Fractional
Differential
Calculus

Volume 2, Number 1 (2012), 99–108

doi:10.7153/fdc-02-07

BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL INCLUSIONS IN BANACH SPACES

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Abstract. This paper is concerned with the existence of solutions of nonlinear fractional differential inclusions with boundary conditions in a Banach space. The main result is obtained by using the set-valued analog of Mönch fixed point theorem combined with the Kuratowski measure of noncompactness.

1. Introduction

The topic of fractional calculus, which deals with derivatives and integrals of arbitrary orders, is enjoying growing interest not only among mathematicians, but also among physicists and engineers. In fact, this branch of calculus has found numerous miscellaneous applications connected with real world problems as they appear in many fields of science and engineering, including fluid flow, signal and image processing, fractals theory, control theory, electromagnetic theory, fitting of experimental data, optics, potential theory, biology, chemistry, diffusion, and viscoelasticity. For some recent developments on the topic, see the papers of El-Sayed et al [20], Gafiychuk et al [22], He [24], Jumarie [28], and Luchko et al [31], and the monographs of Hilfer [26], Kilbas et al [29], and Sabatier et al [34] and references therein.

On the other hand, realistic problems arising in economics, optimal control, and so on, can be modeled by differential inclusions, and so differential inclusions are widely investigated by many authors; see the papers of Agarwal et al [1], Ahmad and Ntouyas [3], Benchohra et al [7, 8, 10, 11], Chang and Nieto [17], Darwish and Ntouyas [18], Frigon [21], and Hamani et al [23], and the monograph of Smirnov [35] and references therein.

In this paper, we consider the existence of solutions for the following fractional differential inclusions with boundary conditions

$$ {}^cD^r y(t) \in F(t, y(t)), \text{ for a.e. } t \in J = [0, T], \quad 1 < r < 2 \tag{1} $$

$$ y(0) = y_0, \quad y(T) = y_T, \tag{2} $$

where ${}^cD$ is the Caputo derivative, $F : J \times E \to \mathscr{P}(E)$ is a multivalued map, $y_0, y_T \in E$ and $(E, |\cdot|)$ denotes a Banach space. Our main tool here is the set-valued analog of

Mathematics subject classification (2010): 26A33, 34A60, 34B15.
Keywords and phrases: differential inclusions, Caputo fractional derivative, boundary value problem, Banach space, existence, fixed point, measure of noncompactness.

Paper FDC-02-07

Mönch's fixed point theorem combined with the technique of measure of noncompactness. Recently, this has proved to be a valued tool in solving fractional differential equations in Banach spaces, for details see the papers of Agarwal et al [2], and Benchohra et al [6, 12, 13, 14, 15, 16]. The main result of the present paper extends the problem (1)-(2) considered in the finite dimensional case by Benchohra et al [7, 9] and Chang and Nieto [17].

2. Preliminaries

In this section we introduce some basic definitions and lemmas which are used throughout this paper. Let $C(J,E)$ the Banach space of continuous functions from J into E with the norm

$$\|y\| = \sup\{|y(t)|, t \in J\}.$$

Let $L^1(J,E)$ be the Banach space of functions $y : J \to E$ which are Bochner integrable and normed by

$$\|y\|_{L^1} = \int_0^T |y(t)|dt.$$

$AC^1(J,E)$ is the space of continuously differentiable functions whose first derivative is absolutely continuous.

We use the notations: 2^E is the collection of all subsets of E and $\mathscr{P}(E) = 2^E \backslash \emptyset$.

$$\mathscr{P}_c(E) = \{A \subset E : A \text{ is nonempty, convex}\},$$
$$\mathscr{P}_{kc}(E) = \{A \subset E : A \text{ is nonempty, compact, convex}\}.$$

Let X, Y be two sets, $N : X \to 2^Y$ a set-valued map, and $A \subset Y$. We define

$$graph(N) = \{(x,y) : \ x \in X, \ y \in N(X)\} \qquad \text{(the graph of } N\text{)}.$$

For more details on multi-valued maps see the books of Deimling [19] and Hu and Papageorgiou [27].

Let $R > 0$, and let

$$B = \{x \in E : |x| \leqslant R\},$$

and

$$U = \{x \in C(J,E) : \|x\| < R\}.$$

Clearly $\overline{U} = C(J,B)$.

DEFINITION 2.1. ([29, 33]) The fractional (arbitrary) order integral of the function $h \in L^1(J,E)$ of order $r \in \mathbb{R}^+$ is defined by

$$I_0^r h(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} h(s)ds,$$

where Γ is the Gamma function.

DEFINITION 2.2. ([29]) For a function h given on the interval J, the Caputo fractional-order derivative of h, is defined by

$$ {}^{c}D_{0+}^{r}h(t) = \frac{1}{\Gamma(n-r)} \int_0^t \frac{h^{(n)}(s)ds}{(t-s)^{1-n+r}} $$

Here $n = [r]+1$ and $[r]$ denotes the integer part of r.

For example for $0 < r \leqslant 1$ and $h : J \to E$ an absolutely continuous function, then the fractional derivative of order r of h exists.

For convenience, we first recall the definition of the Kuratowski measure of noncompactness, and summarize the main properties of this measure.

DEFINITION 2.3. ([4, 5]) Let E be a Banach space and let Ω_E be the family of bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \to [0,\infty]$ defined by

$$ \alpha(M) = \inf\{\varepsilon > 0 : M \subset \bigcup_{j=1}^{m} M_j \text{ and } \mathrm{diam}(M_j) \leqslant \varepsilon\}; \text{ here } M \in \Omega_E. $$

Properties:

(a) $\alpha(M) = 0 \Leftrightarrow \overline{M}$ is compact (M is relatively compact).

(b) $\alpha(M) = \alpha(\overline{M})$.

(c) $M_1 \subset M_2 \Rightarrow \alpha(M_1) \leqslant \alpha(M_2)$.

(d) $\alpha(M_1 + M_2) \leqslant \alpha(M_1) + \alpha(M_2)$.

(e) $\alpha(cM) = |c|\alpha(M); c \in \mathbb{R}$.

(f) $\alpha(convM) = \alpha(M)$.

The details of α and its properties can be found in [4, 5].

DEFINITION 2.4. A multivalued map $F : J \times E \to \mathscr{P}(E)$ is said to be Carathéodory if

(i) $t \to F(t,u)$ is measurable for each $u \in E$.

(ii) $u \to F(t,u)$ is upper semicontinuous for almost all $t \in J$.

For each $y \in C(J,E)$, define the set of selections of F by

$$ S_{F,y} = \{f \in L^1(J,E) : f(t) \in F(t,y(t)) \text{ for a.e. } t \in J\}. $$

THEOREM 2.1. ([25]) *Let E be a Banach space and $C \subset L^1(J,E)$ countable with $|u(t)| \leqslant h(t)$ for a.e. $t \in J$, and every $u \in C$; where $h \in L^1(J,\mathbb{R}_+)$. Then the function $\phi(t) = \alpha(C(t))$ belongs to $L^1(J,\mathbb{R}_+)$ and satisfies*

$$\alpha\left(\left\{\int_0^T u(s)ds : u \in C\right\}\right) \leqslant 2\int_0^T \alpha(C(s))ds.$$

Let us now recall the set-valued analog of Mönch's fixed point theorem.

THEOREM 2.2. ([32]) *Let K be a closed, convex subset of a Banach space E; U a relatively open subset of K, and $N : \overline{U} \to P_c(K)$. Assume graph$(N)$ is closed, N maps compact sets into relatively compact sets, and that for some $x_0 \in U$; the following two conditions are satisfied:*

$$\left.\begin{array}{l} M \subset \overline{U}, M \subset conv(x_0 \cup N(M)) \\ and\ \overline{M} = \overline{C}\ with\ C \subset M countable \end{array}\right\} \quad \Longrightarrow \quad \overline{M} compact. \tag{3}$$

$$x \notin (1-\lambda)x_0 + \lambda N(x) \quad for\ all\ x \in \overline{U}\backslash U, \lambda \in (0,1) \tag{4}$$

Then there exists $x \in \overline{U}$ with $x \in N(x)$.

Let us list the following hypotheses:

(H1) $F : J \times E \to \mathscr{P}_{kc}(E)$ is a Carathéodory multi-valued map.

(H2) For each $R > 0$, there exists a function $p \in L^1(J,\mathbb{R}_+)$ such that

$$\|F(t,y)\|_{\mathscr{P}} = \sup\{|v| : v(t) \in F(t,y)\} \leqslant p(t)$$

for each $(t,y) \in J \times E$ with $|y| \leqslant R$, and

$$\liminf_{R \to +\infty} \frac{\int_0^T p(t)dt}{R} = \delta < \infty. \tag{5}$$

(H3) There is a Carathéodory function $\psi : J \times [0,2R] \to \mathbb{R}_+$ such that

$$\alpha(F(t,M)) \leqslant \psi(t,\alpha(M)),\ \text{a.e. } t \in J,\ \text{and each } M \subset B,$$

and the unique solution $\varphi \in C(J,[0,2R])$ of the inequality

$$\varphi(t) \leqslant 2\left[\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}\psi(s,\varphi(s))ds + \frac{t}{T\Gamma(r)}\int_0^T (T-s)^{r-1}\psi(s,\varphi(s))ds\right], \quad t \in J. \tag{6}$$

is $\varphi \equiv 0$.

LEMMA 2.1. ([30]) *Let J be a compact real interval. Let F be a multivalued map satisfying (H_1) and let Θ be a linear continuous map from $L^1(J,E) \to C(J,E)$. Then the operator*

$$\Theta \circ S_{F,y} : C(J,E) \to \mathscr{P}_{kc}(C(J,E)),\ y \mapsto (\Theta \circ S_{F,y})(y) = \Theta(S_{F,y})$$

is a closed graph operator in $C(J,E) \times C(J,E)$.

3. Existence results

DEFINITION 3.1. A function $y \in AC^1(J,E)$ is said to be a solution of (1)–(2), if there exists a function $f \in L^1(J,E)$ with $f(t) \in F(t,y(t))$, for a.e. $t \in J$, such that

$$^cD^r y(t) = f(t), \quad \text{for a.e. } t \in J, \; 1 < r < 2,$$

and the function y satisfies conditions (2).

For the existence of solutions for the problem (1)–(2), we need the following auxiliary lemma:

LEMMA 3.1. [7] *Let* $1 < r \leqslant 2$ *and let* $f : J \to E$ *be continuous. The unique solution* y *of the linear problem*

$$^cD^r y(t) = f(t), \quad t \in J, \tag{7}$$

$$y(0) = y_0, \quad y(T) = y_T, \tag{8}$$

is given by

$$y(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s)ds - \frac{t}{T\Gamma(r)} \int_0^T (T-s)^{r-1} f(s)ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T. \tag{9}$$

THEOREM 3.1. *Suppose that (H1)-(H3) are satisfied. Then the problem (1)–(2) has at least one solution on* $C(J,B)$, *provided that*

$$\delta < \frac{\Gamma(r)}{2T}. \tag{10}$$

Proof. Transform the problem (1)–(2) into a fixed point problem. Consider the multi-valued map $N : C(J,E) \to \mathscr{P}(C(J,E))$ defined by

$$N(y) = \Big\{ h \in C(J,E) : h(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} v(s)ds - \frac{t}{T\Gamma(r)} \int_0^T (T-s)^{r-1} v(s)ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T, \; v \in S_{F,y} \Big\}.$$

The fixed points of N are solutions to (1)–(2). We shall show that N satisfies the assumptions of the Theorem 2.2. The proof will be given in several steps.

Step 1: $N(y)$ *is convex for each* $y \in C(J,E)$.

If h_1, h_2 belong to $N(y)$, then there exist f_1, $f_2 \in S_{F,y}$ such that for a.e. $t \in J$ we have

$$h_i(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f_i(s)ds - \frac{t}{T\Gamma(r)} \int_0^T (T-s)^{r-1} f_i(s)ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T, \quad i = 1, 2.$$

Let $0 \leqslant \lambda \leqslant 1$. For each $t \in J$, we have

$$\begin{aligned}\big(\lambda h_1 + (1-\lambda)h_2\big)(t) = &\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}(\lambda f_1 + (1-\lambda)f_2)(s)ds \\ &- \frac{t}{T\Gamma(r)}\int_0^T (T-s)^{r-1}(\lambda f_1 + (1-\lambda)f_2)(s)ds \\ &- \Big(\frac{t}{T}-1\Big)y_0 + \frac{t}{T}y_T.\end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$\lambda h_1 + (1-\lambda)h_2 \in N(y)$$

Step 2: $N(M)$ is relatively compact for each compact $M \subset \overline{U}$.

To prove this, let $M \subset \overline{U}$ be a compact set and let (h_n) be any sequence of elements of $N(M)$. We show that (h_n) has a convergent subsequence by using the Arzèla-Ascoli criterion of noncompactness in $C(J,E)$. Since $(h_n) \in N(M)$ there exist $(y_n) \in M$ and $(f_n) \in S_{F,y_n}$ such that

$$\begin{aligned}h_n(t) = &\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} f_n(s)ds - \frac{t}{T\Gamma(r)}\int_0^T (T-s)^{r-1} f_n(s)ds \\ &- \Big(\frac{t}{T}-1\Big)y_0 + \frac{t}{T}y_T.\end{aligned}$$

Using Theorem 2.1 and the properties of the measure of Kuratowski α, we obtain that

$$\alpha\big(\{h_n(t)\}\big) \leqslant 2\left[\frac{1}{\Gamma(r)}\int_0^T \alpha\big(\{(t-s)^{r-1}f_n(s)\}\big)ds + \frac{t}{T\Gamma(r)}\int_0^T \alpha\big(\{(T-s)^{r-1}f_n(s)\}\big)ds\right]. \tag{11}$$

On the other hand, since $M(s)$ is compact in E, the set $\{f_n(s); n \geqslant 1\}$ is compact. Consequently, $\alpha\big(\{f_n(s); n \geqslant 1\}\big) = 0$ for a.e. $s \in J$. Furthermore,

$$\alpha\big(\{(T-s)^{r-1}f_n(s); n \geqslant 1\}\big) = (T-s)^{r-1}\alpha\big(\{f_n(s); n \geqslant 1\}\big) = 0$$

and

$$\alpha\big(\{(t-s)^{r-1}f_n(s); n \geqslant 1\}\big) = (t-s)^{r-1}\alpha\big(\{f_n(s); n \geqslant 1\}\big) = 0$$

for a.e. $t, s \in J$. Now (11) implies that $\{h_n(t); n \geqslant 1\}$ is relatively compact in E, for each $t \in J$.

In addition for each t_1 and t_2 from J, $t_1 < t_2$, we have

$$\begin{aligned}\left|h_n(t_2)-h_n(t_1)\right| &= \Big|\frac{(t_2-t_1)}{T}(y_T-y_0)+\frac{1}{\Gamma(r)}\int_0^{t_1}\left[(t_2-s)^{r-1}-(t_1-s)^{r-1}\right]f_n(s)ds \\ &\quad +\frac{1}{\Gamma(r)}\int_{t_1}^{t_2}(t_2-s)^{r-1}f_n(s)ds+\frac{t_2-t_1}{T\Gamma(r)}\int_0^T(T-s)^{r-1}f_n(s)ds\Big| \\ &\leqslant |y_T-y_0|(t_2-t_1)+\left|\frac{1}{\Gamma(r)}\int_0^{t_1}[(t_2-s)^{r-1}-(t_1-s)^{r-1}]f_n(s)ds\right| \\ &\quad +\left|\frac{1}{\Gamma(r)}\int_{t_1}^{t_2}(t_2-s)^{r-1}f_n(s)ds\right|+\frac{t_2-t_1}{\Gamma(r)}\int_0^T p(s)ds \\ &\leqslant |y_T-y_0|(t_2-t_1)+\frac{1}{\Gamma(r)}\int_0^{t_1}\left|(t_2-s)^{r-1}-(t_1-s)^{r-1}\right|p(s)ds \\ &\quad +\frac{1}{\Gamma(r)}\int_{t_1}^{t_2}\left|(t_2-s)^{r-1}\right|p(s)ds+\frac{t_2-t_1}{\Gamma(r)}\int_0^T p(s)ds. \qquad (12)\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. This shows that $\{h_n; n \geqslant 1\}$ is equicontinuous. Consequently, $\{h_n; n \geqslant 1\}$ is relatively compact in $C(J,E)$.

Step 3: N has a closed graph.

Let $(y_n, h_n) \in graph(N)$, $n \geqslant 1$, with $\|y_n - v\|$, $\|h_n - h\| \to 0$ as $n \to \infty$. We must show that $(y, h) \in graph(N)$.

$(y_n, h_n) \in graph(N)$ means that $h_n \in N(y_n)$ which means that there exists $f_n \in S_{F,y_n}$, such that for each $t \in J$,

$$h_n(t)=\frac{1}{\Gamma(r)}\int_0^t(t-s)^{r-1}f_n(s)ds-\frac{t}{T\Gamma(r)}\int_0^T(T-s)^{r-1}f_n(s)ds-\left(\frac{t}{T}-1\right)y_0+\frac{t}{T}y_T.$$

Consider the continuous linear operator

$$\Theta : L^1(J,E) \to C(J,E)$$
$$f \mapsto \Theta(f)(t)=\frac{1}{\Gamma(r)}\int_0^t(t-s)^{r-1}f(s)ds-\frac{t}{T\Gamma(r)}\int_0^T(T-s)^{r-1}f(s)ds.$$

Clearly,

$$\left\|\left[h_n(t)+\left(\frac{t}{T}-1\right)y_0-\frac{t}{T}y_T\right]-\left[h(t)+\left(\frac{t}{T}-1\right)y_0-\frac{t}{T}y_T\right]\right\| \to 0, \text{ as } n \to \infty.$$

From Lemma 2.1 it follows that $\Theta \circ S_F$ is a closed graph operator. Moreover, we have

$$h_n(t)+\left(\frac{t}{T}-1\right)y_0-\frac{t}{T}y_T \in \Theta(S_{F,y_n}).$$

Since $y_n \to y$, Lemma 2.3 implies that

$$h(t)+\left(\frac{t}{T}-1\right)y_0-\frac{t}{T}y_T=\frac{1}{\Gamma(r)}\int_0^t(t-s)^{r-1}f(s)ds-\frac{t}{T\Gamma(r)}\int_0^T(T-s)^{r-1}f(s)ds$$

for some $f \in S_{F,y}$.

Step 4: Suppose $M \subset \overline{U}$, $M \subset conv(\{0\} \cup N(M))$, and $\overline{M} = \overline{C}$ for some countable set $C \subset M$. Using an estimation of type (12), we see that $N(M)$ is equicontinuous. Then, from $M \subset conv(\{0\} \cup N(M))$, we deduce that M is equicontinuous, too. In order to apply the Arzèla-Ascoli theorem, it remains to show that $M(t)$ is relatively compact in E for each $t \in J$. Since

$$C \subset M \subset conv(\{0\} \cup N(M)) \quad \text{and} \quad C \text{ is countable},$$

we can find a countable set $H = \{h_n : n \geqslant 1\} \subset N(M)$ with $C \subset conv(\{0\} \cup H)$. Then, there exist $y_n \in M$ and $f_n \in S_{F,y_n}$ such that

$$h_n(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f_n(s)ds - \frac{t}{T\Gamma(r)} \int_0^T (T-s)^{r-1} f_n(s)ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T.$$

From $M \subset \overline{C} \subset \overline{conv}(\{0\} \cup H)$, and according to Theorem 2.1, we have

$$\alpha(M(t)) \leqslant (\alpha(\overline{C}(t)) \leqslant \alpha(H(t)) = \alpha(\{h_n(t) : n \geqslant 1\}).$$

Using (11), we obtain

$$\begin{aligned}\alpha(M(t)) \leqslant 2\Bigg[&\frac{1}{\Gamma(r)} \int_0^t \alpha\big(\{(t-s)^{r-1} f_n(s); n \geqslant 1\}\big)ds \\ &+ \frac{t}{T\Gamma(r)} \int_0^T \alpha\big(\{(T-s)^{r-1} f_n(s); n \geqslant 1\}\big)ds\Bigg].\end{aligned}$$

Now, since $f_n(s) \in M(s)$ we have

$$\alpha\big(\{(T-s)^{r-1} f_n(s); n \geqslant 1\}\big) = (T-s)^{r-1} \alpha\big(M(s)\big)$$

and

$$\alpha\big(\{(t-s)^{r-1} f_n(s); n \geqslant 1\}\big) = (t-s)^{r-1} \alpha\big(M(s)\big).$$

It follows that

$$\begin{aligned}\alpha(M(t)) &\leqslant 2\left[\frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \alpha\big(M(s)\big)ds + \frac{t}{T\Gamma(r)} \int_0^T (T-s)^{r-1} \alpha\big(M(s)\big)ds\right] \\ &\leqslant 2\left[\frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \psi(s, \alpha(M(s)))ds + \frac{t}{T\Gamma(r)} \int_0^T (T-s)^{r-1} \psi(s, \alpha(M(s)))ds\right].\end{aligned}$$

Also, the function φ given by $\varphi(t) = \alpha(M(t))$ belongs to $C(J, [0, 2R])$. Consequently by (H3), $\varphi \equiv 0$, that is $\alpha(M(t)) = 0$ for all $t \in J$.

Now, by the Arzèla-Ascoli theorem, M is relatively compact in $C(J, E)$.

Step 5: Let $h \in N(y)$ with $y \in \overline{U}$. Since $|y(s)| \leqslant R$ and (H2), we have $N(\overline{U}) \subseteq \overline{U}$, because if it is not true, there exists a function $y \in \overline{U}$ but $\|N(y)\|_{\mathscr{P}} > R$ and

$$h(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s)ds - \frac{t}{T\Gamma(r)} \int_0^T (T-s)^{r-1} f(s)ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T$$

for some $f \in S_{F,y}$. On the other hand we have

$$\begin{aligned} R &\leqslant \|N(y)\|_{\mathscr{P}} \\ &\leqslant |y_0| + |y_T| + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}|f(s)|ds + \frac{t}{T\Gamma(r)}\int_0^T (T-s)^{r-1}|f(s)|ds \\ &\leqslant |y_0| + |y_T| + \frac{t}{\Gamma(r)}\int_0^T p(s)ds + \frac{tT}{T\Gamma(r)}\int_0^T p(s)ds \\ &\leqslant |y_0| + |y_T| + \frac{2T}{\Gamma(r)}\int_0^T p(s)ds. \end{aligned}$$

Dividing both sides by R and taking the lower limit as $R \to \infty$, we conclude that $\frac{2T}{\Gamma(r)}\delta \geqslant 1$ which contradicts (10). Hence $N(\overline{U}) \subseteq \overline{U}$.

As a consequence of Steps $1-5$ together with Theorem 2.2, we can conclude that N has a fixed point $y \in C(J,B)$ which is a solution of the problem (1)–(2).

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(Received March 20, 2011)

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