Fractional
Differential
Calculus
Volume 2, Number 1 (2012), 109–117
doi:10.7153/fdc-02-08

SOME COMPUTATIONAL RESULTS FOR FUNCTIONS BELONGING TO A FAMILY CONSISTING OF CAUCHY–EULER TYPE DIFFERENTIAL EQUATION

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Abstract. In this study, a family concerning fractional calculus of p-valently analytic functions is first defined and several basic results for the related functions are next given. By making use of these results, several computational results for functions belonging to the family consisting of the (non-homogenous) Cauchy Euler type differential equation are then derived.

1. Introduction, definitions and motivation

Fractional calculus (FC), which is fractional integral or derivative, is an important and also interesting field of mathematics study that qrows out of the traditional definitions of calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value. The concept of FC is not new and it is generally known that integer order derivatives and integrals have clear physical and geometric interpretations. However, in case of fractional-order integration and differentiation, which represent a rapidly qrowing field both in theory and in applications to real world problems, it is not so. Since the appearance of the idea of differentiation and integration of arbitrary (not necessary integer) order there was not any acceptable geometric and physical interpretation of these operations for more than 300 year. In the recent years, its usage has been found in studies of viscoelastic materials, as well as in many fields of sciences and engineering including fluid flow, rheology, diffusive transport, electerical networks, electromagnetic theory, probability and so on. In the literature, it considers different definitions of fractional derivatives and integrals and also differintegrals. For some elementary functions, explicit formula of fractional drevative and integral are presented. For all of them, one may check the works in [2], [7], [13–17], [20–21] and see also [22] in the references. We also want to apply, as an example, some applications of fractional calculus to the certain differential equations which will be used in certain sciences and also engineering.

After all these information, we again want to focus on some applications of both fractional calculus and differential equation in mathematics and to show the way to some researchers who want to use the fractional calculus and Cauchy Euler differential

Mathematics subject classification (2010): 30C50, 30E20, 30C45, 26A33, 30A10.
Keywords and phrases: Unit disk, analytic function, p-valent function, Cauchy-Euler differential equation, complex inequalities, fractional calculus, distortion inequalities, coefficient bounds.

equations in their research fields as novel example. We now begin to investigate necessary definitions for our example consisting of fractional calculus and Cauchy Euler differential equation.

Let

$$k = n+p, n+p+1, n+p+2, \cdots;\ a_k \geqslant 0$$

and also let $\mathscr{A}_n(p)$ denote the family of functions $f(z)$ of the form:

$$f(z) = z^p - a_{n+p}z^{n+p} - a_{n+p+1}z^{n+p+1} - \cdots \tag{1.1}$$

which are *analytic and* p*-valent* in the *open unit disk* $\mathbb{U} = \{z \in \mathbb{C} \ :\ |z| < 1\}$, where $\mathbb{C}$ is the set of complex number, and $n, p \in \mathbb{N} = \{1,2,3,\cdots\} := \mathbb{N}_0 - \{0\}$.

Now, a function $f(z) \in \mathscr{A}_n(p)$ is said to belong to the family $\mathscr{SC}^n_\lambda(\mu, p; \alpha)$, if and only if,

$$\left| z\mathscr{D}_z^{1+\mu} f(z) - (p-\mu)\mathscr{D}_z^{\mu} f(z) \right| < \alpha \left| (1-\lambda) z\mathscr{D}_z^{1+\mu} f(z) + \lambda \mathscr{D}_z^{\mu} f(z) \right| \tag{1.2}$$

$$(0 < \alpha \leqslant p; 0 \leqslant \mu < 1; 0 \leqslant \lambda \leqslant 1; p \in \mathbb{N}; z \in \mathbb{U}),$$

and also a function $f(z) \in \mathscr{A}_n(p)$ is said to belong to the family $\mathscr{H}^n_\lambda(\mu, p, \alpha; \kappa)$, if $w = f(z)$ satisfies the non-homogenous Cauchy-Euler type differential equation:

$$z^2 \frac{d^2w}{dz^2} + 2(1+\kappa)z\frac{dw}{dz} + \kappa(\kappa+1)w = (p+\kappa)(p+\kappa+1)g(z), \tag{1.3}$$

where $g(z) \in \mathscr{SC}^n_\lambda(\mu, p; \alpha)$, $\kappa > -p$, and $\kappa \in \mathbb{R}$.

In (1.2) (and also throughout this paper), $\mathscr{D}_z^{\mu}\{\cdot\}$ denotes an operator of fractional calculus (that is that fractional integral and derivative(s)), which is defined as follows (cf., e.g., [11], [12], [18], [19] and (see also) [13]).

DEFINITION 1. (Fractional Integral Operator) The fractional integral of order μ is defined, for a function $f(z)$, by

$$\mathscr{D}_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\mu}} d\zeta \quad (\mu > 0), \tag{1.4}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\mu-1}$ is removed by requiring $log(z-\zeta)$ to be real when $z-\zeta > 0$.

DEFINITION 2. (Fractional Derivative(s) Operator) The fractional derivative of order μ is defined, for a function $f(z)$, by

$$\mathscr{D}_z^{\mu} f(z) = \begin{cases} \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\mu}} d\zeta & (0 \leqslant \mu < 1) \\ \frac{d^m}{dz^m} \{\mathscr{D}_z^{\mu-m} f(z)\} & (m \leqslant \mu < m+1,\ m \in \mathbb{N}_0) \end{cases}, \tag{1.5}$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\mu}$ is removed, as in Definition 1.

Noting that the inequalities determined by taking $\lambda := 0$ and $\lambda \to 1-$ have very important role for both analytic and geometric functions theory. Besondersly, the related functions families (consisting of p-valently starlike functions of order $p-\alpha$ in $\mathbb{U}$ and p-valently convex functions of order $p-\alpha$ in $\mathbb{U}$ $(0 < \alpha \leqslant p; p \in \mathbb{N})$), *i.e.,* the families $\mathscr{S}_n(p,\alpha,\lambda) := \mathscr{SC}_\lambda^n(0,p;\alpha)$ and $\mathscr{K}_n(p,\alpha,\lambda) := \mathscr{SC}_{1-\lambda}^n(1,p;\alpha)$ which were before defined by Chen et *al.* and several computational results were obtained in [2] and see also [3]. Certian computational results in the papers in [1], [3], [5], [7], [8], [9] and [10], include fractional calculus or differential equations and, for example, thier results can be also checked.

The main purpose of the this investigation is to apply the computational results obtained for the functions in $\mathscr{SC}_\lambda^n(\mu,p;\alpha)$ to the functions in $\mathscr{H}_\lambda^n(\mu,p,\alpha;\kappa)$. For this, certain analytic and geometric characteristics of $\mathscr{SC}_\lambda^n(\mu,p;\alpha)$ are firstly derived and these results are then applied to obtain some comptational results for functions in $\mathscr{H}_\lambda^n(\mu,p,\alpha;\kappa)$. Note that all these results and also their consequences are very important for geometric functions theory. (See, for their details, [4], [6], and see also [13].)

2. Certain computational results for functions in $\mathscr{SC}_\lambda^n(\mu,p;\alpha)$

For the main results, we need to state the following basic computational results for functions in $\mathscr{SC}_\lambda^n(\mu,p;\alpha)$.

THEOREM 2.1. *A function $f(z)$, given by (1.1), is in the family $\mathscr{SC}_\lambda^n(\mu,p;\alpha)$ if and only if*

$$\sum_{k=n+p}^{\infty} [k-p+\alpha\psi(\lambda,\mu;k)]\phi(\mu;k)a_k \leqslant \alpha\psi(\lambda,\mu;p)\phi(\mu;p), \tag{2.1}$$

where, here and throughout this paper,

$$\psi(\lambda,\mu;k) := \lambda + (k-\mu)(1-\lambda) \tag{2.2}$$

and

$$\phi(\mu;k) := \frac{\Gamma(k+1)}{\Gamma(k-\mu+1)} \quad (0 \leqslant \mu < 1; k \geqslant n+p; n,p \in \mathbb{N}). \tag{2.3}$$

The result (2.1) is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{\alpha\psi(\lambda,\mu;p)\phi(\mu;p)}{[n+\alpha\psi(\lambda,\mu;n+p)]\phi(\mu;n+p)} z^{n+p} \quad (n,p \in \mathbb{N}). \tag{2.4}$$

Proof. Let a function $f(z) \in \mathscr{A}_n(p)$ defined by (1.1) satisfy the inequality (2.1). If we let $z \in \partial\mathbb{U}$, then on making use of Definition 2 and also (1.2), we then find that

$$\begin{aligned}
&\left|z\mathscr{D}_z^{1+\mu}f(z)-(p-\mu)\mathscr{D}_z^{\mu}f(z)\right|-\alpha\left|(1-\lambda)z\mathscr{D}_z^{1+\mu}f(z)+\lambda\mathscr{D}_z^{\mu}f(z)\right|\\
&\quad=\left|-\Sigma_{k=n+p}^{\infty}(k-p)a_kz^{k-p}\right|\\
&\qquad-\alpha\left|\psi(\lambda,\mu;p)\phi(\mu;p)-\Sigma_{k=n+p}^{\infty}\psi(\lambda,\mu;k)\phi(\mu;k)a_kz^{k-p}\right|\\
&\quad\leqslant\left[\Sigma_{k=n+p}^{\infty}\left[k-p+\alpha\psi(\lambda,\mu;k)\right]\phi(\mu;k)\right]-\alpha\psi(\lambda,\mu;p)\phi(\mu;p)\\
&\quad\leqslant 0\ \ (0<\alpha\leqslant p;0\leqslant\mu<1;0\leqslant\lambda\leqslant 1;n,p\in\mathbb{N};z\in\partial\mathbb{U}).
\end{aligned}\tag{2.5}$$

Hence, by maximum modulus principal, the function $f(z)$ given by (1.1) belongs to the class $\mathscr{S}\mathscr{C}_{\lambda}^{n}(\mu,p;\alpha)$.

Conversely, suppose that a function $f(z)$ given by (1.1) belongs to the class $\mathscr{S}\mathscr{C}_{\lambda}^{n}(\mu,p;\alpha)$. Then, in view of (1.3) together with using Definition 2, we readily obtain

$$\begin{aligned}
&\left|\frac{z\mathscr{D}_z^{1+\mu}f(z)-(p-\mu)\mathscr{D}_z^{\mu}f(z)}{(1-\lambda)z\mathscr{D}_z^{1+\mu}f(z)+\lambda\mathscr{D}_z^{\mu}f(z)}\right|\\
&\qquad=\left|\frac{-\Sigma_{k=n+p}^{\infty}(k-p)a_kz^{k-p}}{\psi(\lambda,\mu;p)\phi(\mu;p)-\Sigma_{k=n+p}^{\infty}\psi(\lambda,\mu;k)\phi(\mu;k)a_kz^{k-p}}\right|<\alpha.
\end{aligned}\tag{2.6}$$

Finally, by observing the the function $f(z)$ given by (2.4) is indeed an extremal function for the assertion in (2.1), the desired proof of Theorem 2.1 is completed. □

In view of the basic inequality of Theorem 2.1, the inequalities in the following threorems (Theorems 2.2 and 2.3 below) are easily proved. The details are ommitted.

THEOREM 2.2. *Let a function $f(z)$ given by (1.1) be in the $\mathscr{S}\mathscr{C}_{\lambda}^{n}(\mu,p;\alpha)$. Then,*

$$\sum_{k=n+p}^{\infty}a_k\leqslant\frac{\alpha\psi(\lambda,\mu;p)\phi(\mu;p)}{[n+\alpha\psi(\lambda,\mu;n+p)]\phi(\mu;n+p)}\tag{2.7}$$

and

$$\sum_{k=n+p}^{\infty}ka_k\leqslant\frac{\alpha(n+p-\mu)\psi(\lambda,\mu;p)\phi(\mu;p)}{[n+\alpha\psi(\lambda,\mu;n+p)]\phi(\mu;n+p-1)}.\tag{2.8}$$

THEOREM 2.3. *Let a function $f(z)$ given by (1.1) be in the $\mathscr{S}\mathscr{C}_{\lambda}^{n}(\mu,p;\alpha)$. Then,*

$$\left|\left|D_z^{-\delta}f(z)\right|-\phi(-\delta;p)\,|z|^{p+\delta}\right|\leqslant\frac{\alpha\psi(\lambda,\mu;p)\phi(\mu;p)\phi(-\delta;n+p)}{[n+\alpha\psi(\lambda,\mu;n+p)]\phi(\mu;n+p)}|z|^{n+p+\delta},\tag{2.9}$$

$$\left|\left|D_z^{\delta}f(z)\right|-\phi(\delta;p)\,|z|^{p-\delta}\right|\leqslant\frac{\alpha\psi(\lambda,\mu;p)\phi(\mu;p)\phi(\delta;n+p)}{[n+\alpha\psi(\lambda,\mu;n+p)]\phi(\mu;n+p)}|z|^{n+p-\delta}\tag{2.10}$$

and

$$\left|\left|D_z^{1+\delta} f(z)\right| - p\phi(\delta; p-1)\,|z|^{p-\delta-1}\right|$$

$$\leqslant \frac{\alpha(n+p-\mu)\psi(\lambda,\mu;p)\phi(\mu;p)\phi(\delta;n+p-1)}{[n+\alpha\psi(\lambda,\mu;n+p)]\phi(\mu;n+p-1)}|z|^{n+p-\delta-1}. \tag{2.11}$$

The results in (2.9), (2.10), and (2.11) are sharp for the function $f(z)$ given by (2.4).

THEOREM 2.4. *Let a function $f(z)$ given by (1.1) be in the $\mathscr{S}\mathscr{C}_{\lambda}^{n}(\mu,p;\alpha)$. Then, $f(z)$ is p-valently close-to-convex of order β $(0 \leqslant \beta < p; p \in \mathbb{N})$ in $|z| < r_1$, p-valently starlike of order β $(0 \leqslant \beta < p; p \in \mathbb{N})$ in $|z| < r_2$, and p-valently convex of order β $(0 \leqslant \beta < p; p \in \mathbb{N})$ in $|z| < r_3$, where*

$$r_1 := r_1(n,p,\alpha,\beta,\lambda,\mu) = \inf_{k \geqslant n+p} \left(\frac{p-\beta}{k}\cdot\varphi(k)\right)^{\frac{1}{k-p}}, \tag{2.12}$$

$$r_2 := r_2(n,p,\alpha,\beta,\lambda,\mu) = \inf_{k \geqslant n+p} \left(\frac{p-\beta}{k-\beta}\cdot\varphi(k)\right)^{\frac{1}{k-p}} \tag{2.13}$$

and

$$r_3 := r_3(n,p,\alpha,\beta,\lambda,\mu) = \inf_{k \geqslant n+p} \left(\frac{p(p-\beta)}{k(k-\beta)}\cdot\varphi(k)\right)^{\frac{1}{k-p}}, \tag{2.14}$$

where

$$\varphi(k) := \varphi(n,p,\mu,\beta,\alpha,\lambda;k) = \frac{[k-p+\alpha\psi(\lambda,\mu;k)]\phi(\mu;k)}{\alpha\psi(\lambda,\mu;p)\phi(\mu;p)} \tag{2.15}$$

$$(k = n+p, n+p+1, \cdots;\ n,p \in \mathbb{N}).$$

Each of the results in (2.12)-(2.14) is sharp for the function $f(z)$ given by (2.4).

Proof. Let a function $f(z) \in \mathscr{S}\mathscr{C}_{\lambda}^{n}(\mu,p;\alpha)$. Then, in order to show that $f(z) \in \mathscr{A}_n(p)$, given by (1.1), is p-valently close-to-convex of order β $(0 \leqslant \beta < p; p \in \mathbb{N})$, it is sufficient to show that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leqslant p-\beta \quad (|z| < r_1; 0 \leqslant \beta < p; p \in \mathbb{N}). \tag{2.16}$$

By using (1.1) in (2.27), and in the process taking into account the coefficient bound in (2.1), one easily infer that a function $f(z)$, given by (1.1), is p-valently close-to-convex of order β inside the disc $|z| < r_1$, where r_1 is stated with (2.12).

Similarly, by applying the following inequalities

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leqslant p-\beta \quad (|z| < r_2; 0 \leqslant \beta < p; p \in \mathbb{N}) \tag{2.17}$$

and

$$\left|\left(1 + \frac{zf''(z)}{f'(z)}\right) - p\right| \leqslant p-\beta \quad (|z| < r_3; 0 \leqslant \beta < p; p \in \mathbb{N}), \tag{2.18}$$

and proceeding in the same manner as mentioned above, we conclude that a function $f(z)$, given by (1.1), respectively, is p-valently starlike of order β inside the disc $|z| < r_2$, and p-valently convex of order β inside the disc $|z| < r_3$, where r_2 and r_3 are given by (2.13) and (2.14). $\square$

REMARK 2.1. By selecting the parameters μ and/or λ in all theorems above, one can be obtained several computational results. Some of them are comparable with the earlier results generated by Chen et *al.* in [3].

3. Certain computational results for functions in $\mathscr{H}_{\lambda}^{n}(\mu,p,\alpha,\kappa)$

By applying the basic results obtained in Section 2 to the function, which is the solution of the (non-homogenous) Cauchy Euler type equation given in $\mathscr{H}_{\lambda}^{n}(\mu,p,\alpha;\kappa)$, we now derive several results. The first result is given by the following theorem.

THEOREM 3.1. *Let a function $f(z)$ given by (1.1) be in the $\mathscr{H}_{\lambda}^{n}(\mu,p,\alpha;\kappa)$. Then,*

$$||f(z)|-|z|^p| \leqslant \frac{\alpha\psi(\lambda,\mu;n+p)\phi(\mu;p)\theta(n,p;\kappa)}{[n+\alpha\psi(\lambda,\mu;n+p)]\phi(\mu;n+p)}|z|^{n+p} \tag{3.1}$$

and

$$\left||f'(z)|-p\,|z|^{p-1}\right| \leqslant \frac{\alpha(n+p-\mu)\psi(\lambda,\mu;p)\phi(\mu;p)\theta(n,p;\kappa)}{[n+\alpha\psi(\lambda,\mu;p)]\phi(\mu;n+p-1)}|z|^{n+p-1}, \tag{3.2}$$

where

$$\theta(n,p;\kappa) := \frac{(1+\kappa)(2+\kappa)}{n+p+\kappa} \tag{3.3}$$

$$(\kappa \in \mathbb{R}^* := \mathbb{R}-\{-n-p,-n-p-1,\cdots : n,p\in\mathbb{N}\}).$$

The results in (3.1) and (3.2) are sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{\alpha\psi(\lambda,\mu;p)\phi(\mu;p)}{[n+\alpha\psi(\lambda,\mu;n+p)]\phi(\mu;n+p)\theta(n,p;\kappa)}z^{n+p} \quad (n,p\in\mathbb{N}). \tag{3.4}$$

Proof. Assume that $f(z)$ is given by (1.1). Also, let a function $g(z) \in \mathscr{SC}_{\lambda}^{n}(\mu,p;\alpha)$, occurring in the non-homogenous Cauchy-Euler differential type equation (1.3) be of the form:

$$g(z) = z^p - b_{n+p}z^{n+p} - b_{n+p+1}z^{n+p+1} - \cdots \quad (\forall k\in\mathbb{N};\ b_k \geqslant 0). \tag{3.5}$$

Then, we readily find from (1.3) that

$$a_k = \frac{(1+\kappa)(2+\kappa)}{(k+\kappa)(k+\kappa+1)}b_k \quad (k \geqslant n+p; n,p\in\mathbb{N}), \tag{3.6}$$

so that

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k = z^p - \sum_{k=n+p}^{\infty}\frac{(1+\kappa)(2+\kappa)}{(k+\kappa)(k+\kappa+1)}b_k z^k. \tag{3.7}$$

The result (3.7) immediately yields that

$$||f(z)|-|z|^p| \leqslant |z|^{n+p} \sum_{k=n+p}^{\infty} \frac{(1+\kappa)(2+\kappa)}{(k+\kappa)(k+\kappa+1)} b_k \quad (z \in \mathbb{U}). \tag{3.8}$$

Next, because of $g(z) \in \mathscr{SC}_\lambda^n(\mu,p;\alpha)$, therefore, on using the assertion (2.7), we then get the following inequality:

$$b_k \leqslant \frac{\alpha\psi(\lambda,\mu;n+p)\phi(\mu;p)}{[n+\alpha\psi(\lambda,\mu;n+p)]\phi(\mu;n+p)} \quad (k \geqslant n+p; n,p \in \mathbb{N}),$$

which in conjunction with (3.7) and (3.8) yield that

$$||f(z)|-|z|^p| \leqslant \frac{\alpha\psi(\lambda,\mu;n+p)\phi(\mu;p)}{[n+\alpha\psi(\lambda,\mu;n+p)]\phi(\mu;n+p)} \cdot |z|^{n+p} \sum_{k=n+p}^{\infty} \frac{(1+\kappa)(2+\kappa)}{(k+\kappa)(k+\kappa+1)} \quad (z \in \mathbb{U}). \tag{3.9}$$

By also noting the following special result:

$$\left(\sum_{k=n+p}^{\infty} \frac{(1+\kappa)(2+\kappa)}{(k+\kappa)(k+\kappa+1)}\right) = \theta(n,p;\kappa), \tag{3.10}$$

where $\theta(n,p;\kappa)$ is given by (3.3).

The assertion (3.1) of Theorem 3.1 immediately follows from (3.9) and (3.10). The second assertion (3.2) of Theorem 3.1 can be easily established by similarly applying (2.8), (3.6), and (3.10). □

THEOREM 3.2. *Let a function $f(z)$ given by (1.1) be in the $\mathscr{H}_\lambda^n(\mu,p,\alpha;\kappa)$. Then, $f(z)$ is p-valently close-to-convex of order β $(0 \leqslant \beta < p; p \in \mathbb{N})$ in $|z| < r_4$, p-valently starlike of order β $(0 \leqslant \beta < p; p \in \mathbb{N})$ in $|z| < r_5$, and p-valently convex of order β $(0 \leqslant \beta < p; p \in \mathbb{N})$ in $|z| < r_6$, where*

$$r_4 := r_4(n,p,\alpha,\beta,\lambda,\mu,\kappa) = \inf_{k \geqslant n+p} \left(\frac{p-\beta}{k} \cdot \tau(\kappa) \cdot \varphi(k)\right)^{\frac{1}{k-p}}, \tag{3.11}$$

$$r_5 := r_5(n,p,\alpha,\beta,\lambda,\mu,\kappa) = \inf_{k \geqslant n+p} \left(\frac{p-\beta}{k-\beta} \cdot \tau(\kappa) \cdot \varphi(k)\right)^{\frac{1}{k-p}} \tag{3.12}$$

and

$$r_6 := r_6(n,p,\alpha,\beta,\lambda,\mu,\kappa) = \inf_{k \geqslant n+p} \left(\frac{p(p-\beta)}{k(k-\beta)} \cdot \tau(\kappa) \cdot \varphi(k)\right)^{\frac{1}{k-p}}, \tag{3.13}$$

where $\varphi(k)$ is given by (2.22), and $\tau(\kappa)$ is defined by

$$\tau(\kappa) := \frac{(k+\kappa)(k+\kappa+1)}{(1+\kappa)(2+\kappa)};\ k = n+p, n+p+1, \cdots; n,p \in \mathbb{N}; \kappa \in \mathbb{R}^*.$$

Each of those results in (3.11)-(3.13) is sharp for the function $f(z)$ *given by (3.4).*

Proof. Suppose that $f(z) \in \mathbb{A}_n(p)$ is given by (1.1), and also let a function $g(z) \in \mathscr{S}\mathscr{C}_\lambda^n(\mu,p;\alpha)$, occurring in the non-homogenous differential type equation in (1.3). Then, it is sufficient to show that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leqslant p - \beta \ \ for \ \ |z| < r_4.$$

Indeed, we have

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leqslant \sum_{k=n+p}^{\infty} k a_k |z|^{k-1},$$

and by using the coefficient relation (3.6) between the functions $f(z)$ and $g(z)$, we arrive at:

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leqslant \sum_{k=n+p}^{\infty} \frac{(1+\kappa)(2+\kappa)}{(k+\kappa)(k+\kappa+1)} k b_k |z|^{k-1} \leqslant p - \beta. \tag{3.14}$$

Since $g(z) \in \mathscr{S}\mathscr{C}_\lambda^n(\mu,p;\alpha)$ and in view of (2.1) of Theorem 2.1, we also have that

$$\sum_{k=n+p}^{\infty} \frac{[k-p+\alpha\psi(\lambda,\mu;k)]\phi(\mu;k)}{\alpha\psi(\lambda,\mu;p)\phi(\mu;p)} a_k \leqslant 1.$$

Hence, the inequality (3.14) is true if

$$\frac{k}{p-\beta} \cdot \frac{(1+\kappa)(2+\kappa)}{(k+\kappa)(k+\kappa+1)} \cdot |z|^{k-p} \leqslant \varphi(k), \tag{3.15}$$

where $\varphi(k)$ is given by (2.22).

If we solve (3.15) for $|z|$, we easily arrive at the desired result in (3.11). Thus, the proof of (3.11) of Theorem 3.2 is completed. $\square$

For the proofs of (3.12) and (3.13) of Theorem 3.2, by suitable invoking inequalities in (2.17) and (2.18) concerning geometric properties (*i.e., p*-valently starlikeness and *p*-valently convexity) of a function $f(z)$, given by (1.1), belonging to the family $\mathscr{H}_\lambda^n(\mu,p,\alpha;\kappa)$, and making use of (1.1), (3.6), and also the coefficient bound in the inequality (2.1) of Theorem 2.1, appropriately in the process, the given results in (3.12) and (3.13) can be easily proved by similar steps as we used in the proof of (3.11) of Theorem 3.2. We skip further details.

We conclude this paper by remarking that by selecting suitable values of the parameters n, p, μ, and/or λ in all theorems of both sections, one can infer several special and computational results concerning functions in the related families. These obvious considerations are omitted here.

Acknowledgement

The work in this study was supported by the Research Project financed by Tübitak (The Scientific and Technical Research Council of Turkey) with under the project number TBGA-U-105T056.

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(Received February 9, 2011)

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