ such that $t \leqslant l,\ s \leqslant m$. Then*

$$\Sigma_{t,s}^{[q]}(D_a^\gamma v, D_a^\nu v) \leqslant \Sigma_{l,m}^{[q]}(D_a^\gamma v, D_a^\nu v).$$

Proof. The following inequality holds for convex function φ see in [11, p. 4],

$$\frac{\varphi(x_2)-\varphi(x_1)}{x_2-x_1} \leqslant \frac{\varphi(y_2)-\varphi(y_1)}{y_2-y_1}, \tag{3.39}$$

where $x_1 \leqslant y_1$, $x_2 \leqslant y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$. Since by Theorem 20, $\Gamma_s(D_a^\gamma v, D_a^\nu v)$ is log-convex, we can put in (3.39): $\varphi = \log\Gamma_{\varphi_s}(D_a^\gamma v, D_a^\nu v)$, $x_1 = s$, $x_2 = t$, $y_1 = l$, $y_2 = m$. We get for $s \neq t$, $l \neq m$

$$\frac{\log\Gamma_{\varphi_t}(D_a^\gamma v, D_a^\nu v) - \log\Gamma_{\varphi_s}(D_a^\gamma v, D_a^\nu v)}{t-s} \leqslant \frac{\log\Gamma_{\varphi_m}(D_a^\gamma v, D_a^\nu v) - \log\Gamma_{\varphi_l}(D_a^\gamma v, D_a^\nu v)}{t-s},$$

therefore we have

$$\left(\frac{\Gamma_{\varphi_t}(D_a^\gamma v, D_a^\nu v)}{\Gamma_{\varphi_s}(D_a^\gamma v, D_a^\nu v)}\right)^{\frac{1}{t-s}} \leqslant \left(\frac{\Gamma_{\varphi_m}(D_a^\gamma v, v)}{\Gamma_{\varphi_l}(D_a^\gamma v, D_a^\nu v)}\right)^{\frac{1}{m-l}}. \tag{3.40}$$

From (3.40) we get our result for $t \neq s$, $l \neq m$ and for $t = s$, $l = m$; $t \neq s$, $l = m$; $t = s$, $l \neq m$ we can consider limiting cases.

In the following result there is given another approach to identity in Lemma 3(ii), (see, [1]).

THEOREM 22. *Let $\nu > \gamma \geqslant 0$ and let $f \in AC^n[a,b]$ be such that $D_a^\nu f \in L[a,b]$ and $D_a^\gamma f \in L(a,b)$.*

(*i*) *If* $\nu-\gamma \notin \mathbb{N}$ *and* f *is such that* $D_a^{\nu-k}f(0)=0$ *for* $k=1,\dots,[\nu]+1$ *and* $D_a^{\gamma-k}f(0)=0$ *for* $k=1,\dots,[\gamma]+1$*, then*

$$D_a^{\gamma}f(s) = \frac{1}{\Gamma(\nu-\gamma)}\int_a^s (s-t)^{\nu-\gamma-1} D_a^{\nu}f(t)\,dt\,, \quad s\in[a,b]. \tag{3.41}$$

(*ii*) *If* $\nu-\gamma = l\in\mathbb{N}$ *and* f *is such that* $D_a^{\nu-k}f(0)=0$ *for* $k=1,\dots,l$*, then* (3.41) *holds.*

In the following corollary [1], there summarize conditions for identity (3.41).

COROLLARY 2. *Let* $\nu>\gamma\geqslant 0$*,* $n=[\nu]+1$*,* $m=[\gamma]+1$*. Identity* (3.41) *is valid if one of the following conditions holds:*

(*i*) $f\in I_a^{\nu}(L(a,b))$.

(*ii*) $I_a^{n-\nu}f\in AC^n[a,b]$ *and* $D_a^{\nu-k}f(0)=0$ *for* $k=1,\dots n$.

(*iii*) $D_a^{\nu-k}f\in C[a,b]$ *for* $k=1,\dots,n$*,* $D_a^{\nu-1}f\in AC[a,b]$ *and* $D_a^{\nu-k}f(0)=0$ *for* $k=1,\dots n$.

(*iv*) $f\in AC^n[a,b]$*,* $D_a^{\nu}f\in L(a,b)$*,* $D_a^{\gamma}f\in L(a,b)$*,* $\nu-\gamma\notin\mathbb{N}$*,* $D_a^{\nu-k}f(0)=0$ *for* $k=1,\dots,n$ *and* $D_a^{\gamma-k}f(0)=0$ *for* $k=1,\dots,m$.

(*v*) $f\in AC^n[a,b]$*,* $D_a^{\nu}f\in L(a,b)$*,* $D_a^{\gamma}f\in L(a,b)$*,* $\nu-\gamma=l\in\mathbb{N}$*,* $D_a^{\nu-k}f(0)=0$ *for* $k=1,\dots,l$.

(*vi*) $f\in AC^n[a,b]$*,* $D_a^{\nu}f\in L(a,b)$*,* $D_a^{\gamma}f\in L(a,b)$ *and* $f(0)=f'(0)=\cdots=f^{(n-2)}(0)=0$.

(*vii*) $f\in AC^n[a,b]$*,* $D_a^{\nu}f\in L(a,b)$*,* $D_a^{\gamma}f\in L(a,b)$*,* $\nu\notin\mathbb{N}$ *and* $D_a^{\nu-1}f$ *is bounded in a neighborhood of* $t=0$.

By using Theorem 22 and above corollary previous results can be proved. They can be stated as follows:

THEOREM 23. *Let* h*,* q *and* p *be defined as in Theorem* 1*, and* $0<\gamma<\nu-\frac{1}{q}$*. If one of the conditions in Corollary* 2 *is satisfied, then*

$$\int_a^b |D_a^{\gamma}u(x)|^{1-q}h'(|D_a^{\gamma}u(x)|)|D_a^{\nu}v(x)|^q dx \leqslant \frac{q\Gamma^q(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{q}{p}}}{(b-a)^{q(\nu-\gamma-\frac{1}{q})}} h\left(\frac{(b-a)^{\nu-\gamma-\frac{1}{q}}}{\Gamma(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{1}{p}}}\left(\int_a^b |D_a^{\nu}v(x)|^q dx\right)^{\frac{1}{q}}\right), \tag{3.42}$$

If the function $h(x^{\frac{1}{q}})$ *is concave, then the reverse of the inequality* (3.42) *holds.*

THEOREM 24. *Let $h:[0,\infty)\to\mathbb{R}$ be the function with assumptions of Theorem 1. If $h\in C^2(I)$, where $I\subseteq(0,\infty)$ is compact interval, also let $0<\gamma<\nu-\frac{1}{q}$ and one of the conditions in Corollary 2 is satisfied, then there exists $\xi\in I$ such that*

$$\alpha_h(D_a^\gamma v, D_a^\nu v) = \frac{\xi h''(\xi)-(q-1)h'(\xi)}{2q^2\xi^{2q-1}}\left(\frac{(b-a)^{q(\nu-\gamma-\frac{1}{q})}}{\Gamma^q(\nu-\gamma)(p(\nu-\gamma)-p+1)^{\frac{q}{p}}}\cdot\left(\int_a^b |D_a^\nu v(x)|^q dx\right)^2 - 2\int_a^b |D_a^\gamma v(x)|^q |D_a^\nu v(x)|^q dx\right).$$

THEOREM 25. *Let $h_1,h_2:[0,\infty)\to\mathbb{R}$ be the function with assumptions of Theorem 1. If $h_1,h_2\in C^2(I)$, where $I\subseteq(0,\infty)$ is compact interval, let $0<\gamma<\nu-\frac{1}{q}$ and one of the conditions in Corollary 2 is satisfied, then there exists $\xi\in I$ such that*

$$\frac{\alpha_{h_1}(D_a^\gamma v, D_a^\nu v)}{\alpha_{h_2}(D_a^\gamma v, D_a^\nu v)} = \frac{\xi h_1''(\xi)-(q-1)h_1'(\xi)}{\xi h_2''(\xi)-(q-1)h_2'(\xi)}.$$

provided that denominators are not equal to zero.

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