Fractional
Differential
Calculus
Volume 2, Number 1 (2012), 87–98
doi:10.7153/fdc-02-06

APPLICATION OF THE MIXED MONOTONE OPERATOR METHOD TO FRACTIONAL BOUNDARY VALUE PROBLEMS

JOHN R. GRAEF, LINGJU KONG AND QINGKAI KONG

Abstract. The authors use the mixed monotone operator method to study the fractional boundary value problem

$$-D_{0+}^{\nu} u(t) = \lambda f(t,u),\ t \in (0,1),$$

$$u^{(j)}(0) = 0,\ j = 0,\dots,n-2, \quad [D_{0+}^{\alpha} u(t)]_{t=1} = 0.$$

Here, $m \geqslant 1$ and $n \geqslant 3$ are integers, $n-1 < \nu \leqslant n$, $1 \leqslant \alpha \leqslant n-2$, $u(t) = (u_1(t),\dots,u_m(t))^T$, $\lambda = (\lambda_1,\dots,\lambda_m)$, $\lambda f(t,u) = (\lambda_1 f_1(t,u),\dots,\lambda_m f_m(t,u))^T$, and D_{0+}^{ν} is the Riemann-Liouville fractional derivative of order ν. Existence, uniqueness, and dependence of positive solutions on the parameter λ are discussed. An application to a special problem is also presented.

1. Introduction

Let $m \geqslant 1$ and $n \geqslant 3$ be any given integers, and $\lambda_1,\dots,\lambda_m$ be positive parameters. Assume that, for $i = 1,\dots,m$, $f_i : (0,1) \times (0,\infty)^m \to [0,\infty)$ is continuous, and $f_i(t,x_1,\dots,x_m)$ may be singular at $t = 0,\ 1$ and $x_i = 0$. In this paper, we are concerned with positive solutions of the ν-th order fractional boundary value problem (BVP) consisting of the system of fractional differential equations

$$-D_{0+}^{\nu} u(t) = \lambda f(t,u),\ t \in (0,1), \tag{1.1}$$

and the boundary condition

$$u^{(j)}(0) = 0,\ j = 0,\dots,n-2, \quad [D_{0+}^{\alpha} u(t)]_{t=1} = 0, \tag{1.2}$$

where $n-1 < \nu \leqslant n$, $1 \leqslant \alpha \leqslant n-2$,

$$u(t) = (u_1(t),\dots,u_m(t))^T, \quad \lambda = (\lambda_1,\dots,\lambda_m),$$

$$\lambda f(t,u) = (\lambda_1 f_1(t,u),\dots,\lambda_m f_m(t,u))^T,$$

and D_{0+}^{ν} is the Riemann-Liouville fractional derivative of order ν, i.e.,

$$D_{0+}^{\nu} y(t) = \frac{1}{\Gamma(k-\nu)} \frac{d^k}{dt^k} \int_0^t \frac{y(s)}{(t-s)^{\nu+1-k}} ds, \quad k = [\nu]+1.$$

Mathematics subject classification (2010): Primary 34B15; secondary 34B18.
Keywords and phrases: Fractional boundary value problems, uniqueness, dependence on a parameter, mixed monotone operator method.

Paper FDC-02-06

We adopt the convention that $D_{0+}^{\nu} u(t)$ is the Riemann-Liouville fractional derivative of order ν defined component-wise on $u(t)$.

By a *positive solution* of BVP (1.1), (1.2), we mean a function $u : [0,1]^m \to [0,\infty)$ such that $u(t)$ satisfies (1.1) and (1.2) and $u_i(t) > 0$ for $t \in (0,1]$ and $i = 1,\ldots,m$.

The subject of fractional calculus has gained considerable popularity and importance in recent years due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. The monographs [10, 14] are excellent sources for the theory and applications of fractional calculus. Among all the topics, the existence of positive solutions of BVPs of fractional differential equations has been extensively studied by many researchers in recent years; see, for example, [1, 2, 3, 5, 6] and the references therein. In particular, Goodrich [6] studied the scalar BVP consisting of the equation

$$-D_{0+}^{\nu} u(t) = f(t,u),\ t \in (0,1),$$

and the boundary condition

$$u^{(j)}(0) = 0,\ j = 0,\ldots,n-2, \quad [D_{0+}^{\alpha} u(t)]_{t=1} = 0,$$

where $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous. In [6], the author first obtained some properties of the Green's function associated with the problem. Then, applying these properties and the well known Krasnosel'skii fixed point theorem in cones, he derived sufficient conditions for the existence of positive solutions of the problem. The uniqueness of positive solutions is not studied in [6]. To the best of our knowledge, most existing works on fractional BVPs do not even consider the question of uniqueness of positive solutions.

In this paper, we not only investigate the existence and uniqueness of positive solutions of BVP (1.1), (1.2), but also discuss the dependence of positive solutions on the parameter λ. Moreover, as a simple application of our theory, we present some uniqueness and dependence results for the BVP consisting of the system of fractional differential equations

$$-D_{0+}^{\beta} u_i(t) = \lambda_i p_i(t) \left(\sum_{k=1}^{m} a_{ik} u_k^{b_{ik}} + \sum_{k=1}^{m} c_{ik} u_k^{-d_{ik}} \right),\ t \in (0,1),\ i = 1,\ldots,m, \tag{1.3}$$

and the boundary condition

$$u_i^{(j)}(0) = 0,\ [D_{0+}^{\alpha} u_i(t)]_{t=1} = 0, \quad i = 1,\ldots,m,\ j = 0,\ldots,n-2, \tag{1.4}$$

where $p_i : (0,1) \to [0,\infty)$ is continuous and integrable, a_{ik}, b_{ik}, $d_{ik} \geqslant 0$, and $c_{ik} > 0$, i, $k = 1,\ldots,m$. In our proof, part of the analysis relies on some results from mixed monotone operator theory. This technique was first introduced by Guo and Lakshmikantham [9] in 1987. Since then, many authors have investigated such operators and related applications to a variety of problems; see, for example, [4, 7, 11, 8, 12, 13, 15, 16, 17, 18] and the references therein.

The rest of this paper is organized as follows. In Section 2, we present our main results, and the proofs of the main results together with several technical lemmas are given in Section 3.

2. Main results

Throughout this paper we let $0=(0,\dots,0)$ and $\infty=(\infty,\dots,\infty)$. The following notations will be used for any vectors $x=(x_1,\dots,x_m)$ and $y=(y_1,\dots,y_m)$:

- ◇ $x\to y$ if every component of x approaches the corresponding one of y.
- ◇ $x\to y^+$ (y^-) if every component of x approaches the corresponding one of y from the right (left);
- ◇ $x\to\infty$ if every component of x approaches ∞;
- ◇ $x>y$ ($x<y$) if every component of x is strictly larger (smaller) than the corresponding one of y;

We need the following assumptions.

(H1) For $i=1,\dots,m$ and any $x\in(0,\infty)^m$, $f_i(t,x)$ can be written as $f_i(t,x)=w_i(t)(g_i(x)+h_i(x))$, where $w_i:(0,1)\to[0,\infty)$ is continuous and integrable on $(0,1)$, $g_i:[0,\infty)^m\to[0,\infty)$ is continuous and nondecreasing in each of its arguments, and $h_i:(0,\infty)^m\to(0,\infty)$ is continuous and nonincreasing in each of its arguments;

(H2) for $i=1,\dots,m$, there exists $\delta\in(0,1)$ such that

$$g_i(\kappa x)\geqslant\kappa^\delta g_i(x) \tag{2.1}$$

and

$$h_i(\kappa^{-1}x)\geqslant\kappa^\delta h_i(x) \tag{2.2}$$

for $\kappa>0$ and $x>0$;

(H3)

$$0<\int_0^1 s^{\delta(1-\nu)}(1-s)^{\nu-\alpha-1}w_i(s)ds<\infty. \tag{2.3}$$

We let the Banach space $X:=(C[0,1])^m$ be equipped with the norm

$$||u||=\max\{\max_{t\in[0,1]}|u_i(t)|:i=1,\dots,m\},\qquad u=(u_1,\dots,u_m)\in X.$$

We are now in a position to state the main results in this paper.

THEOREM 2.1. *Assume that (H1)–(H3) hold. Then, for any* $\lambda=(\lambda_1,\dots,\lambda_m)>0$, *BVP* (1.1), (1.2) *has a unique positive solution* $u_\lambda(t)=(u_{\lambda,1}(t),\dots,u_{\lambda,m}(t))$. *Furthermore, if* $0<\delta<1/2$, *then such a solution* $u_\lambda(t)$ *satisfies the following properties*

(a) $u_\lambda(t)$ *is strictly increasing in* λ, *i.e.,* $\lambda>\mu>0\Longrightarrow u_\lambda(t)>u_\mu(t)$ *on* $(0,1]$*;*

(b) $\lim_{\mu\to 0^+} ||u_\mu|| = 0$ *and* $\lim_{\lambda\to\infty} ||u_\lambda|| = \infty$*;*

(c) $u_\lambda(t)$ *is continuous in* λ*, i.e.,* $\lambda \to \mu > 0 \Longrightarrow ||u_\lambda - u_\mu|| \to 0$.

Applying Theorem 2.1 to BVP (1.3), (1.4), we have the following result.

COROLLARY 2.1. *Assume that the following conditions hold:*

(A1) $0 < \zeta < 1$*, where* $\zeta = \max\{b_{ik},\ d_{ik} : i,k = 1,\ldots,m\}$*;*

(A2) $0 < \int_0^1 s^{\zeta(1-\nu)}(1-s)^{\nu-\alpha-1} p_i(s)ds < \infty,\ i = 1,\ldots,m$.

Then, for any $\lambda = (\lambda_1,\ldots,\lambda_m) > 0$*, BVP* (1.3)*,* (1.4) *has a unique positive solution* $u_\lambda(t) = (u_{\lambda,1}(t),\ldots,u_{\lambda,m}(t))$*. Furthermore, if* $0 < \zeta < 1/2$*, then such a solution* $u_\lambda(t)$ *satisfies the three properties stated in Theorem* 2.1.

3. Proofs of the main results

The following lemma follows from [6, Theorem 3.1].

LEMMA 3.1. *Let* $l \in L(0,1)$*. Then* $y(t)$ *is a solution of the BVP consisting of the equation*

$$-D_{0+}^{\nu} y(t) = l(t),\ t \in (0,1),$$

and the BC

$$y^{(j)}(0) = 0,\ j = 0,\ldots,n-2, \quad [D_{0+}^{\alpha} y(t)]_{t=1} = 0,$$

if and only if

$$u(t) = \int_0^1 G(t,s)l(s)ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\nu)} \begin{cases} t^{\nu-1}(1-s)^{\nu-\alpha-1} - (t-s)^{\nu-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{\nu-1}(1-s)^{\nu-\alpha-1}, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases} \tag{3.1}$$

Lemma 3.2 below provides some useful bounds on $G(t,s)$.

LEMMA 3.2. *The function* $G(t,s)$ *defined by* (3.1) *satisfies*

$$\frac{1}{\Gamma(\nu)} t^{\nu-1}(1-s)^{\nu-\alpha-1}(1-(1-s)^{\alpha}) \leqslant G(t,s) \leqslant \frac{1}{\Gamma(\nu)} t^{\nu-1}(1-s)^{\nu-\alpha-1} \tag{3.2}$$

for $t,s \in [0,1]$.

Proof. For $0 \leqslant t \leqslant s \leqslant 1$, (3.1) obviously implies (3.2). For $0 < s \leqslant t \leqslant 1$, note from (3.1) that

$$\frac{G(t,s)}{t^{\nu-1}} = \frac{1}{\Gamma(\nu)}\left((1-s)^{\nu-\alpha-1} - \left(1-\frac{s}{t}\right)^{\nu-1}\right).$$

Then, $G(t,s)/t^{\nu-1}$ is decreasing in t for any $0 < s \leqslant t \leqslant 1$, and consequently,

$$\frac{G(t,s)}{t^{\nu-1}} \leqslant \frac{G(s,s)}{s^{\nu-1}} = \frac{1}{\Gamma(\nu)}(1-s)^{\nu-\alpha-1}$$

and

$$\frac{G(t,s)}{t^{\nu-1}} \geqslant G(1,s) = \frac{1}{\Gamma(\nu)}(1-s)^{\nu-\alpha-1}(1-(1-s)^{\alpha})$$

for any $0 < s \leqslant t \leqslant 1$. Thus, we see that (3.2) also holds for $0 \leqslant s \leqslant t \leqslant 1$. This completes the proof of the lemma. □

To prove our theorem, we also need some results from monotone operator theory. The following definition and lemma are well known. For instance, Definition 3.1 can be found in [8, 9, 12, 13, 16, 17, 18] and Lemma 3.3 is a special case of [13, Theorem 2.1]; see also [18, Corollary 4.1].

For any $M > 1$, define

$$P_M = \big\{u = (u_1,\dots,u_m) \in X \;:\; M^{-1}t^{\nu-1} \leqslant u_i(t) \leqslant Mt^{\nu-1} \quad \text{for } t \in [0,1] \text{ and } i = 1,\dots,m\big\}. \tag{3.3}$$

DEFINITION 3.1. Assume that $T : P_M \times P_M \to P_M$. Then, T is called mixed monotone if $T(x,y)$ is nondecreasing in x and nonincreasing in y, i.e., for $x_1, x_2, y_1, y_2 \in P_M$, we have

$$x_1 \leqslant x_2,\ y_1 \geqslant y_2 \Longrightarrow T(x_1,y_1) \leqslant T(x_2,y_2).$$

Moreover, an element $u \in P_M$ is said to be a fixed point of T if $T(u,u) = u$.

LEMMA 3.3. *Assume that $T : P_M \times P_M \to P_M$ is a mixed monotone operator and there exists $\delta \in (0,1)$ such that*

$$T(\kappa u, \kappa^{-1} v) \geqslant \kappa^{\delta} T(u,v) \quad \textit{for } u, v \in P_M \textit{ and } \kappa \in (0,1).$$

Then T has a unique fixed point in P_M.

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. We will prove the theorem in three steps.

Step 1. Let P_M be defined by (3.3). In this step, we show that BVP (1.1), (1.2) has a unique solution in P_M for any $\lambda = (\lambda_1,\dots,\lambda_m) > 0$ if M is large enough (i.e., M satisfies (3.9) below).

For $i=1,\dots,m$, by first letting $x=(x,\dots,x)$ and $\kappa=1/x$, $x>1$, and then letting $x=(1,\dots,1)$, respectively, in (2.1), we obtain

$$g_i(x,\dots,x)\leqslant x^{\delta}g_i(1,\dots,1),\quad x>1, \tag{3.4}$$

and

$$g_i(\kappa,\dots,\kappa)\geqslant \kappa^{\delta}g_i(1,\dots,1),\quad \kappa\in(0,1). \tag{3.5}$$

Similarly, from (2.2) with $x=(1,\dots,1)$ and $x=(x_1,\dots,x_m)=\kappa(y_1,\dots,y_m)$, respectively, we have

$$h_i(\kappa^{-1},\dots,\kappa^{-1})\geqslant \kappa^{\delta}h_i(1,\dots,1),\quad \kappa\in(0,1), \tag{3.6}$$

and

$$h_i(\kappa y_1,\dots,\kappa y_m)\leqslant \kappa^{-\delta}h_i(y_1,\dots,y_m),\quad \kappa\in(0,1),\ y_i>0. \tag{3.7}$$

Choosing $y_j=1,\ j=1,\dots,m$, in (3.7) yields

$$h_i(\kappa,\dots,\kappa)\leqslant \kappa^{-\delta}h_i(1,\dots,1),\quad \kappa\in(0,1). \tag{3.8}$$

For any fixed $\lambda=(\lambda_1,\dots,\lambda_m)>0$, choose $M=M(\lambda)$ large enough so that

$$\begin{aligned}M>\max\Bigg\{1,\ &\left[\frac{\lambda_i}{\Gamma(\nu)}\int_0^1(1-s)^{\nu-\alpha-1}w_i(s)[g_i(1,\dots,1)+s^{\delta(1-\nu)}h(1,\dots,1)]ds\right]^{1/(1-\delta)},\\ &\left[\frac{\lambda_i}{\Gamma(\nu)}\int_0^1(1-s)^{\nu-\alpha-1}(1-(1-s)^{\alpha})w_i(s)\right.\\ &\left.\times[s^{\delta(\nu-1)}g_i(1,\dots,1)+h_i(1,\dots,1)]ds\right]^{-1/(1-\delta)},\ i=1,\dots,m\Bigg\}.\end{aligned} \tag{3.9}$$

Let P_M be defined by (3.3) with the above M. For $u=(u_1,\dots,u_m),\ v=(v_1,\dots,v_m)\in P_M$, define an operator $T_\lambda: P_M\times P_M\to X$ by

$$T_\lambda(u,v)(t)=(T_{\lambda,1}(u,v)(t),\dots,T_{\lambda,m}(u,v)(t)),$$

where

$$T_{\lambda,i}(u,v)(t)=\lambda_i\int_0^1 G(t,s)w_i(s)[g_i(u(s))+h_i(v(s))]ds,\ i=1,\dots,m. \tag{3.10}$$

We will now show that T_λ maps $P_M\times P_M$ into P_M. Let $u=(u_1,\dots,u_m)$, $v=(v_1,\dots,v_m)\in P_M$. For any $i\in\{1,\dots,m\}$ and $s\in(0,1]$, (3.4) implies

$$\begin{aligned}g_i(u(s))=g_i(u_1(s),\dots,u_n(s))&\leqslant g_i(Ms^{\nu-1},\dots,Ms^{\nu-1})\\&\leqslant M^{\delta}g_i(1,\dots,1),\end{aligned}$$

and from (3.8),

$$\begin{aligned}h_i(v(s))=h_i(v_1(s),\dots,v_n(s))&\leqslant h_i(M^{-1}s^{\nu-1},\dots,M^{-1}s^{\nu-1})\\&\leqslant M^{\delta}s^{\delta(1-\nu)}h(1,\dots,1).\end{aligned}$$

Thus,

$$g_i(u(s))+h_i(v(s))\leqslant M^{\delta}[g_i(1,\dots,1)+s^{\delta(1-\nu)}h(1,\dots,1)].$$

Hence, from (3.2), (3.9), and (3.10), we see that

$$\begin{aligned}T_{\lambda,i}(u,v)(t)&\leqslant\frac{\lambda_iM^{\delta}t^{\nu-1}}{\Gamma(\nu)}\int_0^1(1-s)^{\nu-\alpha-1}w_i(s)[g_i(1,\dots,1)+s^{\delta(1-\nu)}h(1,\dots,1)]ds\\&\leqslant Mt^{\nu-1}\quad\text{on }[0,1].\end{aligned}\tag{3.11}$$

On the other hand, for any $i\in\{1,\dots,m\}$ and $s\in(0,1]$, from (3.5),

$$\begin{aligned}g_i(u(s))=g_i(u_1(s),\dots,u_n(s))&\geqslant g_i(M^{-1}s^{\nu-1},\dots,M^{-1}s^{\nu-1})\\&\geqslant M^{-\delta}s^{\delta(\nu-1)}g(1,\dots,1),\end{aligned}$$

and from (3.6),

$$\begin{aligned}h_i(v(s))=h_i(v_1(s),\dots,v_n(s))&\geqslant h_i(Ms^{\nu-1},\dots,Mt^{\nu-1})\\&\geqslant h_i(M,\dots,M)\\&\geqslant M^{-\delta}h(1,\dots,1).\end{aligned}$$

Then,

$$g_i(u(s))+h_i(v(s))\geqslant M^{-\delta}[s^{\delta(\nu-1)}g(1,\dots,1)+h(1,\dots,1)].$$

Again, from (3.2), (3.9), and (3.10), we have

$$\begin{aligned}T_{\lambda,i}(u,v)(t)&\geqslant\frac{\lambda_iM^{-\delta}t^{\nu-1}}{\Gamma(\nu)}\int_0^1(1-s)^{\nu-\alpha-1}(1-(1-s)^{\alpha})w_i(s)\\&\quad\times[s^{\delta(\nu-1)}g_i(1,\dots,1)+h_i(1,\dots,1)]ds\\&\geqslant M^{-1}t^{\nu-1}\quad\text{on }[0,1].\end{aligned}\tag{3.12}$$

From (3.11) and (3.12), we see that $T_{\lambda}(P_M\times P_M)\subseteq P_M$.

Next, for $i=1,\dots,m$, $u=(u_1,\dots,u_m)$, $v=(v_1,\dots,v_m)\in P_M$, $\kappa\in(0,1)$, and $s\in(0,1]$, from (2.1) and (2.2), we have

$$g_i(\kappa u(s))\geqslant\kappa^{\delta}g_i(u(s))\quad\text{and}\quad h_i(\kappa^{-1}v(s))\geqslant\kappa^{\delta}h_i(v(s)).$$

Then, from (3.10),

$$\begin{aligned}T_{\lambda,i}(\kappa u,\kappa^{-1}v)(t)&=\lambda_i\int_0^1G(t,s)w_i(s)[g_i(\kappa u(s))+h_i(\kappa^{-1}v(s))]ds\\&\geqslant\kappa^{\delta}\lambda_i\int_0^1G(t,s)w_i(s)[g_i(u(s))+h_i(v(s))]ds\\&=\kappa^{\delta}T_{\lambda,i}(u,v)(t).\end{aligned}$$

Hence,

$$T_{\lambda}(\kappa u,\kappa^{-1}v)(t)\geqslant\kappa^{\delta}T_{\lambda}(u,v)(t).$$

From the monotonicity of g_i and h_i assumed in (H1), it is easy to verify that $T_{\lambda,i}$ is mixed monotone and so is T_{λ}. Now, we have shown that all the conditions of Lemma 3.3 hold, so there exists a unique $u_{\lambda} = (u_{\lambda,1}\dots,u_{\lambda,m}) \in P_M$ such that $T_\lambda(u_{\lambda}, u_{\lambda}) = u_{\lambda}$. Moreover,

$$\begin{aligned} u_{\lambda,i}(t) &= \lambda_i \int_0^1 G(t,s)w_i(s)[g_i(u_{\lambda}(s)) + h_i(u_{\lambda}(s))]ds \\ &= \lambda_i \int_0^1 G(t,s)f_i(s,u_{\lambda}(s))ds, \quad i = 1,\dots,m, \end{aligned}$$

by (H1). By Lemma 3.1, we see that $u_{\lambda}(t) = (u_{\lambda,1}(t),\dots,u_{\lambda,m}(t))$ is the unique solution of BVP (1.1), (1.2) in P_M for any M satisfying (3.9).

Step 2. In this step, we show that BVP (1.1), (1.2) has at most one positive solution in X for any fixed $\lambda = (\lambda_1,\dots,\lambda_m) > 0$. Assume that BVP (1.1), (1.2) has two positive solutions $u^1_{\lambda}(t) = (u^1_{\lambda,1}(t),\dots,u^1_{\lambda,m}(t))$ and $u^2_{\lambda}(t) = (u^2_{\lambda,1}(t),\dots,u^2_{\lambda,m}(t))$ in X corresponding to the same $\lambda = (\lambda_1,\dots,\lambda_m) > 0$. Then, for $i = 1,\dots,m$ and $j = 1,2$, by Lemma 3.1,

$$u^j_{\lambda,i}(t) = \lambda_i \int_0^1 G(t,s)f_i(s,u^j_{\lambda}(s))ds.$$

From (3.2), we have

$$u^j_{\lambda,i}(t) \geqslant \frac{\lambda_i t^{\nu-1}}{\Gamma(\nu)} \int_0^1 (1-s)^{\nu-\alpha-1}(1-(1-s)^{\alpha})f_i(s,u^j_{\lambda}(s))ds := m_{i,j}t^{\nu-1}$$

and

$$u^j_{\lambda,i}(t) \leqslant \frac{\lambda_i t^{\nu-1}}{\Gamma(\nu)} \int_0^1 (1-s)^{\nu-\alpha-1} f_i(s,u^j_{\lambda}(s))ds := M_{i,j}t^{\nu-1},$$

where

$$m_{i,j} = \frac{\lambda_i}{\Gamma(\nu)} \int_0^1 (1-s)^{\nu-\alpha-1}(1-(1-s)^{\alpha})f_i(s,u^j_{\lambda}(s))ds$$

and

$$M_{i,j} = \frac{\lambda_i}{\Gamma(\nu)} \int_0^1 (1-s)^{\nu-\alpha-1} f_i(s,u^j_{\lambda}(s))ds.$$

Choose M large enough so that M satisfies (3.9) and

$$M \geqslant \max\{1/m_{i,j},\ M_{i,j} : i = 1,\dots,m,\ j = 1,2\}.$$

Then,

$$M^{-1}t^{\nu-1} \leqslant u^j_{\lambda,i}(t) \leqslant Mt^{\nu-1} \quad \text{for } t \in [0,1],\ i = 1,\dots,m, \text{ and } j = 1,2. \tag{3.13}$$

Consequently, u^1_{λ}, $u^2_{\lambda} \in P_M$ with the above M. But by Step 1, we know that BVP (1.1), (1.2) has a unique solution in P_M. Hence, $u^1_{\lambda}(t) \equiv u^2_{\lambda}(t)$ on $[0,1]$. This shows that BVP (1.1), (1.2) has at most one positive solution in X.

Step 3. In this step, we finish the proof of the theorem. Combing Steps 1 and 2, we see that BVP (1.1), (1.2) has a unique solution $u_{\lambda}(t) = (u_{\lambda,1}(t), \dots, u_{\lambda,m}(t))$ for any $\lambda = (\lambda_1, \dots, \lambda_m) > 0$. In the rest of the proof, we show the "furthermore" part of the theorem. For $i = 1, \dots, m$, define an operator $K_i : P_M \times P_M \to X$ by

$$K_i(u, v)(t) = \int_0^1 G(t,s) w_i(s)[g_i(u(s)) + h_i(v(s))]ds. \tag{3.14}$$

Then, from (3.10),

$$T_{\lambda,i}(u, v)(t) = \lambda_i K_i(u, v)(t), \quad i = 1, \dots, m. \tag{3.15}$$

Assume $\lambda := (\lambda_1, \dots, \lambda_m) > \mu := (\mu_1, \dots, \mu_m) > 0$. Let $u_{\lambda} = (u_{\lambda,1}, \dots, u_{\lambda,m})$ and $u_{\mu} = (u_{\mu,1}, \dots, u_{\mu,m})$ be the unique positive solutions of BVP (1.1), (1.2) corresponding to $\lambda = (\lambda_1, \dots, \lambda_m)$ and $\lambda = (\mu_1, \dots, \mu_m)$, respectively. Recall that $0 < \delta < 1/2$. Define a set $B(\lambda, \mu)$ by

$$B(\lambda,\mu) = \left\{ \gamma > 0 \; : \; \frac{1}{\gamma}\left(\frac{\lambda_i}{\mu_i}\right)^{\frac{1}{1-\delta}} u_{\mu,i}(t) \geqslant u_{\lambda,i}(t) \geqslant \gamma \left(\frac{\lambda_i}{\mu_i}\right)^{\frac{1-2\delta}{1-\delta}} u_{\mu,i}(t) \text{ on } [0,1],\ i = 1, \dots, m \right\}.$$

We claim that $B(\lambda, \mu) \neq \emptyset$. To see this, note that, as in obtaining (3.13), there exists $C > 1$ large enough so that

$$C^{-1} t^{\nu-1} \leqslant u_{\lambda,i}(t) \leqslant C t^{\nu-1} \quad \text{and} \quad C^{-1} t^{\nu-1} \leqslant u_{\mu,i}(t) \leqslant C t^{\nu-1}$$

for $t \in [0,1]$ and $i = 1, \dots, m$. Thus,

$$\frac{1}{C^2} \leqslant \frac{u_{\lambda,i}(t)}{u_{\mu,i}(t)} \leqslant C^2 \quad \text{for } t \in (0,1] \text{ and } i = 1, \dots, m.$$

Then, $u_{\lambda,i}(t)/u_{\mu,i}(t)$ can be extended continuously to the interval $[0,1]$. Let

$$l_i(t) = \left(\frac{\mu_i}{\lambda_i}\right)^{\frac{1-2\delta}{1-\delta}} \frac{u_{\lambda,i}(t)}{u_{\mu,i}(t)}, \quad t \in [0,1],\ i = 1, \dots, m.$$

Clearly, $l_i \in C[0,1]$ and $l_i(t) > 0$ on $[0,1]$. Now, it is easy to check that any γ satisfying

$$0 < \gamma < \min\left\{ \min_{t\in[0,1]} l_i(t),\ \min_{t\in[0,1]} (1/l_i(t)),\ i = 1, \dots, m \right\}$$

is in $B(\lambda, \mu)$. Therefore, $B(\lambda, \mu) \neq \emptyset$.

Define $\bar{\gamma} = \sup B(\lambda, \mu)$. Then,

$$\frac{1}{\bar{\gamma}}\left(\frac{\lambda_i}{\mu_i}\right)^{\frac{1}{1-\delta}} u_{\mu,i}(t) \geqslant u_{\lambda,i}(t) \geqslant \bar{\gamma}\left(\frac{\lambda_i}{\mu_i}\right)^{\frac{1-2\delta}{1-\delta}} u_{\mu,i}(t), \quad t \in [0,1],\ i = 1, \dots, m. \tag{3.16}$$

We claim that $\bar{\gamma} \geqslant 1$. In fact, if $0 < \bar{\gamma} < 1$, then, for $i = 1, \ldots, m$, from the monotonicity of g_i and h_i, and (3.16), we have

$$\begin{aligned}&\lambda_i \left[g_i(u_{\lambda,i}(t)) + h_i(u_{\lambda,i}(t))\right] \\ \geqslant\ &\lambda_i \left[g_i\left(\bar{\gamma}\left(\lambda_i\mu_i^{-1}\right)^{(1-2\delta)/(1-\delta)} u_{\mu,i}(t)\right) + h_i\left(\bar{\gamma}^{-1}\left(\lambda_i\mu_i^{-1}\right)^{1/(1-\delta)} u_{\mu,i}(t)\right)\right] \\ \geqslant\ &\lambda_i \left[g_i\left(\bar{\gamma} u_{\mu_i}(t)\right) + h_i\left(\bar{\gamma}^{-1}\left(\lambda_i\mu_i^{-1}\right)^{1/(1-\delta)} u_{\mu,i}(t)\right)\right]\end{aligned}$$

since $\lambda_i/\mu_i > 1$. This, together with (2.1), (2.2), and (3.7) with $\kappa = (\lambda_i^{-1}\mu_i)^{1/(1-\delta)}$, implies that

$$\begin{aligned}&\lambda_i \left[g_i(u_{\lambda,i}(t)) + h_i(u_{\lambda,i}(t))\right] \\ \geqslant\ &\lambda_i \left[\bar{\gamma}^{\delta} g_i\left(u_{\mu,i}(t)\right) + \bar{\gamma}^{\delta} h_i\left(\left(\lambda_i\mu_i^{-1}\right)^{1/(1-\delta)} u_{\mu,i}(t)\right)\right] \\ \geqslant\ &\lambda_i \left[\bar{\gamma}^{\delta} g_i\left(u_{\mu,i}(t)\right) + \bar{\gamma}^{\delta} (\lambda_i^{-1}\mu_i)^{\delta/(1-\delta)} h_i\left(u_{\mu,i}(t)\right)\right] \\ \geqslant\ &\bar{\gamma}^{\delta} (\lambda_i^{-1}\mu_i)^{\delta/(1-\delta)} \lambda_i \left[g_i\left(u_{\mu,i}(t)\right) + h_i\left(u_{\mu,i}(t)\right)\right] \\ =\ &\bar{\gamma}^{\delta} (\lambda_i\mu_i^{-1})^{(1-2\delta)/(1-\delta)} \mu_i \left[g_i\left(u_{\mu,i}(t)\right) + h_i\left(u_{\mu,i}(t)\right)\right].\end{aligned} \tag{3.17}$$

Now, from (3.14), (3.15), and (3.17), it is easy to see that

$$\begin{aligned}u_{\lambda,i}(t) = T_{\lambda,i}(u_{\lambda}, u_{\lambda})(t) &= \lambda_i K_i(u_{\lambda}, u_{\lambda})(t) \\ &\geqslant \bar{\gamma}^{\delta} (\lambda_i\mu_i^{-1})^{(1-2\delta)/(1-\delta)} \mu_i K_i(u_{\mu}, u_{\mu})(t) \\ &= \bar{\gamma}^{\delta} (\lambda_i\mu_i^{-1})^{(1-2\delta)/(1-\delta)} T_{\mu,i}(u_{\lambda}, u_{\lambda})(t) \\ &= \bar{\gamma}^{\delta} (\lambda_i\mu_i^{-1})^{(1-2\delta)/(1-\delta)} u_{\mu,i}(t) \quad \text{on } [0,1].\end{aligned} \tag{3.18}$$

On the other hand, using a similar argument as in verifying (3.17), we obtain

$$\begin{aligned}&\mu_i \left[g_i(u_{\mu,i}(t)) + h_i(u_{\mu,i}(t))\right] \\ \geqslant\ &\bar{\gamma}^{\delta} (\lambda_i^{-1}\mu_i)^{1/(1-\delta)} \lambda_i [g_i(u_{\lambda,i}(t)) + h_i(u_{\lambda,i}(t))].\end{aligned}$$

Then, again from (3.14), (3.15), and (3.17), we have

$$\begin{aligned}u_{\mu,i}(t) = T_{\mu,i}(u_{\mu}, u_{\mu})(t) &= \mu_i K_i(u_{\mu}, u_{\mu})(t) \\ &\geqslant \bar{\gamma}^{\delta} (\lambda_i^{-1}\mu_i)^{1/(1-\delta)} \lambda_i K_i(u_{\lambda}, u_{\lambda})(t) \\ &= \bar{\gamma}^{\delta} (\lambda_i^{-1}\mu_i)^{1/(1-\delta)} T_{\lambda,i}(u_{\lambda}, u_{\lambda})(t) \\ &= \bar{\gamma}^{\delta} (\lambda_i^{-1}\mu_i)^{1/(1-\delta)} u_{\lambda,i}(t) \quad \text{on } [0,1].\end{aligned} \tag{3.19}$$

From (3.18) and (3.19), we get that

$$\frac{1}{\bar{\gamma}^{\delta}} \left(\frac{\lambda_i}{\mu_i}\right)^{\frac{1}{1-\delta}} u_{\mu,i}(t) \geqslant u_{\lambda,i}(t) \geqslant \bar{\gamma}^{\delta} \left(\frac{\lambda_i}{\mu_i}\right)^{\frac{1-2\delta}{1-\delta}} u_{\mu,i}(t), \quad t \in [0,1],\ i = 1, \ldots, m.$$

Since $0<\delta<1/2$, we obtain that $\bar{\gamma}^{\delta}>\bar{\gamma}$. But this contradicts the definition of $\bar{\gamma}$. Therefore, $\bar{\gamma}\geqslant 1$. Then, When $\lambda>\mu$, from (3.16), we have

$$u_{\lambda,i}(t)\geqslant\left(\lambda_i\mu_i^{-1}\right)^{(1-2\delta)/(1-\delta)}u_{\mu,i}(t) \text{ on } [0,1],\ i=1,\dots,m. \tag{3.20}$$

Consequently,

$$u_{\lambda,i}(t)>u_{\mu,i}(t) \quad \text{for } t\in[0,1] \text{ and } i=1,\dots,m.$$

This proves part (a).

Next, we show part (b). When $\lambda>\mu$, from (3.20),

$$u_{\mu,i}(t)\leqslant\left(\lambda_i^{-1}\mu_i\right)^{(1-2\delta)/(1-\delta)}u_{\lambda,i}(t) \quad \text{for } t\in(0,1] \text{ and } i=1,\dots,m,$$

which implies that

$$||u_\mu||\leqslant\max\left\{\left(\lambda_i^{-1}\mu_i\right)^{(1-2\delta)/(1-\delta)},\ i=1,\dots,m\right\}||u_\lambda||.$$

Thus, $||u_\mu||\to 0$ as $\mu\to 0^+$. Similarly, (3.20) also implies that

$$||u_\lambda||\geqslant\min\left\{\left(\lambda_i\mu_i^{-1}\right)^{(1-2\delta)/(1-\delta)},\ i=1,\dots,m\right\}||u_\mu||.$$

Thus, $||u_\lambda||\to\infty$ as $\lambda\to\infty$.

Finally, we prove part (c). When $\lambda>\mu$, from the left hand inequality in (3.16), we have

$$u_{\lambda,i}(t)\leqslant\left(\lambda_i\mu_i^{-1}\right)^{1/(1-\delta)}u_{\mu_i}(t) \quad \text{for } t\in[0,1] \text{ and } i=1,\dots,m. \tag{3.21}$$

Then,

$$||u_\lambda-u_\mu||\leqslant\max\left\{\left(\left(\lambda_i\mu_i^{-1}\right)^{1/(1-\delta)}-1\right),\ i=1,\dots,m\right\}||u_\mu||.$$

As a result, $||u_\lambda-u_\mu||\to 0$ as $\lambda\to\mu^+$. When $\lambda<\mu$, from (3.21) with λ and μ switched, we have

$$u_{\lambda,i}(t)\geqslant\left(\lambda_i\mu_i^{-1}\right)^{1/(1-\delta)}u_{\mu,i}(t) \text{ on } [0,1],\ i=1,\dots,m.$$

This, together with $u_{\lambda,i}(t)<u_{\mu,i}(t)$ on $[0,1]$, implies that

$$||u_\lambda-u_\mu||\leqslant\max\left\{1-\left(\left(\lambda_i\mu_i^{-1}\right)^{1/(1-\delta)}\right),\ i=1,\dots,m\right\}||u_\mu||.$$

Then, $||u_\lambda-u_\mu||\to 0$ as $\lambda\to\mu^-$. Hence, part (c) holds. This completes the proof of the theorem. □

Proof of Corollary 2.1. With $f_i(t,x_1,\dots,x_m)=w_i(t)(g_i(x_1,\dots,x_m)+h_i(x_1,\dots,x_m))$, where $w_i(t)=p_i(t)$, $g_i(x_1,\dots,x_m)=\sum_{k=1}^m a_{ik}x_k^{b_{ik}}$, and $h_i(x_1,\dots,x_m)=\sum_{k=1}^m c_{ik}x_k^{-d_{ik}}$, it is clear that BVP (1.3), (1.4) is of the form of BVP (1.1), (1.2), and (H1) holds. Let ζ be defined in (A1). Then, (A1) and (A2) imply that (H2) and (H3) hold with $\delta=\zeta$. The conclusion now readily follows from Theorem 2.1. □

REFERENCES

[1] R. P. AGARWAL, D. O'REGAN, AND S. STANEK, *Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations*, J. Math. Anal. Appl. **371** (2010), 57–68.

[2] B. AHMAD AND S. SIVASUNDARAM, *On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order*, Appl. Math. Comput. **217** (2010), 480–487.

[3] Z. BAI AND H. LÜ, *Positive solutions for boundary value problem of nonlinear fractional differential equations*, J. Math. Anal. Appl. **311** (2005), 495–505.

[4] A. DOGAN AND J. R. GRAEF, AND L. KONG, *Higher order singular multi-point boundary value problems on time scales*, Proc. Edinburgh Math. Soc., to appear.

[5] M. EL-SHAHED AND J. J. NIETO, *Nontrivial solutions for a nonlinear multi-point boundary value problem of fractional order*, Comput. Math. Appl. **59** (2010), 3438–3443.

[6] C. S. GOODRICH, *Existence of a positive solution to a class of fractional differential equations*, Appl. Math. Lett. **23** (2010), 1050–1055.

[7] J. R. GRAEF AND L. KONG, *Uniqueness and parameter dependence of positive solutions of third order boundary value problems with p-Laplacian*, submitted for publication.

[8] D. GUO, *Existence and uniqueness of positive fixed point for mixed monotone operators and applications*, Appl. Anal. **46** (1992), 91–100.

[9] D. GUO AND V. LAKSHMIKANTHAM, *Coupled fixed points of nonlinear operators with applications*, Nonlinear Anal. **11** (1987), 623–632.

[10] A. A. KILBAS, H. M. SRIVASTAVA, AND J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, Elsevier, Boston, 2006.

[11] L. KONG, *Second order singular boundary value problems with integral boundary conditions*, *Nonlinear Anal.* **72** (2010), 2628–2638

[12] X. LIAN AND Y. LI, *Fixed point theorems for a class of mixed monotone operators with applications*, Nonlinear Anal. **67** (2007), 2752–2762.

[13] Z. LIANG, L. ZHANG, AND S. LI, *Fixed point theorems for a class of mixed monotone operators*, Z. Anal. Anwend. **22** (2003), 529–542.

[14] I. PODLUBNY, *Fractional Differential Equations*, Academic Press, New York, 1999.

[15] Z. WEI, W. DONG, AND J. CHE, *Periodic boundary value problems for fractional differential equations involving a Riemann-Liouville fractional derivative*, Nonlinear Anal. **73** (2010), 3232–3238.

[16] Y. WU, *New fixed point theorems and applications of mixed monotone operator*, J. Math. Anal. Appl. **341** (2008), 883–893.

[17] Z. ZHANG AND K. WANG, *On fixed point theorems of mixed monotone operators and applications*, Nonlinear Anal. **70** (2009), 3279–3284.

[18] Z. ZHAO, *Existence and uniqueness of fixed points for some mixed monotone operators*, Nonlinear Anal. **73** (2010), 1481–1490.

(Received November 8, 2010)

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37403, USA
e-mail: John-Graef@utc.edu

Lingju Kong
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37403, USA
e-mail: Lingju-Kong@utc.edu

Qingkai Kong
Department of Mathematics
Northern Illinois University
DeKalb, IL 60115, USA
e-mail: kong@math.niu.edu