Fractional Differential Calculus
Volume 3, Number 1 (2013), 1–20
doi:10.7153/fdc-03-01

BOUNDARY VALUE PROBLEMS FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS AND INCLUSIONS WITH NONLOCAL AND INTEGRAL BOUNDARY CONDITIONS

SOTIRIS K. NTOUYAS

Abstract. In this paper, we study a class of boundary value problems of nonlinear fractional differential equations and inclusions with nonlocal and integral boundary conditions. Some new existence and uniqueness results are obtained by using a variety of fixed point theorems. Examples are given to illustrate the results.

1. Introduction

In recent years, boundary value problems for nonlinear fractional differential equations have been addressed by several researchers. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. These characteristics of the fractional derivatives make the fractional-order models more realistic and practical than the classical integer-order models. As a matter of fact, fractional differential equations arise in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc. [25, 31, 32, 33]. For some recent development on the topic, see [1]-[8], [10]-[13], [15, 27, 28, 34, 35] and the references therein.

Recently, Benchohra *et al.* in [14] investigated the following first order nonlocal boundary value problem:

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \ \ 0 < t < T, \ \ 0 < q < 1, \\ x(0) + g(x) = x_0, \end{cases} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative. They proved existence and uniqueness results by using Banach's contraction principle and Schaefer's fixed point theorem.

Nonlocal conditions were initiated by Bitsadze [16]. As remarked by Byszewski [18, 19, 20], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena. For example, $g(x)$ may be given by $g(x) = \sum_{i=1}^{p} c_i x(t_i)$ where $c_i, i = 1, \ldots, p$, are given constants and $0 < t_1 < \ldots < t_p \leqslant T$.

Mathematics subject classification (2010): 26A33, 34A08, 34A60.
Keywords and phrases: Fractional differential equations, fractional differential inclusions, nonlocal conditions, integral boundary conditions, existence, fixed point theorems.

Paper FDC-03-01

Later, Benchohra *et al.* in [15] investigated the following nonlocal boundary value problem:

$$\begin{cases} {}^cD^q x(t) = f(t,x(t)), \ \ 0 < t < T, \ \ 1 < q \leqslant 2, \\ x(0) = g(x), \quad x(T) = x_T, \end{cases} \tag{1.2}$$

where ${}^cD^q$ denotes the Caputo fractional derivative. Using Schaefer's fixed point theorem they provided sufficient criteria for the existence of at least one solution for the problem (1.2) with the conditions that $f(t,x)$ is uniformly bounded on $[0,T]\times\mathbb{R}$ and that the set $g(C([0,T]))$ is bounded. Also, they established criteria for the uniqueness of solutions by virtue of the Banach's fixed point theorem.

Zhong and Lin [36] investigated the existence and uniqueness of solutions for the following nonlocal and multiple-point boundary value problem:

$$\begin{cases} {}^cD^q x(t) = f(t,x(t)), \ \ 0 < t < 1, \ \ 1 < q \leqslant 2, \\ x(0) = x_0 + g(x), \quad x'(1) = x_1 + \displaystyle\sum_{i=1}^{m-2} b_i x'(\xi_i), \end{cases} \tag{1.3}$$

where ${}^cD^q$ denotes the Caputo fractional derivative, $x_0, x_1 \in \mathbb{R}$, $b_i \geqslant 0, 0 < \xi_i < 1, i = 1,2,\ldots,m-2$, and $d = \sum_{i=1}^{m-2} b_i < 1$. They give some sufficient conditions for the uniqueness of solutions and for the existence of at least one solution of the problem (1.3) by means of the contraction principle in the Banach space and by the fixed point theorem attributed to D. O'Regan [29].

More recently, Ahmad *et al.* in [9] investigated the existence and uniqueness of solutions for the following boundary value problem with three-point integral boundary conditions:

$$\begin{cases} {}^cD^q x(t) = f(t,x(t)), \ \ 0 < t < 1, \ \ 1 < q \leqslant 2, \\ x(0) = 0, \qquad x(1) = \alpha \displaystyle\int_0^\eta x(s)ds, \ \ 0 < \eta < 1, \end{cases} \tag{1.4}$$

where ${}^cD^q$ denotes the Caputo fractional derivative, and $\alpha \in \mathbb{R}$, $\alpha \neq 2/\eta^2$. Some new existence and uniqueness results are obtained by using standard fixed point theorems, such as Banach's contraction principle, Krasnoselskii's fixed point theorem and Leray-Schauder degree theory.

Motivated by the aforementioned papers, we intend in this paper to discuss the existence and uniqueness of solutions for a boundary value problem of nonlinear fractional differential equations and inclusions of order $q \in (1,2]$ with nonlocal and integral boundary conditions. Thus, in the first part of the paper we discuss the boundary value problem of nonlinear fractional differential equations given by:

$$\begin{cases} {}^cD^q x(t) = f(t,x(t)), \ \ 0 < t < 1, \ \ 1 < q \leqslant 2, \\ x(0) = x_0 + g(x), \quad x(1) = \alpha \displaystyle\int_0^\eta x(s)ds, \ \ 0 < \eta < 1, \end{cases} \tag{1.5}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous, $g : C([0,1],\mathbb{R}) \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ is such that $\alpha \neq 2/\eta^2$. Two results are given, one based on Banach's contraction principle and another one based on a fixed point theorem due to D. O'Regan [29]. Examples are also provided to illustrate the possible application of the established analytical results.

In the second part, we cover the multivalued case, considering the following boundary value problem for fractional order differential inclusions with nonlocal and integral boundary conditions:

$$\begin{cases} {}^cD^q x(t) \in F(t,x(t)), \;\; 0<t<1, \;\; 1<q\leqslant 2, \\ x(0)=x_0+g(x), \quad x(1)=\alpha \int_0^\eta x(s)ds, \;\; 0<\eta<1, \end{cases} \tag{1.6}$$

where $F:[0,1]\times\mathbb{R}\to\mathscr{P}(\mathbb{R})$ is a multivalued map, $\mathscr{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$. We present two existence results for the problem (1.6), when the right hand side is convex as well as nonconvex valued. The first result relies on the Nonlinear Alternative for contractive maps, while in the second result, we combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values.

The remainder of the paper is organized as follows. Section 2 preliminarily provides some definitions and lemmas which are crucial to the following discussion. Section 3 contains the existence and uniqueness results for the problem (1.5), while the results for the problem (1.6) are contained in Section 4.

2. Preliminaries from fractional calculus

Let us recall some basic definitions of fractional calculus [25, 31, 33].

DEFINITION 2.1. If $g(t)\in AC^n[a,b]$, then the Caputo derivative of fractional order q is defined as

$${}^cD^q g(t)=\frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1}g^{(n)}(s)ds, \;\; n-1<q<n, n=[q]+1,$$

where $[q]$ denotes the integer part of the real number q. Here $AC^n[a,b]$ denote the space of real valued functions $g(t)$ which have continuous derivatives up to order $n-1$ on $[a,b]$ such that $g^{n-1}(t)\in AC[a,b]$.

DEFINITION 2.2. The Riemann-Liouville fractional integral of order q is defined as

$$I^q g(t)=\frac{1}{\Gamma(q)}\int_0^t \frac{g(s)}{(t-s)^{1-q}}ds, \;\; q>0,$$

provided the right hand side is pointwise defined on $(0,\infty)$.

LEMMA 2.3. *For $q > 0$, the general solution of the fractional differential equation ${}^cD^q x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$ ($n = [q] + 1$).

In view of Lemma 2.3, it follows that

$$I^q\, {}^cD^q x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}, \tag{2.1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$ $(n = [q] + 1)$.

To define the solution for the problem (1.5), we find the solution for its associated linear problem.

LEMMA 2.4. *Assume that $\alpha\eta^2 \neq 2$. For a given $y \in C([0,1], \mathbb{R})$ the unique solution of the boundary value problem*

$$\begin{cases} {}^cD^q x(t) = y(t), \ \ 0 < t < 1, \ \ 1 < q \leqslant 2, \\ x(0) = x_0 + g(x), \quad x(1) = \alpha \displaystyle\int_0^\eta x(s)ds, \ \ 0 < \eta < 1, \end{cases} \tag{2.2}$$

is given by

$$\begin{aligned} u(t) = & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s) ds - \frac{2t}{(2-\alpha\eta^2)\Gamma(q)} \int_0^1 (1-s)^{q-1} y(s) ds \\ & + \frac{2\alpha t}{(2-\alpha\eta^2)\Gamma(q)} \int_0^\eta \Big(\int_0^s (s-m)^{q-1} y(m) dm \Big) ds \\ & + \left[\frac{2(\alpha\eta - 1)t}{2 - \alpha\eta^2} + 1 \right] (x_0 + g(x)), \quad t \in [0,1]. \end{aligned} \tag{2.3}$$

Proof. For some constants $c_0, c_1 \in \mathbb{R}$, we have [25]

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s) ds - c_0 - c_1 t. \tag{2.4}$$

From $x(0) = x_0 + g(x)$ we have $c_0 = -(x_0 + g(x))$. Applying the other boundary condition we get

$$\begin{aligned} c_1 = & \frac{-2\alpha}{2-\alpha\eta^2} \int_0^\eta \int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} y(m) dm ds + \frac{2}{2-\alpha\eta^2} \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} y(s) ds \\ & + \frac{2(1-\alpha\eta)}{2-\alpha\eta^2} (x_0 + g(x)). \end{aligned}$$

Substituting in (2.4), the values of c_0 and c_1 we obtain (2.3). □

3. Existence results-The single-valued case

We denote by $\mathscr{C} = C([0,1],\mathbb{R})$ the Banach space of all continuous functions from $[0,1] \to \mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$.

In view of Lemma 2.4, we define an operator $F : \mathscr{C} \to \mathscr{C}$ by

$$\begin{aligned}(Fx)(t) = &\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,x(s))ds - \frac{2t}{(2-\alpha\eta^2)\Gamma(q)}\int_0^1 (1-s)^{q-1} f(s,x(s))ds \\ &+\frac{2\alpha t}{(2-\alpha\eta^2)\Gamma(q)}\int_0^\eta \Big(\int_0^s (s-m)^{q-1} f(m,x(m))dm\Big)ds \\ &+\left[\frac{2(\alpha\eta-1)t}{2-\alpha\eta^2}+1\right](x_0+g(x)),\ \ t\in[0,1].\end{aligned}$$

Define two operators from $\mathscr{C} \to \mathscr{C}$, respectively, by

$$\begin{aligned}(F_1x)(t) = &\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,x(s))ds \\ &-\frac{2t}{(2-\alpha\eta^2)\Gamma(q)}\int_0^1 (1-s)^{q-1} f(s,x(s))ds \\ &+\frac{2\alpha t}{(2-\alpha\eta^2)\Gamma(q)}\int_0^\eta \Big(\int_0^s (s-m)^{q-1} f(m,x(m))dm\Big)ds,\ \ t\in[0,1],\end{aligned} \tag{3.1}$$

and

$$(F_2x)(t) = \left[\frac{2(\alpha\eta-1)t}{2-\alpha\eta^2}+1\right](x_0+g(x)),\ \ t\in[0,1]. \tag{3.2}$$

Clearly

$$(Fx)(t) = (F_1x)(t) + (F_2x)(t),\ \ t\in[0,1]. \tag{3.3}$$

THEOREM 3.1. *Let $f : [0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Assume that:*

(A_1) $|f(t,x)-f(t,y)| \leqslant L|x-y|, \forall t\in[0,1],\ L>0,\ x,y\in\mathbb{R}$*;*

(A_2) *there exist a positive constant* $\ell < \left(\frac{2|\alpha\eta-1|}{|2-\alpha\eta^2|}+1\right)^{-1}$ *and a continuous function* $\phi : [0,\infty)\to[0,\infty)$ *such that* $\phi(z)\leqslant \ell z$ *and* $|g(u)-g(v)|\leqslant \phi(\|u-v\|)$ *for all* $u,v\in C([0,1])$.

(A_3) $\gamma = \frac{L}{\Gamma(q+1)}\left(1+\frac{2[(q+1)+|\alpha|\eta^{q+1}]}{|2-\alpha\eta^2|(q+1)}\right)+\ell\left(\frac{2|\alpha\eta-1|}{|2-\alpha\eta^2|}+1\right) < 1.$

Then the boundary value problem (1.5) has a unique solution.

Proof. For $x,y\in\mathscr{C}$ and for each $t\in[0,1]$, from the definition of F and assumptions (A_1) and (A_2), we obtain

$$
\begin{aligned}
&|(Fx)(t)-(Fy)(t)| \\
&\leqslant \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,x(s))-f(s,y(s))|ds \\
&\quad +\left|\frac{2t}{(2-\alpha\eta^2)\Gamma(q)}\right|\int_0^1 (1-s)^{q-1}|f(s,x(s))-f(s,y(s))|ds \\
&\quad +\left|\frac{2\alpha t}{(2-\alpha\eta^2)\Gamma(q)}\right|\int_0^\eta \Big(\int_0^s (s-m)^{q-1}|f(m,x(m))-f(m,y(m))|dm\Big)ds \\
&\quad +\left|\frac{2(\alpha\eta-1)t}{2-\alpha\eta^2}+1\right||g(x)-g(y)| \\
&\leqslant L\|x-y\|\left[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds+\left|\frac{2t}{(2-\alpha\eta^2)\Gamma(q)}\right|\int_0^1 (1-s)^{q-1}ds\right. \\
&\quad \left.+\frac{2|\alpha|}{|2-\alpha\eta^2|\Gamma(q)}\int_0^\eta \Big(\int_0^s (s-m)^{q-1}dm\Big)ds\right]+\left(\frac{2|\alpha\eta-1|}{|2-\alpha\eta^2|}+1\right)\ell\|x-y\| \\
&\leqslant \left\{\frac{L}{\Gamma(q+1)}\left(1+\frac{2[(q+1)+|\alpha|\eta^{q+1}]}{|2-\alpha\eta^2|(q+1)}\right)+\ell\left(\frac{2|\alpha\eta-1|}{|2-\alpha\eta^2|}+1\right)\right\}\|x-y\|,
\end{aligned}
$$

and hence

$$\|Fx-Fy\|\leqslant \gamma\|x-y\|.$$

As $\gamma<1$, by (A_3), F is a contraction map from the Banach space $\mathscr{C}$ into itself. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

Next, we introduce the fixed point theorem which was established by O'Regan in [29]. This theorem will be adopted to prove the next main result.

LEMMA 3.2. *Denote by U an open set in a closed, convex set C of a Banach space E. Assume $0\in U$. Also assume that $F(\bar U)$ is bounded and that $F:\bar U\to C$ is given by $F=F_1+F_2$, in which $F_1:\bar U\to E$ is continuous and completely continuous and $F_2:\bar U\to E$ is a nonlinear contraction (i.e., there exists a nonnegative nondecreasing function $\phi:[0,\infty)\to[0,\infty)$ satisfying $\phi(z)<z$ for $z>0$, such that $\|F_2(x)-F_2(y)\|\leqslant\phi(\|x-y\|)$ for all $x,y\in\bar U$). Then, either*

(C1) F has a fixed point $u\in\bar U$; or

(C2) there exist a point $u\in\partial U$ and $\lambda\in(0,1)$ with $u=\lambda F(u)$, where $\bar U$ and ∂U, respectively, represent the closure and boundary of U.

Let

$$\Omega_r=\{x\in C([0,1],\mathbb{R}):\|x\|<r\},$$

and denote the maximum number by

$$M_r=\max\{|f(t,x)|:(t,x)\in[0,1]\times[-r,r]\}.$$

THEOREM 3.3. *Let* $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ *be a continuous function. Suppose that* $(A_1),(A_2)$ *hold. In addition we assume that:*

(A_4) $g(0)=0$;

(A_5) *there exists a nonnegative function* $p\in C([0,1],\mathbb{R})$ *and a nondecreasing function* $\psi:[0,\infty)\to[0,\infty)$ *such that*

$$|f(t,u)|\leqslant p(t)\psi(|u|) \text{ for any } (t,u)\in[0,1]\times\mathbb{R};$$

(A_6) $\sup_{t\in(0,\infty)}\dfrac{r}{k_0|x_0|+p_0\psi(r)}>\dfrac{1}{1-k_0\ell}$, *where*

$$p_0=\frac{1}{\Gamma(q)}\int_0^1(1-s)^{q-1}p(s)ds+\frac{2}{|2-\alpha\eta^2|\Gamma(q)}\int_0^1(1-s)^{q-1}p(s)ds \\ +\frac{2|\alpha|}{|2-\alpha\eta^2|\Gamma(q)}\int_0^\eta\Big(\int_0^s(s-m)^{q-1}p(m)dm\Big)ds,$$

and

$$k_0=\frac{2|\alpha\eta-1|}{|2-\alpha\eta^2|}+1.$$

Then the boundary value problem (1.5) has at least one solution on $[0,1]$.

Proof. Consider the operator $F:\mathscr{C}\to\mathscr{C}$ as that defined in (3.3), that is,

$$(Fx)(t)=(F_1x)(t)+(F_2x)(t),\ \ t\in[0,1],$$

where the operators F_1 and F_2 are defined respectively in (3.1) and (3.2).

From (A_6) there exists a number $r_0>0$ such that

$$\frac{r_0}{k_0|x_0|+p_0\psi(r_0)}>\frac{1}{1-k_0\ell}. \tag{3.4}$$

We shall prove that the operators F_1 and F_2 satisfy all the conditions in Lemma 3.2.

Step 1. *The operator* F_1 *is continuous and completely continuous.* We first show that $F_1(\bar{\Omega}_{r_0})$ is bounded. For any $x\in\bar{\Omega}_{r_0}$ we have

$$\|F_1x\|\leqslant\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}|f(s,x(s))|ds+\frac{2t}{|2-\alpha\eta^2|\Gamma(q)}\int_0^1(1-s)^{q-1}|f(s,x(s))|ds \\ +\frac{2|\alpha|t}{|2-\alpha\eta^2|\Gamma(q)}\int_0^\eta\Big(\int_0^s(s-m)^{q-1}|f(m,x(m))|dm\Big)ds \\ \leqslant\frac{M_{r_0}}{\Gamma(q+1)}\left(1+\frac{2[(q+1)+|\alpha|\eta^{q+1}]}{|2-\alpha\eta^2|(q+1)}\right).$$

This proves that $F_1(\bar{\Omega}_{r_0})$ is uniformly bounded.

In addition, for any $t_1, t_2 \in [0,1], t_1 < t_2$, we have:

$$\begin{aligned}
&|(F_1x)(t_2) - (F_1x)(t_1)| \\
\leqslant\ & \frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]|f(s,x(s))|ds \\
&+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}|f(s,x(s))|ds \\
&+\frac{2|t_2-t_1|}{|2-\alpha\eta^2|\Gamma(q)}\int_0^1(1-s)^{q-1}|f(s,x(s))|ds \\
&+\frac{2|\alpha||t_2-t_1|}{|2-\alpha\eta^2|\Gamma(q)}\int_0^{\eta}\Big(\int_0^s(s-m)^{q-1}|f(m,x(m))|dm\Big)ds \\
\leqslant\ & \frac{M_{r_0}}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]ds+\frac{M_{r_0}}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}ds \\
&+\frac{2M_{r_0}}{|2-\alpha\eta^2|\Gamma(q+1)}\left(1+\frac{|\alpha|\eta^{q+1}}{q+1}\right)|t_2-t_1|,
\end{aligned}$$

which is independent of x, and tends to zero as $t_2 - t_1 \to 0$. Thus, F_1 is equicontinuous. Hence, by the Arzelá-Ascoli Theorem, $F_1(\bar{\Omega}_{r_0})$ is a relatively compact set. Now, let $x_n \subset \bar{\Omega}_{r_0}$ with $\|x_n - x\| \to 0$. Then the limit $\|x_n(t) - x(t)\| \to 0$ uniformly valid on $[0,1]$. From the uniform continuity of $f(t,x)$ on the compact set $[0,1]\times[-r_0,r_0]$ if follows that $\|f(t,x_n(t)) - f(t,x(t))\| \to 0$ is uniformly valid on $[0,1]$. Hence $\|F_1x_n - F_1x\| \to 0$ as $n \to \infty$ which proves the continuity of F_1. Hence Step 1 is completely proved.

Step 2. *The operator $F_2 : \bar{\Omega}_{r_0} \to C([0,1],\mathbb{R})$ is contractive*. This is a consequence of (A_2).

Step 3. *The set $F(\bar{\Omega}_{r_0})$ is bounded.* By (A_2) and (A_4) imply that

$$\|F_2(x)\| \leqslant \left(\frac{2|\alpha\eta-1|}{|2-\alpha\eta^2|}+1\right)(|x_0|+\ell r_0),$$

for any $x \in \bar{\Omega}_{r_0}$. This, with the boundedness of the set $F_1(\bar{\Omega}_{r_0})$ implies that the set $F(\bar{\Omega}_{r_0})$ is bounded.

Step 4. *Finally, it is to show that the case (C2) in Lemma 3.2 does not occur*. To this end, we suppose that (C2) holds. Then, we have that there exist $\lambda \in (0,1)$ and $x \in \partial\Omega_{r_0}$ such that $x = \lambda Fx$. So, we have $\|x\| = r_0$ and

$$\begin{aligned}
x(t) = \lambda\Bigg[&\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f(s,x(s))ds-\frac{2t}{(2-\alpha\eta^2)\Gamma(q)}\int_0^1(1-s)^{q-1}f(s,x(s))ds \\
&+\frac{2\alpha t}{(2-\alpha\eta^2)\Gamma(q)}\int_0^{\eta}\Big(\int_0^s(s-m)^{q-1}f(m,x(m))dm\Big)ds \\
&+\left[\frac{2(\alpha\eta-1)t}{2-\alpha\eta^2}+1\right](x_0+g(x))\Bigg].
\end{aligned}$$

With hypotheses $(A_4)-(A_6)$, we have

$$\begin{aligned} r_0 \leqslant & \frac{\psi(r_0)}{\Gamma(q)} \int_0^t (t-s)^{q-1} p(s)ds + \frac{2\psi(r_0)}{|2-\alpha\eta^2|\Gamma(q)} \int_0^1 (1-s)^{q-1} p(s)ds \\ & + \frac{2|\alpha|\psi(r_0)}{|2-\alpha\eta^2|\Gamma(q)} \int_0^\eta \Big(\int_0^s (s-m)^{q-1} p(m)dm \Big) ds \\ & + \left(\frac{2|\alpha\eta - 1|}{|2-\alpha\eta^2|} + 1 \right) (|x_0| + \ell r_0), \end{aligned}$$

which implies

$$r_0 \leqslant k_0 \ell r_0 + k_0 |x_0| + p_0 \psi(r_0).$$

Thus,

$$\frac{r_0}{k_0|x_0| + p_0\psi(r_0)} \leqslant \frac{1}{1-k_0\ell},$$

which contradicts to (3.4). Consequently, we have proved that the operators F_1 and F_2 satisfy all the conditions in Lemma 3.2. Hence, the operator F has at least one fixed point $x \in \bar{\Omega}_{r_0}$, which is the solution of the boundary value problem (1.5). The proof is completed. □

We illustrate the above obtained results by concrete examples.

EXAMPLE 3.4. Consider the following boundary value problem

$$\begin{cases} {}^cD^{3/2}x(t) = \dfrac{1}{(t+2)^2} \cdot \dfrac{|x|}{1+|x|} + 1 + \sin^2 t, \ \ 0<t<1, \\ x(0) = 1 + \dfrac{1}{16} x(\xi), \quad x(1) = 4 \displaystyle\int_0^{1/2} x(s)ds, \ \ 0<\xi<1. \end{cases} \tag{3.5}$$

Here $q = 3/2, \ell = 1/16, \alpha = 4, \eta = 1/2$, and $f(t,x) = \dfrac{1}{(t+2)^2} \cdot \dfrac{|x|}{1+|x|} + 1 + \sin^2 t$. As $|f(t,x) - f(t,y)| \leqslant \dfrac{1}{4}|x-y|$, therefore (A_1) is satisfied with $L = \dfrac{1}{4}$. Further

$$\gamma = \frac{1}{3\sqrt{\pi}} \left(3 + \frac{2\sqrt{2}}{5} \right) + \frac{3}{16} \approx 0.8582441 < 1.$$

Thus, by Theorem 3.1, problem (3.5) has a unique solution on $[0,1]$.

EXAMPLE 3.5. Let $\beta > 0$ and consider the following boundary value problem

$$\begin{cases} {}^cD^{3/2}x(t) = \beta t^2 \sin^2 x, \ \ 0<t<1, \\ x(0) = \dfrac{1}{3} + \ell x(\xi), \quad x(1) = 4 \displaystyle\int_0^{1/2} x(s)ds, \ \ 0<\xi<1. \end{cases} \tag{3.6}$$

We shall prove that the problem (3.6) admits at least one solution provided that $|\ell|<1$ and $0<\beta<\frac{5\sqrt{\pi}}{33}(1-|\ell|)^2$.

In order to show the validity of this claim, we need to verify that all conditions in Theorem 3.3 are satisfied. Note that here $q=3/2, f(t,x)=\beta t^2\sin^2 x, x_0=1/3, g(x)=\ell x(\xi), \alpha=4, \eta=1/2, \alpha\neq 2/\eta^2$.

The function $g(x)=\ell x(\xi)$ is contractive because $|g(u)-g(v)|<|\ell|\cdot\|u-v\|$ for any $u,v\in C([0,1])$. Moreover $g(0)=0$. Hence, the condition (A_4) is satisfied. Set $p(t)=\beta t$ and $\psi(x)=x^2$. We have

$$|f(t,x)|\leqslant|\beta t^2\sin^2 x|\leqslant\beta t x^2,\ \text{ for any }\ (t,x)\in[0,1]\times\mathbb{R}.$$

Thus condition (A_5) is satisfied. We find

$$p_0=\frac{2}{\sqrt{\pi}}\cdot\frac{4}{15}\beta+\frac{2}{\sqrt{\pi}}\cdot\frac{8}{15}\beta+\frac{16}{\sqrt{\pi}}\cdot\frac{1}{320}\beta=\frac{33\beta}{20\sqrt{\pi}}.$$

We also have $k_0=3$ and arrive the estimation:

$$\sup_{t\in(0,\infty)}\frac{r}{k_0|x_0|+p_0\psi(r)}=\sup_{t\in(0,\infty)}\frac{r}{1+\frac{33\beta}{20\sqrt{\pi}}r^2}=\frac{1}{2}\sqrt{\frac{20\sqrt{\pi}}{33\beta}}>\frac{1}{1-|\ell|},$$

provided $|\ell|<1$ and $0<\beta<\frac{5\sqrt{\pi}}{33}(1-|\ell|)^2$. This means that (A_6) is satisfied as long as both $|\ell|<1$ and $0<\beta<\frac{5\sqrt{\pi}}{33}(1-|\ell|)^2$ hold. Therefore, according to Theorem 3.3, we can conclude that problem (3.6) has at least one solution on $[0,1]$.

4. Existence results-The multi-valued case

Let us recall some basic definitions on multi-valued maps [21], [23].

For a normed space $(X,\|\cdot\|)$, let $P_{cl}(X)=\{Y\in\mathscr{P}(X): Y \text{ is closed}\}$, $P_b(X)=\{Y\in\mathscr{P}(X): Y \text{ is bounded}\}$, $P_{cp}(X)=\{Y\in\mathscr{P}(X): Y \text{ is compact}\}$, and $P_{cp,c}(X)=\{Y\in\mathscr{P}(X): Y \text{ is compact and convex}\}$. A multi-valued map $G:X\to\mathscr{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x\in X$. The map G is bounded on bounded sets if $G(\mathbb{B})=\cup_{x\in\mathbb{B}}G(x)$ is bounded in X for all $\mathbb{B}\in P_b(X)$ (i.e.

$$\sup_{x\in\mathbb{B}}\{\sup\{|y|: y\in G(x)\}\}<\infty).$$

G is called upper semi-continuous (u.s.c.) on X if for each $x_0\in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathscr{N}_0$ of x_0 such that $G(\mathscr{N}_0)\subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B}\in P_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n\to x_*,\ y_n\to y_*,\ y_n\in G(x_n)$ imply $y_*\in G(x_*)$. G has a fixed point if there is $x\in X$ such that $x\in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : [0;1] \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y-z| : z \in G(t)\}$$

is measurable.

Let $L^1([0,1],\mathbb{R})$ be the Banach space of measurable functions $x : [0,1] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^1 |x(t)|dt$.

DEFINITION 4.1. A function $x \in AC^2([0,1],\mathbb{R})$ is a solution of the problem (1.6) if $x(0) = x_0 + g(x)$, $x(1) = \alpha \int_0^\eta x(s)ds$, and there exists a function $f \in L^1([0,1],\mathbb{R})$ such that $f(t) \in F(t,x(t))$ a.e. on $[0,1]$ and

$$\begin{aligned} x(t) = & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds - \frac{2t}{(2-\alpha\eta^2)\Gamma(q)} \int_0^1 (1-s)^{q-1} f(s)ds \\ & + \frac{2\alpha t}{(2-\alpha\eta^2)\Gamma(q)} \int_0^\eta \Big(\int_0^s (s-m)^{q-1} f(m)dm \Big) ds \\ & + \left[\frac{2(\alpha\eta-1)t}{2-\alpha\eta^2} + 1 \right] (x_0 + g(x)), \;\; t \in [0,1]. \end{aligned}$$

4.1. The Carathéodory case

DEFINITION 4.2. A multivalued map $F : [0,1] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \longmapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;

(ii) $x \longmapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,1]$;

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1([0,1],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leqslant \varphi_\alpha(t)$$

for all $\|x\| \leqslant \alpha$ and for a.e. $t \in [0,1]$.

For each $y \in C([0,1],\mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0,1],\mathbb{R}) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in [0,1]\}.$$

The following lemma will be used in the sequel.

LEMMA 4.3. *([26]) Let X be a Banach space. Let $F:[0,1]\times\mathbb{R}\to\mathscr{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,1],X)$ to $C([0,1],X)$. Then the operator*

$$\Theta\circ S_F : C([0,1],X)\to P_{cp,c}(C([0,1],X)),\ \ x\mapsto(\Theta\circ S_F)(x)=\Theta(S_{F,x})$$

is a closed graph operator in $C([0,1],X)\times C([0,1],X)$.

To prove our main result in this section we will use the following form of the Nonlinear Alternative for contractive maps [30, Corollary 3.8].

THEOREM 4.4. *Let X be a Banach space, and D a bounded neighborhood of $0\in X$. Let $Z_1 : X\to\mathscr{P}_{cp,c}(X)$ and $Z_2:\bar{D}\to\mathscr{P}_{cp,c}(X)$ two multi-valued operators satisfying*

(a) Z_1 is contraction, and

(b) Z_2 is u.s.c and compact.

Then, if $G=Z_1+Z_2$, either

(i) G has a fixed point in $\bar{D}$ or

(ii) there is a point $u\in\partial D$ and $\lambda\in(0,1)$ with $u\in\lambda G(u)$.

THEOREM 4.5. *Assume that*

(H_1) *$F:[0,1]\times\mathbb{R}\to\mathscr{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory multivalued map;*

(H_2) *there exists a continuous nondecreasing function $\psi:[0,\infty)\to(0,\infty)$ and a function $p\in L^1([0,1],\mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathscr{P}}:=\sup\{|y|: y\in F(t,x)\}\leqslant p(t)\psi(\|x\|)\ \text{for each}\ (t,x)\in[0,1]\times\mathbb{R};$$

(H_3) *there exists a constant $L_g<\left[\frac{2|\alpha\eta-1|}{|2-\alpha\eta^2|}+1\right]^{-1}$ such that*

$$|g(x)-g(y)|\leqslant L_g|x-y|,\ \ \forall x,y\in\mathbb{R};$$

(H_4) *there exists a number $M>0$ such that*

$$\frac{\left(1-L_g\left[\frac{2|\alpha\eta-1|}{|2-\alpha\eta^2|}+1\right]\right)M}{\frac{1}{\Gamma(q)}\left(1+\frac{2(1+|\alpha|\eta^q)}{|2-\alpha\eta^2|}\right)\psi(M)\|p\|_{L^1}+\left[\frac{2|\alpha\eta-1|}{|2-\alpha\eta^2|}+1\right]|x_0|}>1. \qquad (4.1)$$

Then the boundary value problem (1.6) has at least one solution on $[0,1]$.

Proof. Transform the problem (1.6) into a fixed point problem. Consider the operator $\mathscr{N} : C([0,1],\mathbb{R}) \longrightarrow \mathscr{P}(C([0,1],\mathbb{R}))$ defined by:

$$\mathscr{N}(x) = \left\{ \begin{array}{l} h \in C([0,1],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} f(s)ds \\ -\dfrac{2t}{(2-\alpha\eta^2)\Gamma(q)} \displaystyle\int_0^1 (1-s)^{q-1} f(s)ds \\ +\dfrac{2\alpha t}{(2-\alpha\eta^2)\Gamma(q)} \displaystyle\int_0^\eta \Big(\int_0^s (s-m)^{q-1} f(m)dm \Big) ds \\ + \left[\dfrac{2(\alpha\eta-1)t}{2-\alpha\eta^2} + 1 \right] (x_0 + g(x)), \end{array} \right. \end{array} \right\}$$

for $f \in S_{F,x}$.

Now, we define two operators as follows. $\mathscr{A} : C([0,1],\mathbb{R}) \longrightarrow C([0,1],\mathbb{R})$ by

$$\mathscr{A}x(t) = \left[\frac{2(\alpha\eta-1)t}{2-\alpha\eta^2} + 1 \right] (x_0 + g(x)), \tag{4.2}$$

and the multi-valued operator $\mathscr{B} : C([0,1],\mathbb{R}) \longrightarrow \mathscr{P}(C([0,1],\mathbb{R}))$ by

$$\mathscr{B}(x) = \left\{ \begin{array}{l} h \in C([0,1],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} f(s)ds \\ -\dfrac{2t}{(2-\alpha\eta^2)\Gamma(q)} \displaystyle\int_0^1 (1-s)^{q-1} f(s)ds \\ +\dfrac{2\alpha t}{(2-\alpha\eta^2)\Gamma(q)} \displaystyle\int_0^\eta \Big(\int_0^s (s-m)^{q-1} f(m)dm \Big) ds. \end{array} \right. \end{array} \right\} \tag{4.3}$$

Then $\mathscr{N} = \mathscr{A} + \mathscr{B}$. We shall show that the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Theorem 4.4 on $[0,1]$. For better readability, we break the proof into a sequence of steps and claims.

Step 1: We show that $\mathscr{A}$ *is a contraction on* $C([0,1],\mathbb{R})$. Let $x,y \in C([0,1],\mathbb{R})$. Then

$$\begin{aligned} |\mathscr{A}x(t) - \mathscr{A}y(t)| &= \left| \left[\frac{2(\alpha\eta-1)t}{2-\alpha\eta^2} + 1 \right] (g(x) - g(y)) \right| \\ &\leqslant \left[\frac{2|\alpha\eta-1|}{|2-\alpha\eta^2|} + 1 \right] |g(x) - g(y)| \\ &\leqslant L_g \left[\frac{2|\alpha\eta-1|}{|2-\alpha\eta^2|} + 1 \right] |x-y|. \end{aligned}$$

Taking supremum over t,

$$\|\mathscr{A}x - \mathscr{A}y\| \leqslant L_0 \|x-y\|, \quad L_0 = L_g \left[\frac{2|\alpha\eta-1|}{|2-\alpha\eta^2|} + 1 \right] < 1.$$

This shows that $\mathscr{A}$ is a contraction, since $L_0 < 1$.

Step 2: We shall show that the operator $\mathscr{B}$ *is compact and convex valued and it is completely continuous*. This will be given in several claims.

CLAIM I: $\mathscr{B}$ *maps bounded sets into bounded sets in* $C([0,1],\mathbb{R})$. To see this, let $B_r = \{x \in C([0,1],\mathbb{R}) : \|x\| \leqslant r\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then, for each $h \in \mathscr{B}(x), x \in B_r$, there exists $f \in S_{F,x}$ such that

$$\begin{aligned}
|h(t)| &\leqslant \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s)|ds + \frac{2}{|2-\alpha\eta^2|\Gamma(q)}\int_0^1 (1-s)^{q-1}|f(s)|ds \\
&\quad + \frac{2|\alpha|}{|2-\alpha\eta^2|\Gamma(q)}\int_0^\eta \Big(\int_0^s (s-m)^{q-1}|f(m)|dm\Big)ds \\
&\leqslant \psi(\|x\|)\left(\frac{1}{\Gamma(q)} + \frac{2}{|2-\alpha\eta^2|\Gamma(q)} + \frac{2|\alpha|\eta^q}{|2-\alpha\eta^2|\Gamma(q)}\right)\int_0^1 p(s)ds \\
&= \frac{\psi(\|x\|)}{\Gamma(q)}\left(1 + \frac{2(1+|\alpha|\eta^q)}{|2-\alpha\eta^2|}\right)\int_0^1 p(s)ds.
\end{aligned}$$

Thus,

$$\|h\| \leqslant \frac{\psi(r)}{\Gamma(q)}\left(1 + \frac{2(1+|\alpha|\eta^q)}{|2-\alpha\eta^2|}\right)\|p\|_{L^1},$$

and consequently for each $h \in \mathscr{B}(B_q)$ we have

$$\|h\| \leqslant \frac{\psi(r)}{\Gamma(q)}\left(1 + \frac{2(1+|\alpha|\eta^q)}{|2-\alpha\eta^2|}\right)\|p\|_{L^1} := \ell.$$

CLAIM II: Next we show that $\mathscr{B}$ *maps bounded sets into equi-continuous sets*. Let B_r be, as above, a bounded set and let $h \in \mathscr{B}(x)$ for $x \in B_r$. Let $t', t'' \in [0,1]$ with $t' < t''$ and $x \in B_r$, where B_r. For each $h \in B(x)$, we obtain

$$\begin{aligned}
|h(t'') - h(t')| &\leqslant \left|\frac{1}{\Gamma(q)}\int_0^{t'} [(t''-s)^{q-1} - (t'-s)^{q-1}]f(s)ds\right| \\
&\quad + \left|\frac{1}{\Gamma(q)}\int_{t'}^{t''} (t''-s)^{q-1}f(s)ds\right| + \left|\frac{2(t''-t')}{(2-\alpha\eta^2)\Gamma(q)}\int_0^1 (1-s)^{q-1}f(s)ds\right| \\
&\quad + \left|\frac{2\alpha(t''-t')}{(2-\alpha\eta^2)\Gamma(q)}\int_0^\eta \Big(\int_0^s (s-m)^{q-1}f(m)dm\Big)ds\right| \\
&\leqslant \frac{\psi(r)}{\Gamma(q)}\int_0^{t'} [(t''-s)^{q-1} - (t'-s)^{q-1}]p(s)ds \\
&\quad + \frac{\psi(r)}{\Gamma(q)}\int_{t'}^{t''} (t''-s)^{q-1}p(s)ds + \frac{2(t''-t')}{|2-\alpha\eta^2|\Gamma(q)}\int_0^1 (1-s)^{q-1}p(s)ds \\
&\quad + \frac{2|\alpha|\psi(r)(t''-t')}{|2-\alpha\eta^2|\Gamma(q)}\int_0^\eta \Big(\int_0^s (s-m)^{q-1}p(m)dm\Big)ds.
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_r$ as $t'' - t' \to 0$. As $\mathscr{B}$ satisfies the above three assumptions, therefore it follows by the Arzelá-Ascoli theorem that $\mathscr{B} : C([0,1],\mathbb{R}) \to \mathscr{P}(C([0,1],\mathbb{R}))$ is completely continuous.

CLAIM III: Next we prove that $\mathscr{B}$ *has a closed graph.* Let $x_n \to x_*, h_n \in \mathscr{B}(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathscr{B}(x_*)$. Associated with $h_n \in \mathscr{B}(x_n)$, there exists $f_n \in S_{F,x_n}$ such that for each $t \in [0,1]$,

$$\begin{aligned} h_n(t) = &\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f_n(s)ds - \frac{2t}{(2-\alpha\eta^2)\Gamma(q)}\int_0^1 (1-s)^{q-1} f_n(s)ds \\ &+ \frac{2\alpha t}{(2-\alpha\eta^2)\Gamma(q)}\int_0^\eta \Big(\int_0^s (s-m)^{q-1} f_n(m)dm\Big)ds. \end{aligned}$$

Thus we have to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [0,1]$,

$$\begin{aligned} h_*(t) = &\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f_*(s)ds - \frac{2t}{(2-\alpha\eta^2)\Gamma(q)}\int_0^1 (1-s)^{q-1} f_*(s)ds \\ &+ \frac{2\alpha t}{(2-\alpha\eta^2)\Gamma(q)}\int_0^\eta \Big(\int_0^s (s-m)^{q-1} f_*(m)dm\Big)ds. \end{aligned}$$

Let us consider the continuous linear operator $\Theta : L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ given by

$$\begin{aligned} f \mapsto \Theta(f)(t) = &\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s)ds - \frac{2t}{(2-\alpha\eta^2)\Gamma(q)}\int_0^1 (1-s)^{q-1} f(s)ds \\ &+ \frac{2\alpha t}{(2-\alpha\eta^2)\Gamma(q)}\int_0^\eta \Big(\int_0^s (s-m)^{q-1} f(m)dm\Big)ds. \end{aligned}$$

Observe that

$$\begin{aligned} \|h_n(t) - h_*(t)\| = &\Bigg\| \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}(f_n(s) - f_*(s))ds \\ &- \frac{2t}{(2-\alpha\eta^2)\Gamma(q)}\int_0^1 (1-s)^{q-1}(f_n(s) - f_*(s))ds \\ &+ \frac{2\alpha t}{(2-\alpha\eta^2)\Gamma(q)}\int_0^\eta \Big(\int_0^s (s-m)^{q-1}(f_n(s) - f_*(s))dm\Big)ds\Bigg\| \to 0, \end{aligned}$$

as $n \to \infty$. Thus, it follows by Lemma 4.3 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$\begin{aligned} h_*(t) = &\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f_*(s)ds - \frac{2t}{(2-\alpha\eta^2)\Gamma(q)}\int_0^1 (1-s)^{q-1} f_*(s)ds \\ &+ \frac{2\alpha t}{(2-\alpha\eta^2)\Gamma(q)}\int_0^\eta \Big(\int_0^s (s-m)^{q-1} f_*(m)dm\Big)ds, \end{aligned}$$

for some $f_* \in S_{F,x_*}$. Hence $\mathscr{B}$ has a closed graph (and therefore has closed values). As a result $\mathscr{B}$ is compact valued.

Therefore the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Theorem 4.4 and hence an application of it yields that either condition (i) or condition (ii) holds. We show that the conclusion (ii) is not possible. If $x \in \lambda\mathscr{A}(x)+\lambda\mathscr{B}(x)$ for $\lambda \in (0,1)$, then there exists $f \in S_{F,x}$ such that

$$\begin{aligned} x(t) &= \lambda \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s)ds - \lambda \frac{2t}{(2-\alpha\eta^2)\Gamma(q)}\int_0^1 (1-s)^{q-1}f(s)ds \\ &+\lambda \frac{2\alpha t}{(2-\alpha\eta^2)\Gamma(q)}\int_0^\eta \Big(\int_0^s (s-m)^{q-1}f(m)dm\Big)ds \\ &+\lambda \left[\frac{2(\alpha\eta-1)t}{2-\alpha\eta^2}+1\right](x_0+g(x)), \;\; t\in[0,1]. \end{aligned}$$

Consequently, we have

$$\begin{aligned} |x(t)| &\leqslant \frac{1}{\Gamma(q)}\left(1+\frac{2(1+|\alpha|\eta^q)}{|2-\alpha\eta^2|}\right)\int_0^1 f(s)ds + \left[\frac{2|\alpha\eta-1|}{|2-\alpha\eta^2|}+1\right][|x_0|+|g(x)|] \\ &\leqslant \frac{1}{\Gamma(q)}\left(1+\frac{2(1+|\alpha|\eta^q)}{|2-\alpha\eta^2|}\right)\psi(\|x\|)\int_0^1 p(s)ds \\ &+\left[\frac{2|\alpha\eta-1|}{|2-\alpha\eta^2|}+1\right][|x_0|+L_g\|x\|]. \end{aligned}$$

If condition (ii) of Theorem 3.1 holds, then there exists $\lambda \in (0,1)$ and $x \in \partial B_M$ with $x=\lambda\mathscr{N}(x)$. Then, x is a solution of (2.3) with $\|x\|=M$. Now, the previous inequality implies

$$\frac{\left(1-L_g\left[\frac{2|\alpha\eta-1|}{|2-\alpha\eta^2|}+1\right]\right)M}{\frac{1}{\Gamma(q)}\left(1+\frac{2(1+|\alpha|\eta^q)}{|2-\alpha\eta^2|}\right)\psi(M)\|p\|_{L^1}+\left[\frac{2|\alpha\eta-1|}{|2-\alpha\eta^2|}+1\right]|x_0|} \leqslant 1$$

which contradicts to (4.1). Hence, $\mathscr{N}$ has a fixed point in $[0,1]$ by Theorem 4.4, and consequently the boundary value problem (1.6) has a solution. This completes the proof. □

REMARK 4.6. We would like point out that the condition $L_0<1$ can be deleted if we use the well-known Bielecki's renorming method. Of course assumption (H_3) has to be adjusted slightly for the new norm.

REMARK 4.7. If ψ satisfies a sublinear condition or more generally

$$\lim_{\xi\to\infty}\frac{\xi}{\frac{1}{\Gamma(q)}\left(1+\frac{2(1+|\alpha|\eta^q)}{|2-\alpha\eta^2|}\right)\psi(\xi)\|p\|_{L^1}+\left[\frac{2|\alpha\eta-1|}{|2-\alpha\eta^2|}+1\right]|x_0|} > 1-L_g\left[\frac{2|\alpha\eta-1|}{|2-\alpha\eta^2|}+1\right]$$

then the existence of M in (H_4) is guaranteed.

EXAMPLE 4.8. Consider the following fractional boundary value problem

$$\begin{cases} {}^cD^{3/2}x(t) \in F(t,x(t)), \ \ 0<t<1, \\ x(0) = \dfrac{1}{3} + \dfrac{1}{5}x(\xi), \ x(1) = 4\displaystyle\int_0^{1/2} x(s)ds. \end{cases} \tag{4.4}$$

Here, $q = 3/2$, $x_0 = 1/3, \ell = 1/5$, $\alpha = 4$, $\eta = 1/2$, and $F : [0,1]\times\mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map given by

$$x \to F(t,x) = \left[\frac{1}{9}\frac{|x|^3}{|x|^3+3}, \frac{1}{20}\frac{|x|}{|x|+1}\right].$$

For $f \in F$, we have

$$|f| \leqslant \max\left(\frac{1}{9}\frac{|x|^3}{|x|^3+3}, \frac{1}{20}\frac{|x|}{|x|+1}\right) \leqslant \frac{1}{9}, \ \ x \in \mathbb{R}.$$

Thus,

$$\|F(t,x)\|_{\mathscr{P}} := \sup\{|y| : y \in F(t,x)\} \leqslant \frac{1}{9} = p(t)\psi(\|x\|), \ \ x \in \mathbb{R},$$

with $p(t) = 1$, $\psi(\|x\|) = \dfrac{1}{9}$.

Further, using the condition (H_4) we find that $M > 5.2678719$. Clearly, all the conditions of Theorem 4.5 are satisfied. So there exists at least one solution of the problem (4.4) on $[0,1]$.

4.2. The lower semi-continuous case

As a next result, we study the case when F is not necessarily convex valued. Our strategy to deal with this problems is based on the nonlinear alternative of Leray-Schauder type together with the selection theorem of Bressan and Colombo [17] for lower semi-continuous maps with decomposable values.

Let us mention some auxiliary facts. Let X be a nonempty closed subset of a Banach space E and $G : X \to \mathscr{P}(E)$ be a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E. Let A be a subset of $[0,1]\times\mathbb{R}$. A is $\mathscr{L}\otimes\mathscr{B}$ measurable if A belongs to the $\sigma-$algebra generated by all sets of the form $\mathscr{J}\times\mathscr{D}$, where $\mathscr{J}$ is Lebesgue measurable in $[0,1]$ and $\mathscr{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathscr{A}$ of $L^1([0,1],\mathbb{R})$ is decomposable if for all $u,v \in \mathscr{A}$ and measurable $\mathscr{J} \subset [0,1] = J$, the function $u\chi_{\mathscr{J}} + v\chi_{J-\mathscr{J}} \in \mathscr{A}$, where $\chi_{\mathscr{J}}$ stands for the characteristic function of $\mathscr{J}$.

DEFINITION 4.9. Let Y be a separable metric space and let $N : Y \to \mathscr{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator. We say N has a property (BC) if N is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.

Let $F : [0,1] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathscr{F} : C([0,1] \times \mathbb{R}) \to \mathscr{P}(L^1([0,1],\mathbb{R}))$ associated with F as

$$\mathscr{F}(x) = \{w \in L^1([0,1],\mathbb{R}) : w(t) \in F(t,x(t)) \text{ for a.e. } t \in [0,1]\},$$

which is called the Nemytskii operator associated with F.

DEFINITION 4.10. Let $F : [0,1] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nemytskii operator $\mathscr{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

LEMMA 4.11. *([17]) Let Y be a separable metric space and let a multivalued operator $N : Y \to \mathscr{P}(L^1([0,1],\mathbb{R}))$ satisfying the property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1([0,1],\mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

THEOREM 4.12. *Assume that $(H_2),(H_3),(H_4)$ and the following condition holds:*

(H_5) *$F : [0,1] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that*

- (a) *$(t,x) \longmapsto F(t,x)$ is $\mathscr{L} \otimes \mathscr{B}$ measurable,*
- (b) *$x \longmapsto F(t,x)$ is lower semicontinuous for each $t \in [0,1]$;*

Then the boundary value problem (1.6) has at least one solution on $[0,1]$.

Proof. It follows from (H_2) and (H_5) that F is of l.s.c. type ([22]). Then from Lemma 4.11, there exists a continuous function $f : C([0,1],\mathbb{R}) \to L^1([0,1],\mathbb{R})$ such that $f(x) \in \mathscr{F}(x)$ for all $x \in C([0,1],\mathbb{R})$.

Consider the problem

$$\begin{cases} {}^cD^q x(t) = f(x(t)), \ \ 0 < t < 1, \ \ 1 < q \leqslant 2, \\ x(0) = x_0 + g(x), \quad x(1) = \alpha \displaystyle\int_0^\eta x(s)ds, \ \ 0 < \eta < 1, \ \alpha \neq 2/\eta^2. \end{cases} \tag{4.5}$$

Observe that if $x \in AC^2([0,1])$ is a solution of (4.5), then x is a solution to the problem (1.6). Now, we define two operators as follows: $\mathscr{A}' : C([0,1],\mathbb{R}) \longrightarrow C([0,1],\mathbb{R})$ by

$$\mathscr{A}'x(t) = \left[\frac{2(\alpha\eta - 1)t}{2 - \alpha\eta^2} + 1\right](x_0 + g(x)), \tag{4.6}$$

and the operator $\mathscr{B}' : C([0,1],\mathbb{R}) \longrightarrow C([0,1],\mathbb{R})$ by

$$\mathscr{B}'x(t) = \begin{cases} \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} f(x(s))ds \\ -\dfrac{2t}{(2-\alpha\eta^2)\Gamma(q)} \displaystyle\int_0^1 (1-s)^{q-1} f(x(s))ds \\ +\dfrac{2\alpha t}{(2-\alpha\eta^2)\Gamma(q)} \displaystyle\int_0^\eta \Big(\int_0^s (s-m)^{q-1} f(x(m))dm\Big)ds. \end{cases} \tag{4.7}$$

Now $\mathscr{A}', \mathscr{B}' : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ are continuous. Also the argument in Theorem 3.1 guarantees that $\mathscr{A}'$ and $\mathscr{B}'$ satisfy all the conditions of the Nonlinear Alternative for contractive maps in the single valued setting [24] and hence the problem 4.5 has a solution. □

REMARK 4.13. The results of this paper can be easily extended to more general integral boundary conditions. Thus, for example, we can study the following fractional boundary value problem

$$\begin{cases} {}^cD^q x(t) = f(t,x(t)), \;\; 0 < t < 1, \;\; 1 < q \leqslant 2, \\ x(0) = x_0 + g(x), \quad x(1) = \alpha \displaystyle\int_\mu^\nu x(s)ds, \;\; 0 < \mu < \nu < 1, \end{cases} \tag{4.8}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $g : C([0,1],\mathbb{R}) \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ is such that $\alpha \neq 2/(\nu^2 - \mu^2)$. We omit the details.

REFERENCES

[1] R. P. AGARWAL, B. ANDRADE, C. CUEVAS, *Weighted pseudo-almost periodic solutions of a class of semilinear fractional differential equations*, Nonlinear Anal. Real World Appl. **11** (2010), 3532–3554.

[2] R.P. AGARWAL, V. LAKSHMIKANTHAM, J. J. NIETO, *On the concept of solution for fractional differential equations with uncertainty*, Nonlinear Anal. **72** (2010), 2859–2862.

[3] R.P. AGARWAL, Y. ZHOU, Y. HE, *Existence of fractional neutral functional differential equations*, Comput. Math. Appl. **59** (2010), 1095–1100.

[4] B. AHMAD, S. SIVASUNDARAM, *On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order*, Appl. Math. Comput. **217** (2010), 480–487.

[5] B. AHMAD, *Existence of solutions for irregular boundary value problems of nonlinear fractional differential equations*, Appl. Math. Lett. **23** (2010), 390–394.

[6] B. AHMAD, *Existence of solutions for fractional differential equations of order $q \in (2,3]$ with anti-periodic boundary conditions*, J. Appl. Math. Comput. **34** (2010), 385–391.

[7] B. AHMAD, *On nonlocal boundary value problems for nonlinear integro-differential equations of arbitrary fractional order*, Results Math. **63** (2013), 183–194.

[8] B. AHMAD, A. ALSAEDI, *Existence and uniqueness of solutions for coupled systems of higher order nonlinear fractional differential equations*, Fixed Point Theory Appl. **2010** (2010), Article ID 364560, 17 pp.

[9] B. AHMAD, S.K. NTOUYAS, A. ALSAEDI, *New existence results for nonlinear fractional differential equations with three-point integral boundary conditions*, Adv. Differ. Equ. **2011**, Article ID 107384, 11 pp.

[10] B. AHMAD, J.J. NIETO, *Existence results for nonlinear boundary value problems of fractional integrodifferential equations with integral boundary conditions*, Bound. Value Probl. **2009**, Art. ID 708576, 11 pp.

[11] B. AHMAD, J.J. NIETO, *Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions*, Comput. Math. Appl. **58** (2009), 1838–1843.

[12] Z.B. BAI, *On positive solutions of a nonlocal fractional boundary value problem*, Nonlinear Anal. **72** (2010), 916–924.

[13] K. BALACHANDRAN, J.J. TRUJILLO, *The nonlocal Cauchy problem for nonlinear fractional integrodifferential equations in Banach spaces*, Nonlinear Anal. **72** (2010), 4587–4593.

[14] M. BENCHOHRA, S. HAMANI, S.K. NTOUYAS, *Boundary value problems for differential equations with fractional order*, Surv. Math. Appl. **3** (2008), 1–12.

[15] M. BENCHOHRA, S. HAMANI, S.K. NTOUYAS, *Boundary value problems for differential equations with fractional order and nonlocal conditions*, Nonlinear Anal. **71** (2009), 2391–2396.

[16] A.V. BITSADZE, *On the Theory of Nonlocal Boundary Value Problems*, Dokl. Akad. Nauk SSSR **277** (1984), 17–19.

[17] A. BRESSAN, G. COLOMBO, *Extensions and selections of maps with decomposable values*, Studia Math. **90** (1988), 69–86.

[18] L. BYSZEWSKI, V.LAKSHMIKANTHAM, *Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space*, Appl. Anal. **40** (1991), 11–19.

[19] L. BYSZEWSKI, *Theorems about existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, J. Math. Anal. Appl. **162** (1991), 494–505.

[20] L. BYSZEWSKI, *Existence and uniqueness of mild and classical solutions of semilinear functional-differential evolution nonlocal Cauchy problem*, Selected problems of mathematics, 50th Anniv. Cracow Univ. Technol. Anniv., Issue 6, Cracow Univ. Technol., Krakow, 1995, pp. 25-33.

[21] K. DEIMLING, *Multivalued Differential Equations*, Walter De Gruyter, Berlin-New York, 1992.

[22] M. FRIGON, *Théorèmes d'existence de solutions d'inclusions différentielles*, Topological Methods in Differential Equations and Inclusions (edited by A. Granas and M. Frigon), NATO ASI Series C, Vol. **472**, Kluwer Acad. Publ., Dordrecht, 1995, 51–87.

[23] SH. HU, N. PAPAGEORGIOU, *Handbook of Multivalued Analysis, Theory I*, Kluwer, Dordrecht, 1997.

[24] A. GRANAS AND J. DUGUNDJI, *Fixed Point Theory*, Springer-Verlag, New York, 2005.

[25] A.A. KILBAS, H.M. SRIVASTAVA, J.J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[26] A. LASOTA AND Z. OPIAL, *An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **13** (1965), 781–786.

[27] M.P. LAZAREVIĆ, A.M. SPASIĆ, *Finite-time stability analysis of fractional order time-delay systems: Gronwall's approach*, Math. Comput. Model. **49** (2009), 475–481.

[28] J.J. NIETO, *Maximum principles for fractional differential equations derived from Mittag-Leffler functions*, Appl. Math. Lett. **23** (2010), 1248–1251.

[29] D. O'REGAN, *Fixed-point theory for the sum of two operators*, Appl. Math. Lett. **9** (1996), 1–8.

[30] W. V. PETRYSHYN AND P.M. FITZPATRIC, *A degree theory, fixed point theoremsand mapping theorems for multivalued noncompact maps*, Trans. Amer. Math. Soc. **194** (1974), 1–25.

[31] I. PODLUBNY, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[32] J. SABATIER, O.P. AGRAWAL, J.A.T. MACHADO (Eds.), *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, 2007.

[33] S. G. SAMKO, A.A. KILBAS, O.I. MARICHEV, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[34] Z. WEI, Q. LI, J. CHE, *Initial value problems for fractional differential equations involving RiemannLiouville sequential fractional derivative*, J. Math. Anal. Appl. **367** (2010), 260–272.

[35] S. Q. ZHANG, *Positive solutions to singular boundary value problem for nonlinear fractional differential equation*, Comput. Math. Appl. **59** (2010), 1300–1309.

[36] W. ZHONG, W. LIN, *Nonlocal and multiple-point boundary value problem for fractional differential equations*, Comput. Math. Appl. **39** (2010), 1345–1351.

Sotiris K. Ntouyas
Department of Mathematics
University of Ioannina
451 10 Ioannina, GREECE
e-mail: sntouyas@uoi.gr