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WEIGHTED HARDY–TYPE INEQUALITIES FOR MONOTONE CONVEX FUNCTIONS WITH SOME APPLICATIONS

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Abstract. In this paper, we establish some new refined weighted Hardy-type inequalities involving monotone convex functions. We give the results for some special kernels of Riemann-Liouville and Weyl's operators as applications. Also we discuss some related dual cases. At the end, we prove some refined G. H. Hardy-type inequalities for different kinds of fractional integrals and fractional derivatives.

1. Introduction

The first work in the field of fractional calculus is the book by Oldham and Spanier [15] published in 1974. One of the most recent works on the subject of fractional calculus is the book of Podlubny [17] published in 1999, which deals principally with fractional differential equations. These books draw the considerable attention of the different mathematician in this field to explore the new ideas in fractional calculus.

Fractional calculus deals with the study of fractional order integral and derivative operators over real or complex domains and their applications. Fractional calculus have been of great importance during the last few decades. This follows from the intensive development of the theory of fractional calculus, followed by the applications of its methods in various sciences and engineering. For further details and literature about the fractional calculus we refer [5], [6], [13] and the reference cited there in.

We start with the inequality of G. H. Hardy. Let $[a,b], -\infty < a < b < \infty$ be a finite interval on real axis $\mathbb{R}$ and $1 \leqslant p \leqslant \infty$, then

$$\|I^{\alpha}_{a^+} f\|_p \leqslant K\|f\|_p, \quad \|I^{\alpha}_{b^-} f\|_p \leqslant K\|f\|_p \tag{1.1}$$

holds, where

$$K = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)},$$

$I^{\alpha}_{a^+} f$ and $I^{\alpha}_{b^-} f$ of order $\alpha > 0$ denote the Riemann-Liouville fractional integrals defined by

$$I^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(y)(x-y)^{\alpha-1} dy, \quad (x > a)$$

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and

$$I_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b f(y)(y-x)^{\alpha-1}dy,\ \ (x<b),$$

where Γ is the Gamma function, i.e. $\Gamma(\alpha)=\int_0^\infty e^t t^{\alpha-1}dt$.

G. H. Hardy proved the inequality (1.1) involving left-sided fractional integral in one of his initial paper, see [10]. The calculation for the constant K is hidden inside the proof.

In this paper, our particular interest is to give the Hardy-type inequality and prove some new inequalities involving monotone convex function using different kinds of fractional integrals and fractional derivative like fractional integral of a function with respect to an increasing function, Riemann-Liouville fractional integrals, Caputo fractional derivative, Erdelyi-Kóber fractional integrals and Hadamard-type fractional integrals. Numerous mathematicians obtained new Hardy-type inequalities for different fractional integrals and fractional derivatives. For details we refer to [7], [9], [11], [12], [14], [16].

Let $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_2,\Sigma_2,\mu_2)$ be measure spaces with σ-finite measures and A_k be an integral operator defined by

$$A_kf(x):=\frac{1}{K(x)}\int_{\Omega_2}k(x,y)f(y)d\mu_2(y), \tag{1.2}$$

where $k:\Omega_1\times\Omega_2\to\mathbb{R}$ is measurable and non-negative kernel, f is measurable function on Ω_2 and

$$K(x):=\int_{\Omega_2}k(x,y)d\mu_2(y),\quad x\in\Omega_1. \tag{1.3}$$

Throughout the paper, we consider that $K(x)>0$ a.e. on Ω_1.

The following theorem is given in [8].

THEOREM 1.1. *Let $0<p\leqslant q<\infty$, or $-\infty<q\leqslant p<0$, $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_2,\Sigma_2,\mu_2)$ be measure spaces with σ-finite measures, u be a weight function on Ω_1, k be a non-negative measurable function on $\Omega_1\times\Omega_2$, K be defined on Ω_1 by (1.3) and that the function $x\mapsto u(x)\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}}$ is integrable on Ω_1 for each fixed $y\in\Omega_2$, and that v is defined on Ω_2 by*

$$v(y):=\left(\int_{\Omega_1}u(x)\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}}d\mu_1(x)\right)^{\frac{p}{q}}<\infty.$$

If Φ is a non-negative convex function on the interval $I\subseteq\mathbb{R}$ and $\varphi:I\to\mathbb{R}$ is any function, such that $\varphi\in\partial\Phi(x)$ for all $x\in \mathrm{Int}I$, then the inequality

$$\left(\int_{\Omega_2} v(y)\Phi(f(y))\,d\mu_2(y)\right)^{\frac{q}{p}} - \int_{\Omega_1} u(x)[\Phi(A_kf(x))]^{\frac{q}{p}}d\mu_1(x)$$
$$\geqslant \frac{q}{p}\int_{\Omega_1}\frac{u(x)}{K(x)}\Phi^{\frac{q}{p}-1}(A_kf(x))\int_{\Omega_2} k(x,y)r(x,y)d\mu_2(y)d\mu_1(x) \tag{1.4}$$

holds for all measurable functions $f:\Omega_2\to\mathbb{R}$*, such that* $f(y)\in I$ *for all* $y\in\Omega_2$*, where* A_k *is defined by* (1.2) *and* $r:\Omega_1\times\Omega_2\to\mathbb{R}$ *is a non-negative function defined by*

$$r(x,y) = ||\Phi(f(y)) - \Phi(A_kf(x))| - |\varphi(A_kf(x))||f(y) - A_kf(x)||.$$

REMARK 1.2. For $p=q$, the Theorem 1.1 becomes [7, Theorem 2.1] and convex function Φ need not to be non-negative.

Throughout this paper, all measures are assumed to be positive, all functions are assumed to be positive and measurable and expressions of the form $0\cdot\infty$, $\frac{\infty}{\infty}$ and $\frac{0}{0}$ are taken to be equal to zero. Moreover, by a weight $u=u(x)$ we mean a non-negative measurable function on the actual interval or more general set. $B(\cdot;\cdot,\cdot)$ denotes the incomplete Beta functions, defined by

$$B(x;a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt, \quad x\in[0,1],\ \ a,b>0. \tag{1.5}$$

As usual, $B(a,b)=B(1;a,b)$ stands for the standard Beta function.

The paper is organized in the following pattern: After introduction, in Section 2, we construct some inequalities of Hardy-type involving monotone convex with their related dual cases. We give related results for special kernels of Riemman-Liouville and Weyle's operators. Also, we give the results by considering power and exponential functions. In Section 3, we give some improvements of G. H. Hardy-type inequalities for different kind of fractional integrals like fractional integral of a function with respect to an increasing function, Riemann-Liouville fractional integrals, Hadamard-type fractional integrals and Erdelyi-Kóber fractional integrals. In Section 4, we give the improvement for Canavati-type fractional derivative and Caputo fractional derivative.

2. The main results

Here we provide the more general results related to Hardy-type inequalities for monotone convex functions.

THEOREM 2.1. *Let* $0<p\leqslant q<\infty$*, or* $-\infty<q\leqslant p<0$ *and let the assumptions of Theorem* 1.1 *be satisfied. If* Φ *is a non-negative monotone convex on the interval* $I\subseteq\mathbb{R}$*,* $f(y)>A_kf(x)$ *for* $y\in\Omega_2{}'$ $(\Omega_2{}'\subset\Omega_2)$ *and* $\varphi:I\to\mathbb{R}$ *is any function, such that* $\varphi(x)\in\partial\Phi(x)$ *for all* $x\in \operatorname{Int} I$*, then the inequality*

$$
\begin{aligned}
&\left(\int_{\Omega_2} v(y)\Phi(f(y))\,d\mu_2(y)\right)^{\frac{q}{p}} - \int_{\Omega_1} u(x)\Phi^{\frac{q}{p}}(A_k f(x))\,d\mu_1(x) \\
&\geqslant \frac{q}{p}\left|\int_{\Omega_1} \frac{u(x)}{K(x)}\Phi^{\frac{q}{p}-1}(A_k f(x)) \int_{\Omega_2} sgn(f(y)-A_k f(x))k(x,y)\Big[\Phi(f(y))-\Phi(A_k f(x)) \right. \\
&\qquad\qquad \left. -|\varphi(A_k f(x))|\cdot(f(y)-A_k f(x))\Big]\,d\mu_2(y)\,d\mu_1(x)\right| \qquad (2.1)
\end{aligned}
$$

holds for all measurable functions $f:\Omega_2 \to \mathbb{R}$, such that $f(y)\in I$, for all fixed $y \in \Omega_2$ where $A_k f$ is defined by (1.2).

If Φ is a non-negative monotone concave, then the order of terms on the left-hand side of (2.1) is reversed.

Proof. Consider the case, when Φ is non-decreasing on the interval I. For a fixed $x\in\Omega_1$, let $\Omega_2' = \{y\in\Omega_2 : f(y) > A_k f(x)\}$. Then

$$
\begin{aligned}
&\int_{\Omega_2} k(x,y)|\Phi(f(y))-\Phi(A_k f(x))|d\mu_2(y) \\
&= \int_{\Omega_2'} k(x,y)[\Phi(f(y))-\Phi(A_k f(x)))]d\mu_2(y) \\
&\quad + \int_{\Omega_2\setminus\Omega_2'} k(x,y)[\Phi(A_k f(x))-\Phi(f(y))]d\mu_2(y) \\
&= \int_{\Omega_2'} k(x,y)\Phi(f(y))d\mu_2(y) - \int_{\Omega_2\setminus\Omega_2'} k(x,y)\Phi(f(y))d\mu_2(y) \\
&\quad -\Phi(A_k f(x))\int_{\Omega_2'} k(x,y)d\mu_2(y) + \Phi(A_k f(x))\int_{\Omega_2\setminus\Omega_2'} k(x,y)d\mu_2(y) \\
&= \int_{\Omega_2} sgn(f(y)-A_k f(x))k(x,y)[\Phi(f(y))-\Phi(A_k f(x)]d\mu_2(y). \qquad (2.2)
\end{aligned}
$$

Similarly, we can write

$$
\begin{aligned}
&\int_{\Omega_2} k(x,y)|f(y)-A_k f(x)|d\mu_2(y) \\
&\qquad = \int_{\Omega_2} sgn(f(y)-A_k f(x))k(x,y)(f(y)-A_k f(x))d\mu_2(y). \qquad (2.3)
\end{aligned}
$$

From (1.4), (2.2) and (2.3), we get (2.1).

The case, when Φ is non-increasing can be discussed in the similar way. $\square$

For $p=q$, we get the following result which is in fact the new version of the [7, Theorem 2.1] involving monotone convex function and the function Φ not need to be non-negative.

COROLLARY 2.2. *Let $\Omega_1,\Omega_2,\mu_1,\mu_2,u,k,K$ and v be as in Theorem 1.1. If Φ is a monotone convex on the interval $I\subseteq\mathbb{R}$, $f(y)>A_kf(x)$ for $y\in{\Omega_2}'$ $({\Omega_2}'\subset\Omega_2)$ and $\varphi:I\to\mathbb{R}$ is any function, such that $\varphi(x)\in\partial\Phi(x)$ for all $x\in Int\,I$, then the inequality*

$$\begin{aligned}\int_{\Omega_2} & v(y)\Phi(f(y))\,d\mu_2(y)-\int_{\Omega_1}u(x)\Phi(A_kf(x))\,d\mu_1(x)\\ &\geqslant\left|\int_{\Omega_1}\frac{u(x)}{K(x)}\int_{\Omega_2}sgn(f(y)-A_kf(x))k(x,y)\Big[\Phi(f(y))-\Phi(A_kf(x))\right.\\ &\qquad\left.-|\varphi(A_kf(x))|\cdot(f(y)-A_kf(x))\Big]\,d\mu_2(y)\,d\mu_1(x)\right|\end{aligned}\tag{2.4}$$

holds for all measurable functions $f:\Omega_2\to\mathbb{R}$, such that $f(y)\in I$, for all fixed $y\in\Omega_2$ where A_kf is defined by (1.2).

If Φ is a monotone concave, then the order of terms on the left-hand side of (2.4) is reversed.

Although the (2.1), holds for non-negative monotone convex functions some choices of Φ are of our particular interest. Here, we consider the power and exponential functions. Let the function $\Phi:\mathbb{R}_+\to\mathbb{R}$ be defined by $\Phi(x)=x^p$. It is non-negative and monotone function. Obviously, $\varphi(x)=\Phi'(x)=px^{p-1}$, $x\in\mathbb{R}_+$, so Φ is convex for $p\in\mathbb{R}\setminus[0,1)$, concave for $p\in(0,1]$, and affine, that is, both convex and concave for $p=1$.

COROLLARY 2.3. *Let $\Omega_1,\Omega_2,\mu_1,\mu_2,u,k,K$ and v be as in Theorem 1.1. Let $p\in\mathbb{R}$ be such that $p\neq0$, $f:\Omega_2\to\mathbb{R}$ be a non-negative measurable function (positive for $p<0$), A_kf be defined by (1.2) and*

$$M_{p,k}f(x,y)=f^p(y)-A_k^pf(x)-|p|\cdot|A_kf(x)|^{p-1}(f(y)-A_kf(x))\tag{2.5}$$

for $x\in\Omega_1$, $y\in\Omega_2$. If $p\geqslant1$ or $p<0$, then the inequality

$$\begin{aligned}&\left(\int_{\Omega_2}v(y)f^p(y)d\mu_2(y)\right)^{\frac{q}{p}}-\int_{\Omega_1}u(x)A_k^qf(x)d\mu_1(x)\\ &\geqslant\frac{q}{p}\left|\int_{\Omega_1}\frac{u(x)}{K(x)}(A_kf(x))^{q-p}\int_{\Omega_2}sgn(f(y)-A_kf(x))k(x,y)M_{p,k}f(x,y)\,d\mu_2(y)\,d\mu_1(x)\right|\end{aligned}\tag{2.6}$$

holds. If $p \in (0,1)$ *relation* (2.6) *holds with*

$$\int_{\Omega_1} u(x) A_k^q f(x) d\mu_1(x) - \left(\int_{\Omega_2} v(y) f^p(y) d\mu_2(y) \right)^{\frac{q}{p}}$$

on its left hand-side.

For the monotone convex function $\Phi : \mathbb{R}_+ \to \mathbb{R}$ defined by $\Phi(x) = e^x$, $x \in \mathbb{R}_+$ the following result follows.

COROLLARY 2.4. *Let* $\Omega_1, \Omega_2, \mu_1, \mu_2, u, k, K$ *and* v *be defined as in Theorem* 1.1 *and let* $p > 0$. *Let* $G_k f(x)$ *be defined by*

$$G_k f(x) := \exp\left(\frac{1}{K(x)} \int_{\Omega_2} k(x,y) \ln f(y) d\mu_2(y) \right), \tag{2.7}$$

$$P_{p,k} f(x,y) = f^p(y) - G_k^p f(x) - p|G_k^p(x)| \ln \frac{f(y)}{G_k f(x)} \tag{2.8}$$

and $f : \Omega_2 \to \mathbb{R}$ *be a positive measurable function,* $f(y) > G_k f(x)$ *for* $y \in \Omega_2{}'$ $(\Omega_2{}' \subset \Omega_2)$. *Then the following inequality holds:*

$$\left(\int_{\Omega_2} v(y) f^p(y) d\mu_2(y) \right)^{\frac{q}{p}} - \int_{\Omega_1} u(x) G_k^q f(x) d\mu_1(x)$$
$$\geqslant \frac{q}{p} \left| \int_{\Omega_1} \frac{u(x)}{K(x)} G_k^{q-p} f(x) \int_{\Omega_2} sgn(f(y) - G_k f(x)) k(x,y) P_{p,k} f(x,y) d\mu_2(y)\, d\mu_1(x) \right|. \tag{2.9}$$

Proof. Apply (2.1) with $\Phi : \mathbb{R} \to \mathbb{R}$, $\Phi(x) = e^x$, and replace the function f with $p \ln f$. Note that $G_k f = \exp(A_k(\ln f))$. $\square$

Here we give the results for one dimensional settings, with intervals in $\mathbb{R}$ and Lebesgue measures. Also we give the related dual results.

THEOREM 2.5. *Let* $0 < b \leqslant \infty$ *and* $k : (0,b) \times (0,b) \to \mathbb{R}$ *be a non-negative measurable function, such that*

$$K(x) =: \int_0^x k(x,y) dy, \quad x \in (0,b). \tag{2.10}$$

Let u *be a weight function such that the function* $x \mapsto \frac{u(x)}{x} \cdot \left(\frac{k(x,y)}{K(x)} \right)^{\frac{q}{p}}$ *is integrable on* (y,b) *for each fixed* $y \in (0,b)$ *and let the function* $w : (0,b) \to \mathbb{R}$ *be defined by*

$$w(y) = y\left(\int_y^b \left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}} u(x)\frac{dx}{x}\right)^{\frac{p}{q}}. \tag{2.11}$$

If Φ is a non-negative monotone convex on the interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is that $\varphi(x) \in \partial\Phi(x)$ for all $x \in IntI$, then the following inequality

$$\begin{aligned}&\left(\int_0^b w(y)\Phi\left(f(y)\right)\frac{dy}{y}\right)^{\frac{q}{p}} - \int_0^b u(x)\Phi^{\frac{q}{p}}\left(A_kf(x)\right)\frac{dx}{x}\\ &\geqslant \frac{q}{p}\left|\int_0^b \frac{u(x)}{K(x)}\Phi^{\frac{q}{p}-1}\left(A_kf(x)\right)\int_0^x sgn(f(y)-A_kf(x))k(x,y)\Big[\Phi(f(y)-\Phi(A_kf(x))\right.\\ &\left.\qquad -|\varphi(A_kf(x))|.(f(y)-A_kf(x))\Big]dy\frac{dx}{x}\right|\end{aligned} \tag{2.12}$$

holds for all measurable functions $f : (0,b) \to \mathbb{R}$, $f(y) > A_kf(x)$ for $y \in I'$ $(I' \subset (0,b))$, such that $f(y) \in I$, for all fixed $y \in (0,b)$ where A_kf is defined by

$$A_kf(x) := \frac{1}{K(x)}\int_0^x k(x,y)f(y)dy, \quad x \in (0,b). \tag{2.13}$$

Proof. Denote $D_1 = \{(x,y) \in \mathbb{R}_+^2 : 0 < y \leqslant x < b\}$ and set $\Omega_1 = \Omega_2 = (0,b)$. Replace $d\mu_1(x)$, $d\mu_2(y)$, $u(x)$ and $k(x,y)$ by dx, dy, $\frac{u(x)}{x}$ and $k(x,y)\chi_{D_1}(x,y)$ respectively in (2.1) to get (2.12). Moreover, $w(y) = yv(y)$, $y \in (0,b)$. □

By considering the power and exponential functions, we can give the following results.

COROLLARY 2.6. *Let $0 < b \leqslant \infty$, u,k,K and w be defined in Theorem 2.5. Let $p \in \mathbb{R}$, $p \neq 0$, f be a non-negative measurable function on $(0,b)$, $f(y) > A_kf(x)$ for $y \in I'$ $(I' \subset (0,b))$, where A_kf and $M_{p,k}$ be defined by (2.13) and (2.5) respectively. If $p > 1$ or $p < 0$, then the following inequality holds:*

$$\begin{aligned}&\left(\int_0^b w(y)f^p(y)\frac{dy}{y}\right)^{\frac{q}{p}} - \int_0^b u(x)\left(A_kf(x)\right)^q\frac{dx}{x}\\ &\geqslant \frac{q}{p}\left|\int_0^b \frac{u(x)}{K(x)}\left(A_kf(x)\right)^{q-p}\int_0^x sgn(f(y)-A_kf)k(x,y)M_{p,k}f(x,y)\,dy\frac{dx}{x}\right|.\end{aligned} \tag{2.14}$$

If $p \in (0,1)$, then the order of terms on the left-hand side of relation (2.14) is reversed.

COROLLARY 2.7. *Let $0 < b \leqslant \infty$ u,k,K and w be defined in Theorem 2.5 and $P_{p,k}$ by (2.8). Let $p > 1$ and f be positive measurable function on $(0,b)$, $f(y) > G_k f(x)$ for $y \in I'$ $(I' \subset (0,b))$. Then the following inequality holds:*

$$\left(\int_0^b w(y) f^p(y) \frac{dy}{y}\right)^{\frac{q}{p}} - \int_0^b u(x) G_k^q f(x) \frac{dx}{x}$$
$$\geqslant \frac{q}{p} \left| \int_0^b \frac{u(x)}{K(x)} G_k^{q-p} f(x) \int_0^x sgn(f(y) - G_k f(x)) k(x,y) P_{p,k} f(x,y) dy \frac{dx}{x} \right|, \tag{2.15}$$

where $G_k f(x)$ is defined by

$$G_k f(x) := \exp\left(\frac{1}{K(x)} \int_0^b k(x,y) \ln f(y) d\mu_2(y)\right), \quad x \in (0,b). \tag{2.16}$$

The above results can be applied to some important particular kernels. Here we consider the result related to the Riemman-Liouville operator

$$R_\gamma f(x) = \frac{\gamma}{x^\gamma} \int_0^x (x-y)^{\gamma-1} f(y) dy, \tag{2.17}$$

where $\gamma \in \mathbb{R}_+$. Obviously, for $\gamma = 1$, in (2.17), we have $R_1 = H$, which is classical Hardy's integral operator

$$Hf(x) = \frac{1}{x} \int_0^x f(y) dy, \quad x \in (0,b).$$

EXAMPLE 2.8. Suppose $0 < b \leqslant \infty$, $\gamma \in \mathbb{R}_+$, $f(y) > R_\gamma f$ for $y \in I'$ $(I' \subset (0,b))$ and D_1 is defined in the proof of Theorem 2.5. If $u(x) = \frac{1}{x}$, we get $v(y) = \left(\frac{\gamma}{y}\right) B^{\frac{p}{q}}(y)$, where $B(y) = B\left(1 - \frac{y}{b}; 1 + \frac{(\gamma-1)q}{p}, \frac{q}{p}\right)$, for $\frac{(\gamma-1)q}{p} < 1$, $k(x,y) = \frac{\gamma}{x^\gamma}(x-y)^{\gamma-1}\chi_{D_1}(x,y)$ and $R_\gamma f$ as in (2.17), then (2.12) becomes

$$\left(\gamma \int_0^b B^{\frac{p}{q}}(y) \Phi(f(y)) \frac{dy}{y}\right)^{\frac{q}{p}} - \int_0^b \Phi^{\frac{q}{p}}\left(R_\gamma f(x)\right) \frac{dx}{x}$$
$$\geqslant \frac{\gamma q}{p} \left| \int_0^b \int_0^x sgn(f(y) - R_\gamma f(x)) \Phi\left(R_\gamma f(x)\right)^{\frac{q}{p}-1} (x-y)^{\gamma-1} \Big[\Phi(f(y) - \Phi(R_\gamma f(x)) \right.$$
$$\left. - |\varphi(R_\gamma f(x))|.(f(y) - R_\gamma f(x))\Big] dy \frac{dx}{x^{\gamma+2}} \right|, \tag{2.18}$$

where $B(y)$ is incomplete Beta function defined in (1.5).

Now we give the dual results to Theorem 2.5 with some corollaries.

THEOREM 2.9. *Let $0 < b \leqslant \infty$ and let $k : (b,\infty) \times (b,\infty) \to \mathbb{R}$ be a non-negative measurable function and $\widetilde{K}(x)$ be defined by*

$$\widetilde{K}(x) := \int_x^\infty k(x,y)dy, \quad x \in (b,\infty). \tag{2.19}$$

Let u be a weight function such that the function $x \mapsto \frac{u(x)}{x} \cdot \left(\frac{k(x,y)}{\widetilde{K}(x)}\right)^{\frac{q}{p}}$ is integrable on (b,y) for each fixed $y \in (b,\infty)$ and let the function $\widetilde{w} : (b,\infty) \to \mathbb{R}$ be defined by

$$\widetilde{w}(y) = y \left(\int_b^y \left(\frac{k(x,y)}{\widetilde{K}(x)}\right)^{\frac{q}{p}} u(x) \frac{dx}{x} \right)^{\frac{p}{q}}. \tag{2.20}$$

If Φ is a non-negative monotone convex on the interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is that $\varphi(x) \in \partial\Phi(x)$ for all $x \in Int I$, then the following inequality

$$\begin{aligned} &\left(\int_b^\infty \widetilde{w}(y)\Phi\left(f(y)\right) \frac{dy}{y} \right)^{\frac{q}{p}} - \int_b^\infty u(x)\Phi^{\frac{q}{p}}\left(\widetilde{A}_k f(x)\right) \frac{dx}{x} \\ &\geqslant \frac{q}{p} \left| \int_b^\infty \frac{u(x)}{\widetilde{K}(x)} \Phi^{\frac{q}{p}-1}\left(\widetilde{A}_k f(x)\right) \int_x^\infty sgn(f(y) - \widetilde{A}_k f) k(x,y) \left[\Phi(f(y)) - \Phi(\widetilde{A}_k f(x)) \right.\right. \\ &\qquad \left.\left. - |\varphi(\widetilde{A}_k f(x))|.(f(y) - \widetilde{A}_k f(x)) \right] dy \frac{dx}{x} \right| \end{aligned} \tag{2.21}$$

holds for all measurable functions $f : (b,\infty) \to \mathbb{R}$, such that $f(y) \in I$, for all fixed $y \in (b,\infty)$, $f(y) > \widetilde{A}_k f$ for $y \in I''$ $(I'' \subset (b,\infty))$, where $\widetilde{A}_k f$ is defined by

$$\widetilde{A}_k f(x) := \frac{1}{\widetilde{K}(x)} \int_x^\infty k(x,y) f(y) dy, \quad x \in (b,\infty). \tag{2.22}$$

Proof. Denote $D_2 = \{(x,y) \in \mathbb{R}^2_+ : b < x \leqslant y < \infty\}$ and set $\Omega_1 = \Omega_2 = (b,\infty)$. Replace $d\mu_1(x)$, $d\mu_2(y)$, $u(x)$ and $k(x,y)$ by dx, dy, $\frac{u(x)}{x}$ and $k(x,y)\chi_{D_2}(x,y)$. respectively in (2.1) to get (2.21) . Also Moreover, $\widetilde{w}(y) = yv(y)$, $y \in (b,\infty)$. □

COROLLARY 2.10. *Let $0 < b \leqslant \infty$ $u, k, \widetilde{K}$ and $\widetilde{w}$ be defined in Theorem 2.9. Let $p > 1$, f be a non-negative measurable function on (b,∞), $f(y) > \widetilde{A}_k f$ for $y \in I''$ $(I'' \subset (b,\infty))$, where $\widetilde{A}_k f$ be defined by (2.22). Then the following inequality holds:*

$$\left(\int_b^\infty \widetilde{w}(y) f^p(y) \frac{dy}{y}\right)^{\frac{q}{p}} - \int_b^\infty u(x) \left(\widetilde{A}_k f(x)\right)^q \frac{dx}{x}$$
$$\geqslant \frac{q}{p} \left| \int_b^\infty \frac{u(x)}{\widetilde{K}(x)} \left(\widetilde{A}_k f(x)\right)^{q-p} \int_x^\infty sgn(f(y) - \widetilde{A}_k f) k(x,y) \left[f^p(y) - (\widetilde{A}_k f(x))^p \right.\right.$$
$$\left.\left. - p|\widetilde{A}_k f(x)|^{p-1}.(f(y) - \widetilde{A}_k f(x)) \right] dy \frac{dx}{x} \right|. \tag{2.23}$$

COROLLARY 2.11. *Let* $0 < b \leqslant \infty$ $u, k, \widetilde{K}$ *and* $\widetilde{w}$ *be defined in Theorem* 2.9. *Let* f *be a positive measurable function on* (b, ∞), $f(y) > \widetilde{G}_k f$ *for* $y \in I''$ $(I'' \subset (b, \infty))$. *Then the following inequality holds:*

$$\left(\int_b^\infty \widetilde{w}(y) f^p(y) \frac{dy}{y}\right)^{\frac{q}{p}} - \int_b^\infty u(x) \widetilde{G}_k^q f(x) \frac{dx}{x}$$
$$\geqslant \frac{q}{p} \left| \int_b^\infty \frac{u(x)}{\widetilde{K}(x)} \widetilde{G}_k^{q-p} f(x) \int_x^\infty sgn(f(y) - \widetilde{G}_k f(x)) k(x,y) \left[f^p(y) - \widetilde{G}_k^p f(x) \right.\right.$$
$$\left.\left. - p|\widetilde{G}_k^p f(x)|.(\ln f(y) - \ln \widetilde{G}_k f(x)) \right] dy \frac{dx}{x} \right|, \tag{2.24}$$

where $\widetilde{G}_k f(x)$ *is defined by*

$$\widetilde{G}_k f(x) := \exp\left(\frac{1}{\widetilde{K}(x)} \int_x^\infty k(x,y) \ln f(y) dy \right) \quad x \in (b, \infty). \tag{2.25}$$

Now we give Hardy-type inequality for Weyl's fractional integral operator.

$$W_\gamma f(x) = \gamma x \int_x^\infty (y-x)^{\gamma-1} \frac{f(y)}{y^{\gamma+1}} dy, \tag{2.26}$$

where $\gamma \in \mathbb{R}_+$. It is quite clear that $W_1 = \widetilde{H}$ for $\gamma = 1$ we get the classical dual Hardy's operator.

EXAMPLE 2.12. Suppose $0 < b \leqslant \infty$, $\gamma \in \mathbb{R}_+$, $f(y) > W_\gamma f$ for $y \in I''$ $(I'' \subset (b, \infty))$ and D_2 is defined in the proof of Theorem 2.9. If $u(x) = \frac{1}{x}$, we get $v(y) = \left(\frac{\gamma}{y}\right) B^{\frac{p}{q}}(y)$, where $B(y) = B\left(1 - \frac{b}{y}; 1 + \frac{(\gamma-1)q}{p}, \frac{q}{p}\right)$ for $\frac{(\gamma-1)q}{p} < 1$, $k(x,y) = \frac{\gamma x}{y^{\gamma+1}}(y - x)^{\gamma-1} \chi_{D_2}(x,y)$, where $W_\gamma f$ as in (2.26), then (2.21) becomes

$$\left(\gamma\int_b^\infty B^{\frac{p}{q}}(y)\Phi\left(f(y)\right)\frac{dy}{y}\right)^{\frac{q}{p}}-\int_b^\infty \Phi^{\frac{q}{p}}\left(W_\gamma f(x)\right)\frac{dx}{x}$$
$$\geqslant \frac{\gamma q}{p}\left|\int_b^\infty \Phi^{\frac{q}{p}-1}\left(W_\gamma f(x)\right)\int_x^\infty sgn(f(y)-W_\gamma f)(y-x)^{\gamma-1}\Big[\Phi(f(y)-\Phi(W_\gamma f(x))\right.$$
$$\left.-|\varphi(W_\gamma f)|.(f(y)-W_\gamma f(x))\Big]\frac{dy}{y^{\gamma+1}}\frac{dx}{x}\right|. \tag{2.27}$$

Observe that $B(y)$ is incomplete Beta function defined in (1.5).

REMARK 2.13. Set $k(x,y)=\frac{1}{y^2}$, $x,y\in(b,\infty)$ in (2.22), and denote

$$\widetilde{H}f(x)=x\int_x^\infty f(y)\frac{dy}{y^2},\quad x\in(b,\infty),$$

and if $u(x)\equiv 1$, then the inequality given in (2.21) becomes

$$\frac{p}{q}\left(\int_b^\infty\left(y^{\frac{q}{p}}-b^{\frac{q}{p}}\right)\Phi\left(f(y)\right)\frac{dy}{y^2}\right)^{\frac{q}{p}}-\int_b^\infty\Phi^{\frac{q}{p}}\left(\widetilde{H}f(x)\right)\frac{dx}{x}$$
$$\geqslant\frac{q}{p}\left|\int_b^\infty\Phi^{\frac{q}{p}-1}\left(\widetilde{H}f(x)\right)\int_x^\infty sgn(f(y)-\widetilde{H}f(x))\Big[\Phi(f(y))-\Phi(\widetilde{H}f(x))\right.$$
$$\left.-|\varphi(\widetilde{H}f(x))|.(f(y)-\widetilde{H}f(x))\Big]\frac{dy}{y^2}dx\right|. \tag{2.28}$$

3. G. H. Hardy-type inequalities for fractional integrals

We continue our analysis about improvements by taking the non-negative difference of the left-hand side and the right-hand side of the inequality given in (2.1) by taking $\Phi:\mathbb{R}_+\to\mathbb{R}_+$, $\Phi(x)=x^s, s\geqslant 1$ as:

$$\pi(s)=\left(\int_{\Omega_2}v(y)f^s(y)d\mu_2(y)\right)^{\frac{q}{p}}-\int_{\Omega_1}u(x)\left(A_kf(x)\right)^{\frac{sq}{p}}d\mu_1(x)$$
$$-\frac{q}{p}\left|\int_{\Omega_1}\frac{u(x)}{K(x)}\Phi^{\frac{q}{p}-1}(A_kf(x))\int_{\Omega_2}sgn(f(y)-A_kf(x))k(x,y)\Big[f^s(y)-(A_kf(x))^s\right.$$
$$\left.-s|A_kf(x)|^{s-1}\cdot(f(y)-A_kf(x))\Big]d\mu_2(y)\,d\mu_1(x)\right|. \tag{3.1}$$

We continue with definitions and some properties of the *fractional integrals of a function f with respect to given function g*. For details see e.g. [13, p. 99]:

Let (a,b), $-\infty \leqslant a < b \leqslant \infty$ be a finite or infinite interval of the real line $\mathbb{R}$ and $\alpha > 0$. Also let g be an increasing function on $(a,b]$ and g' be a continuous function on (a,b). The left- and right-sided fractional integrals of a function f with respect to another function g in $[a,b]$ are given by

$$(I^{\alpha}_{a+;g}f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t)f(t)dt}{[g(x)-g(t)]^{1-\alpha}}, \quad x > a$$

and

$$(I^{\alpha}_{b-;g}f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t)f(t)dt}{[g(t)-g(x)]^{1-\alpha}}, \quad x < b$$

respectively.

Here we give the general result for the fractional integrals of a function f with respect to given function g and then we extract some special cases for the Riemman-Liouville fractional integrals and Hadamard-type fractional integrals.

Our first result involve fractional integral of f with respect to another increasing function g.

THEOREM 3.1. *Let $0 < p \leqslant q < \infty$, $s \geqslant 1$, $f \geqslant 0$, $\alpha > 1 - \frac{p}{q}$, g be increasing function on $(a,b]$ such that g' be continuous on (a,b), $f(y) > \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g}f(x)$ for $y \in I$ $(I \subset (a,b))$. Then the following inequality holds true:*

$$\pi_1(s) \leqslant \overline{H}_1(s) - B_1(s) \leqslant \overline{H}_1(s),$$

where

$$\begin{aligned}\pi_1(s) = {} & \frac{\alpha^{\frac{q}{p}}}{((\alpha-1)\frac{q}{p}+1)} \left(\int_a^b g'(y)(g(b)-g(y))^{\alpha-1+\frac{p}{q}} f^s(y)dy \right)^{\frac{q}{p}} \\ & -(\Gamma(\alpha+1))^{\frac{sq}{p}} \int_a^b g'(x)(g(x)-g(a))^{\frac{\alpha q}{p}(1-s)} \left(I^{\alpha}_{a+;g}f(x)\right)^{\frac{sq}{p}} dx - B_1(s),\end{aligned}$$

$$\begin{aligned}B_1(s) = {} & \frac{\alpha q(\Gamma(\alpha+1))^{s(\frac{q}{p}-1)}}{p} \left| \int_a^b \int_a^x sgn\left(f(y) - \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g}f(x) \right) g'(x)g'(y) \right. \\ & \times \frac{(g(x)-g(a))^{\frac{\alpha(q-p)(1-s)}{p}}}{(g(x)-g(y))^{1-\alpha}} \left(I^{\alpha}_{a+;g}f(x)\right)^{s(\frac{q}{p}-1)} \left[f^s(y) - \left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g}f(x) \right)^s \right. \\ & \left.\left. -s \left| \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g}f(x) \right|^{s-1} \left(f(y) - \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g}f(x) \right) \right] dydx \right|,\end{aligned}$$

and

$$\overline{H}_1(s) = (g(b)-g(a))^{\frac{\alpha q}{p}(1-s)} \left[\frac{\alpha^{\frac{q}{p}} (g(b)-g(a))^{\frac{q(\alpha s-1)+p}{p}}}{(\alpha-1)\frac{q}{p}+1} \left(\int_a^b g'(y) f^s(y) dy \right)^{\frac{q}{p}} - (\Gamma(\alpha+1))^{\frac{sq}{p}} \int_a^b g'(x) (I^{\alpha}_{a+;g} f(x))^{\frac{sq}{p}} dx \right].$$

Proof. Applying Theorem 2.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{g'(y)}{\Gamma(\alpha)(g(x)-g(y))^{1-\alpha}}, & a \leqslant y \leqslant x; \\ 0, & x < y \leqslant b, \end{cases}$$

we get that $K(x) = \frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^{\alpha}$ and $A_k f(x) = \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x)$.

For particular weight function $u(x) = g'(x)(g(x)-g(a))^{\frac{\alpha q}{p}}$, $x \in (a,b)$, we get $v(y) = (\alpha g'(y)(g(b)-g(y))^{\alpha-1+\frac{p}{q}})/(((\alpha-1)\frac{q}{p}+1)^{\frac{p}{q}})$, then (2.1) takes the form

$$\pi_1(s) = \frac{\alpha^{\frac{q}{p}}}{(\alpha-1)\frac{q}{p}+1} \left(\int_a^b g'(y)(g(b)-g(y))^{\alpha-1+\frac{p}{q}} f^s(y) dy \right)^{\frac{q}{p}} - (\Gamma(\alpha+1))^{\frac{sq}{p}} \int_a^b g'(x)(g(x)-g(a))^{\frac{\alpha q}{p}(1-s)} \left(I^{\alpha}_{a+;g} f(x) \right)^{\frac{sq}{p}} dx - B_1(s).$$

Since $\frac{\alpha q}{p}(1-s) \leqslant 0$, g is increasing and $B_1(s) \geqslant 0$, we obtain that

$$\begin{aligned} 0 \leqslant \rho_1(s) &\leqslant \frac{\alpha^{\frac{q}{p}} (g(b)-g(a))^{(\alpha-1)\frac{q}{p}+1}}{(\alpha-1)\frac{q}{p}+1} \left(\int_a^b g'(y) f^s(y) dy \right)^{\frac{q}{p}} \\ &\quad - (g(b)-g(a))^{\frac{\alpha q}{p}(1-s)} (\Gamma(\alpha+1))^{\frac{sq}{p}} \int_a^b g'(x) (I^{\alpha}_{a+;g} f(x))^{\frac{sq}{p}} dx - B_1(s) \\ &= \overline{H}_1(s) - M_1(s) \\ &\leqslant \overline{H}_1(s). \end{aligned}$$

This completes the proof. □

If $g(x) = x$, then $I^{\alpha}_{a+;x} f(x)$ reduces to $I^{\alpha}_{a+} f(x)$ left-sided Riemann–Liouville fractional integral and the following result follows.

COROLLARY 3.2. *Let $0 < p \leqslant q < \infty$, $\alpha > 1 - \frac{p}{q}$, $s \geqslant 1$, $f \geqslant 0$, $f(y) > \frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} I^{\alpha}_{a+} f(x)$, for $y \in I$ $(I \subset (a,b))$. Then the following inequality holds true:*

$$\pi_2(s) \leqslant \overline{H}_2(s) - B_2(s) \leqslant \overline{H}_2(s),$$

where

$$\pi_2(s) = \frac{\alpha^{\frac{q}{p}}}{(\alpha-1)\frac{q}{p}+1}\left(\int_a^b (b-y)^{\alpha-1+\frac{p}{q}} f^s(y)dy\right)^{\frac{q}{p}}$$
$$-(\Gamma(\alpha+1))^{\frac{sq}{p}}\int_a^b (x-a)^{\frac{\alpha q}{p}(1-s)}\left(I^\alpha_{a^+}f(x)\right)^{\frac{sq}{p}}dx - B_2(s),$$

$$B_2(s) = \frac{\alpha q(\Gamma(\alpha+1))^{s(\frac{q}{p}-1)}}{p}\left|\int_a^b\int_a^x sgn\left(f(y)-\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}I^\alpha_{a^+}f(x)\right)(x-a)^{\frac{\alpha(q-p)(1-s)}{p}}\right.$$
$$\times\left(I^\alpha_{a^+}f(x)\right)^{s(\frac{q}{p}-1)}(x-y)^{\alpha-1}\left[f^s(y)-\left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}I^\alpha_{a^+}f(x)\right)^s\right.$$
$$\left.\left.-s\left|\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}I^\alpha_{a_+}f(x)\right|^{s-1}\left(f(y)-\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}I^\alpha_{a^+}f(x)\right)\right]dy\,dx\right|$$

and

$$\overline{H}_2(s) = (b-a)^{\frac{\alpha q}{p}(1-s)}\left[\frac{\alpha^{\frac{q}{p}}(b-a)^{\frac{q(\alpha s-1)+p}{p}}}{(\alpha-1)\frac{q}{p}+1}\left(\int_a^b f^s(y)dy\right)^{\frac{q}{p}}\right.$$
$$\left.-(\Gamma(\alpha+1))^{\frac{sq}{p}}\int_a^b (I^\alpha_{a^+}f(x))^{\frac{sq}{p}}dx\right].$$

REMARK 3.3. If we take $g(x) = \log x$, then $I^\alpha_{a+;x}f(x)$ reduces to $J^\alpha_{a_+}f(x)$ left-sided Hadamard-type fractional integral that is defined for $\alpha > 0$ by

$$(J^\alpha_{a_+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x\left(\log\frac{x}{y}\right)^{\alpha-1}\frac{f(y)dy}{y}, \quad x > a$$

and we obtain the following inequality:

$$\pi_3(s) \leqslant H_3(s) - B_3(s) \leqslant H_3(s),$$

where

$$\pi_3(s) = \frac{\alpha^{\frac{q}{p}}}{(\alpha-1)\frac{q}{p}+1}\left(\int_a^b(\log b-\log y)^{\alpha-1+\frac{p}{q}}f^s(y)\frac{dy}{y}\right)^{\frac{q}{p}}$$
$$-(\Gamma(\alpha+1))^{\frac{sq}{p}}\int_a^b(\log x-\log a)^{\frac{\alpha q}{p}(1-s)}\left(J^\alpha_{a_+}f(x)\right)^{\frac{sq}{p}}\frac{dx}{x} - B_3(s),$$

$$B_3(s) = \frac{\alpha q(\Gamma(\alpha+1))^{s(\frac{q}{p}-1)}}{p}\left|\int_a^b\int_a^x sgn\left(f(y)-\frac{\Gamma(\alpha+1)}{(\log x-\log a)^{\alpha}}J^{\alpha}_{a_+}f(x)\right)\right.$$
$$\times\left(J^{\alpha}_{a_+}f(x)\right)^{s(\frac{q}{p}-1)}\frac{(\log x-\log a)^{\frac{\alpha(q-p)(1-s)}{p}}}{(\log x-\log y)^{1-\alpha}}\left[f^s(y)-\left(\frac{\Gamma(\alpha+1)}{(\log x-\log a)^{\alpha}}J^{\alpha}_{a_+}f(x)\right)^s\right.$$
$$\left.\left.-s\left|\frac{\Gamma(\alpha+1)}{(\log x-\log a)^{\alpha}}J^{\alpha}_{a_+}f(x)\right|^{s-1}\left(f(y)-\frac{\Gamma(\alpha+1)}{(\log x-\log a)^{\alpha}}J^{\alpha}_{a_+}f(x)\right)\right]\frac{dy}{y}\frac{dx}{x}\right|$$

and

$$\overline{H}_3(s) = (\log b-\log a)^{\frac{\alpha q}{p}(1-s)}\left[\frac{\alpha^{\frac{q}{p}}(\log b-\log a)^{\frac{q(\alpha s-1)+p}{p}}}{(\alpha-1)\frac{q}{p}+1}\left(\int_a^b f^s(y)\frac{dy}{y}\right)^{\frac{q}{p}}\right.$$
$$\left.-(\Gamma(\alpha+1))^{\frac{sq}{p}}\int_a^b\left(J^{\alpha}_{a_+}f(x)\right)^{\frac{sq}{p}}\frac{dx}{x}\right].$$

Now we present definitions and some properties of the *Erdélyi-Kober type fractional integrals*. Some of these definitions and results were presented in Samko et al. in [18].

Let (a,b), $(0\leqslant a<b\leqslant\infty)$ be a finite or infinite interval of the half-axis $\mathbb{R}^+$. Also let $\alpha>0$, $\sigma>0$, and $\eta\in\mathbb{R}$. The left-sided Erdélyi–Kober type fractional integrals of order $\alpha\in\mathbb{R}$ are defined by

$$(I^{\alpha}_{a_+;\sigma;\eta}f)(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\int_a^x\frac{t^{\sigma\eta+\sigma-1}f(t)dt}{(x^{\sigma}-t^{\sigma})^{1-\alpha}},\quad x>a$$

Now, we give the following result.

THEOREM 3.4. *Let* $0<p\leqslant q<\infty$, $s\geqslant 1$, $\alpha>1-\frac{p}{q}$, $f\geqslant 0$, $f(y)>\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^{\sigma}\right)^{\alpha}{}_2F_1(x)}I^{\alpha}_{a_+;\sigma;\eta}f(x)$ *for* $y\in I$ $(I\subset(a,b))$ *and* ${}_2F_1(a,b;c;z)$ *denotes the hypergeometric function. Then the following inequality holds true:*

$$0\leqslant\pi_4(s)\leqslant\overline{H}_4(s)-B_4(s)\leqslant\overline{H}_4(s),$$

where

$$\pi_4(s) = \frac{\alpha^{\frac{q}{p}}\sigma^{\frac{p}{q}-1}}{(\alpha-1)\frac{q}{p}+1}\left(\int_a^b y^{\sigma-1}{}_2F_1(y)(b^{\sigma}-y^{\sigma})^{\alpha-1+\frac{p}{q}}f^s(y)dy\right)^{\frac{q}{p}}$$
$$-(\Gamma(\alpha+1))^{\frac{sq}{p}}\int_a^b x^{\frac{\sigma\alpha sq}{p}+\sigma-1}\left((x^{\sigma}-a^{\sigma})^{\alpha}{}_2F_1(x)\right)^{\frac{q}{p}(1-s)}\left(I^{\alpha}_{a_+;\sigma;\eta}f(x)\right)^{\frac{sq}{p}}dx-B_4(s),$$

$$B_4(s) = \frac{\alpha\sigma q(\Gamma(\alpha+1))^{s(\frac{q}{p}-1)}}{p}\left|\int_a^b\int_a^x sgn\left(f(y)-\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(x)}I^\alpha_{a+;\sigma;\eta}f(x)\right)\right.$$
$$\times x^{\sigma\alpha s(\frac{q}{p}-1)-\sigma\eta+\sigma-1}\left((x^\sigma-a^\sigma)^\alpha 2F_1(x)\right)^{(\frac{q}{p}-1)(1-s)}\left(I^\alpha_{a+;\sigma;\eta}f(x)\right)^{s(\frac{q}{p}-1)}$$
$$\times\frac{y^{\sigma\eta+\sigma-1}}{(x^\sigma-y^\sigma)^{1-\alpha}}\left[f^s(y)-\left(\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(x)}I^\alpha_{a+;\sigma;\eta}f(x)\right)^s\right.$$
$$-s\left|\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(x)}I^\alpha_{a+;\sigma;\eta}f(x)\right|^{s-1}$$
$$\left.\left.\times\left(f(y)-\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(x)}I^\alpha_{a+;\sigma;\eta}f(x)\right)\right]dy\,dx\right|$$

and

$$\overline{H}_4(s) = (b^\sigma-a^\sigma)^{\alpha\frac{q}{p}(1-s)}\left[\frac{\alpha^{\frac{q}{p}}\sigma^{\frac{q}{p}-1}b^{\sigma-1}(b^\sigma-a^\sigma)^{\frac{q(\alpha s-1)+p}{p}}}{(\alpha-1)\frac{q}{p}+1}\left(\int_a^b {}_2F_1(y)f^s(y)dy\right)^{\frac{q}{p}}\right.$$
$$\left.-a^{\sigma\alpha s\frac{q}{p}+\sigma-1}(\Gamma(\alpha+1))^{\frac{sq}{p}}\int_a^b\left({}_2F_1(x)\right)^{\frac{q}{p}(1-s)}\left(I^\alpha_{a+;\sigma;\eta}f(x)\right)^{\frac{sq}{p}}dx\right]$$

$${}_2F_1(x) = {}_2F_1\left(-\eta,\alpha;\alpha+1;1-\left(\frac{a}{x}\right)^\sigma\right)\ \textit{and}\ {}_2F_1(y) = {}_2F_1\left(\eta,\alpha;\alpha+1;1-\left(\frac{b}{y}\right)^\sigma\right).$$

Proof. Applying Theorem 2.1 with $\Omega_1=\Omega_2=(a,b)$, $d\mu_1(x)=dx$, $d\mu_2(y)=dy$,

$$k(x,y)=\begin{cases}\frac{1}{\Gamma(\alpha)}\frac{\sigma x^{-\sigma(\alpha+\eta)}}{(x^\sigma-y^\sigma)^{1-\alpha}}y^{\sigma\eta+\sigma-1}, & a\leqslant y\leqslant x;\\ 0, & x<y\leqslant b,\end{cases}$$

we get that $K(x)=\frac{1}{\Gamma(\alpha+1)}\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(-\eta,\alpha;\alpha+1;1-\left(\frac{a}{x}\right)^\sigma)$ and $A_kf(x)=\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(x)}I^\alpha_{a+;\sigma;\eta}f(x)$.

For particular weight function $u(x)=x^{\sigma-1}\left((x^\sigma-a^\sigma)^\alpha {}_2F_1(x)\right)^{\frac{q}{p}}$, $x\in(a,b)$ we get $v(y)=(\alpha\sigma^{1-\frac{p}{q}}y^{\sigma-1}{}_2F_1(y)(b^\sigma-y^\sigma)^{\alpha-1+\frac{p}{q}})/(((\alpha-1)\frac{q}{p}+1)^{\frac{p}{q}})$, then (2.1) becomes

$$\pi_4(s) = \frac{\alpha^{\frac{q}{p}}\sigma^{\frac{p}{q}-1}}{(\alpha-1)\frac{q}{p}+1}\left(\int_a^b y^{\sigma-1}\,{}_2F_1(y)(b^\sigma-y^\sigma)^{\alpha-1+\frac{p}{q}}f^s(y)dy\right)^{\frac{q}{p}}$$
$$-(\Gamma(\alpha+1))^{\frac{sq}{p}}\int_a^b x^{\frac{\sigma\alpha sq}{p}+\sigma-1}\left((x^\sigma-a^\sigma)^\alpha\,{}_2F_1(x)\right)^{\frac{q}{p}(1-s)}\left(I^\alpha_{a+;\sigma;\eta}f(x)\right)^{\frac{sq}{p}}dx - B_4(s).$$

Since $\frac{q}{p}(1-s)\leqslant 0$ and $M_4(s)\geqslant 0$, we get that

$$\begin{aligned}\pi_4(s) \leqslant{}& \frac{\alpha^{\frac{q}{p}}\sigma^{\frac{q}{p}-1}b^{(\sigma-1)\frac{q}{p}}(b^\sigma-a^\sigma)^{\frac{q}{p}(\alpha-1)+1}}{(\alpha-1)\frac{q}{p}+1}\left(\int_a^b {}_2F_1(y)f^s(y)dy\right)^{\frac{q}{p}}\\
&-a^{\frac{\sigma\alpha sq}{p}+\sigma-1}(\Gamma(\alpha+1))^{\frac{sq}{p}}(b^\sigma-a^\sigma)^{\frac{\alpha q}{p}(1-s)}\int_a^b({}_2F_1(x))^{\frac{q}{p}(1-s)}\left(I^\alpha_{a+;\sigma;\eta}f(x)\right)^{\frac{sq}{p}}dx\\
&-B_4(s)\\
={}&\overline{H}_4(s)-B_4(s)\\
\leqslant{}&\overline{H}_4(s)\end{aligned}$$

This complete the proof. □

REMARK 3.5. Similar result can be obtained for the right sided fractional integral of a function with respect to another increasing function, the right-sided Riemann-Liouville fractional integral,the right-sided Hadamard-type fractional integral and also for the right-sided Erdélyi-Kober type fractional integral, but here we omit the details.

4. G. H. Hardy-type inequalities for fractional derivatives

Here we give the improvements for different fractional derivatives.

We define Canavati-type fractional derivative (ν-fractional derivative of f), for details see [1] and [3]. We consider

$$C_a^\nu([a,b]) = \{f\in C^n([a,b]) : I^{1-\bar{\nu}}f^{(n)}\in C^1([a,b])\},$$

$\nu>0$, $n=[\nu]$, $[.]$ is the integral part, and $\bar{\nu}=\nu-n$, $0\leqslant\bar{\nu}<1$.

For $f\in C_a^\nu([a,b])$, the Canavati-ν fractional derivative of f is defined by

$$D_a^\nu f = DI_a^{1-\bar{\nu}}f^{(n)},$$

where $D=d/dx$.

LEMMA 4.1. *Let* $\nu>\gamma>0$, $n=[\nu]$, $m=[\gamma]$. *Let* $f\in C_a^\nu([a,b])$, *be such that* $f^{(i)}(a)=0$, $i=m,m+1,\dots,n-1$. *Then*

$(i) \qquad f \in C_a^\gamma([a,b])$

$(ii) \qquad (D_a^\gamma f)(x) = \frac{1}{\Gamma(\nu-\gamma)} \int_a^x (x-t)^{\nu-\gamma-1} (D_a^\nu f)(t)dt,$

for every $x \in [a,b]$.

THEOREM 4.2. *Let* $0 < p \leqslant q < \infty$, $s \geqslant 1$, $\nu - \gamma > 1 - \frac{p}{q}$, $D_a^\nu f(y) > \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_a^\gamma f(x)$ *for* $y \in I$ $(I \subset (a,b))$ *and let the assumptions in Lemma 4.1 be satisfied. Then for the non-negative functions* $D_a^\nu f$ *and* $D_a^\gamma f$ *the following inequality holds true:*

$$0 \leqslant \pi_5(s) \leqslant \overline{H}_5(s) - B_5(s) \leqslant \overline{H}_5(s),$$

where

$$\pi_5(s) = \frac{(\nu-\gamma)^{\frac{q}{p}}}{(\nu-\gamma-1)\frac{q}{p}+1} \left(\int_a^b (b-y)^{\nu-\gamma-1+\frac{p}{q}} (D_a^\nu f(y))^s dy \right)^{\frac{q}{p}}$$
$$-(\Gamma(\nu-\gamma+1))^{\frac{sq}{p}} \int_a^b (x-a)^{\frac{(\nu-\gamma)q(1-s)}{p}} (D_a^\gamma f(x))^{\frac{sq}{p}} dx - B_5(s),$$

$$B_5(s) = \frac{q(\nu-\gamma)(\Gamma(\nu-\gamma+1))^{s(\frac{q}{p}-1)}}{p} \left| \int_a^b \int_a^x sgn \left(D_a^\nu f(y) - \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_a^\gamma f(x) \right) \right.$$
$$\times (x-a)^{\frac{(\nu-\gamma)(q-p)(1-s)}{p}} (D_a^\gamma f(x))^{s(\frac{q}{p}-1)} (x-y)^{\nu-\gamma-1}$$
$$\times \left[(D_a^\nu f(y))^s - \left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^\gamma} D_a^\gamma f(x) \right)^s \right.$$
$$\left. \left. -s \left| \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_a^\gamma f(x) \right|^{s-1} \left(D_a^\nu f(y) - \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_a^\gamma f(x) \right) \right] dy dx \right|$$

and

$$\overline{H}_5(s) = (b-a)^{(\nu-\gamma)\frac{q}{p}(1-s)} \left[\frac{(\nu-\gamma)^{\frac{q}{p}} (b-a)^{\frac{q((\nu-\gamma)s-1)+p}{p}}}{(\nu-\gamma-1)\frac{q}{p}+1} \left(\int_a^b (D_a^\nu f(y))^s dy \right)^{\frac{q}{p}} \right.$$
$$\left. - (\Gamma(\nu-\gamma+1))^{\frac{sq}{p}} \int_a^b (D_a^\gamma f(x))^{\frac{sq}{p}} dx \right].$$

Proof. Applying Theorem 2.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{(x-y)^{\nu-\gamma-1}}{\Gamma(\nu-\gamma)}, & a \leqslant y \leqslant x; \\ 0, & x < y \leqslant b, \end{cases}$$

we get that $K(x)=\frac{(x-a)^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}$ and $A_k f(x)=\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D_a^\gamma f(x)$. Replace f by $D_a^\nu f$. For particular weight function $u(x)=(x-a)^{\frac{(\nu-\gamma)q}{p}}, x\in(a,b)$ we get $v(y)=((\nu-\gamma)(b-y)^{\nu-\gamma-1+\frac{p}{q}})/(((\nu-\gamma-1)\frac{q}{p}+1)^{\frac{p}{q}})$, then (2.1) takes the form

$$\begin{aligned}\pi_5(s)=&\frac{(\nu-\gamma)^{\frac{q}{p}}}{(\nu-\gamma-1)\frac{q}{p}+1}\left(\int_a^b (b-y)^{\nu-\gamma-1+\frac{p}{q}}(D_a^\nu f(y))^s dy\right)^{\frac{q}{p}}\\&-(\Gamma(\nu-\gamma+1))^{\frac{sq}{p}}\int_a^b (x-a)^{\frac{(\nu-\gamma)q(1-s)}{p}}(D_a^\gamma f(x))^{\frac{sq}{p}}dx-B_5(s),\end{aligned}$$

Since $\frac{(\nu-\gamma)q}{p}(1-s)\leqslant 0$ and $B_5(s)\geqslant 0$, we obtain that

$$\begin{aligned}\pi_5(s)\leqslant&\frac{(\nu-\gamma)^{\frac{q}{p}}(b-a)^{(\nu-\gamma-1)\frac{q}{p}+1}}{(\nu-\gamma-1)\frac{q}{p}+1}\left(\int_a^b (D_a^\nu f(y))^s dy\right)^{\frac{q}{p}}\\&-(b-a)^{\frac{(\nu-\gamma)q}{p}(1-s)}(\Gamma(\nu-\gamma+1))^{\frac{sq}{p}}\int_a^b (D_a^\gamma f(x))^{\frac{sq}{p}}dx-B_5(s)\\=&\overline{H}_5(s)-B_5(s)\\\leqslant&\overline{H}_5(s).\end{aligned}$$

This complete the proof. □

Next, we define Caputo fractional derivative, for details see [1, p. 449]. First, let $AC([a,b])$ be space of all absolutely continuous function on $[a,b]$. By $AC^n([a,b])$ we denote the space of all functions $g\in C^{n-1}([a,b])$ with $g^{(n-1)}\in AC([a,b])$.

Let $\gamma\geqslant 0$, $n=\lceil\gamma\rceil$, $g\in AC^n([a,b])$. The Caputo fractional derivative is given by

$$D_{*a}^\gamma g(t)=\frac{1}{\Gamma(n-\gamma)}\int_a^x\frac{g^{(n)}(y)}{(x-y)^{\gamma-n+1}}dy,$$

for all $x\in[a,b]$. The above function exists almost everywhere for $x\in[a,b]$.

We continue with the following lemma that is given [4].

LEMMA 4.3. *Let $\nu>\gamma\geqslant 0$, $n=[\nu]+1$, $m=[\gamma]+1$ and $f\in AC^n([a,b])$. Suppose that one of the following conditions hold:*

(a) $\nu,\gamma\notin\mathbb{N}_0$ *and* $f^i(0)=0$ *for* $i=m,\dots,n-1$.
(b) $\nu\in\mathbb{N}_0,\gamma\notin\mathbb{N}_0$ *and* $f^i(0)=0$ *for* $i=m,\dots,n-2$.
(c) $\nu\notin\mathbb{N}_0,\gamma\in\mathbb{N}_0$ *and* $f^i(0)=0$ *for* $i=m-1,\dots,n-1$.
(d) $\nu\in\mathbb{N}_0,\gamma\in\mathbb{N}_0$ *and* $f^i(0)=0$ *for* $i=m-1,\dots,n-2$.

Then

$$D^{\gamma}_{*a}f(x) = \frac{1}{\Gamma(\nu-\gamma)} \int_a^x (x-y)^{\nu-\gamma-1} D^{\nu}_{*a}f(y)dy$$

for all $a \leqslant x \leqslant b$.

THEOREM 4.4. *Let* $0 < p \leqslant q < \infty$, $s \geqslant 1$, $\nu - \gamma > 1 - \frac{p}{q}$, $D^{\nu}_{*a}f(y) > \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D^{\gamma}_{*a}f(x)$ *for* $y \in I \subset (a,b)$ *and let the assumptions in Lemma 4.3 be satisfied. Then for the non-negative functions* $D^{\nu}_{*a}f$ *and* $D^{\gamma}_{*a}f$ *the following inequality holds true:*

$$\pi_6(s) \leqslant \overline{H}_6(s) - B_6(s) \leqslant \overline{H}_6(s),$$

where

$$\pi_6(s) = \frac{(\nu-\gamma)^{\frac{q}{p}}}{(\nu-\gamma-1)\frac{q}{p}+1} \left(\int_a^b (b-y)^{\nu-\gamma-1+\frac{p}{q}} (D^{\nu}_{*a}f(y))^s dy \right)^{\frac{q}{p}}$$
$$-(\Gamma(\nu-\gamma+1))^{\frac{sq}{p}} \int_a^b (x-a)^{\frac{(\nu-\gamma)q(1-s)}{p}} \left(D^{\gamma}_{*a}f(x)\right)^{\frac{sq}{p}} dx - B_6(s),$$

$$B_6(s) = \frac{q(\nu-\gamma)(\Gamma(\nu-\gamma+1))^{s(\frac{q}{p}-1)}}{p} \left| \int_a^b \int_a^x sgn\left(D^{\nu}_{*a}f(y) - \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D^{\gamma}_{*a}f(x) \right) \right.$$
$$\times (x-a)^{\frac{(\nu-\gamma)(q-p)(1-s)}{p}} \left(D^{\gamma}_{*a}f(x)\right)^{s(\frac{q}{p}-1)} (x-y)^{\nu-\gamma-1}$$
$$\times \left[(D^{\nu}_{*a}f(y))^s - \left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\gamma}} D^{\gamma}_{*a}f(x) \right)^s \right.$$
$$\left. \left. -s \left| \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D^{\gamma}_{*a}f(x) \right|^{s-1} \left(D^{\nu}_{*a}f(y) - \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D^{\gamma}_{*a}f(x) \right) \right] dydx \right|$$

and

$$\overline{H}_6(s) = (b-a)^{(\nu-\gamma)\frac{q}{p}(1-s)} \left[\frac{(\nu-\gamma)^{\frac{q}{p}} (b-a)^{\frac{q((\nu-\gamma)s-1)+p}{p}}}{(\nu-\gamma-1)\frac{q}{p}+1} \left(\int_a^b (D^{\nu}_{*a}f(y))^s dy \right)^{\frac{q}{p}} \right.$$
$$\left. - (\Gamma(\nu-\gamma+1))^{\frac{sq}{p}} \int_a^b (D^{\gamma}_{*a}f(x))^{\frac{sq}{p}} dx \right].$$

Proof. Similar to the proof of Theorem 4.2. □

Next we give results with respect to the generalized Riemann–Liouville fractional derivative. Let us recall the definition, for details see [2].

Let $\alpha > 0$ and $n = [\alpha] + 1$ where $[\cdot]$ is the integral part and we define the *generalized Riemann-Liouville fractional derivative of* f of order α by

$$(D_a^\alpha f)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-y)^{n-\alpha-1} f(y)\, dy.$$

In addition, we stipulate

$$D_a^0 f := f =: I_a^0 f, \quad I_a^{-\alpha} f := D_a^\alpha f \quad \text{if} \quad \alpha > 0.$$

If $\alpha \in \mathbb{N}$ then $D_a^\alpha f = \frac{d^\alpha f}{dx^\alpha}$, the ordinary α-order derivative.

The space $I_a^\alpha(L(a,b))$ is defined as the set of all functions f on $[a,b]$ of the form $f = I_a^\alpha \varphi$ for some $\varphi \in L(a,b)$, [18, Chapter 1, Definition 2.3]. According to Theorem 2.3 in [18, p. 43], the latter characterization is equivalent to the condition

$$I_a^{n-\alpha} f \in AC^n[a,b], \tag{4.1}$$

$$\frac{d^j}{dx^j} I_a^{n-\alpha} f(a) = 0, \quad j = 0, 1, \ldots, n-1.$$

A function $f \in L(a,b)$ satisfying (4.1) is said to have an *integrable fractional derivative* $D_a^\alpha f$, [18, Chapter 1, Definition 2.4].

The following lemma is given in [2].

LEMMA 4.5. *Let $\beta > \alpha \geqslant 0$, $n = [\beta] + 1$, $m = [\alpha] + 1$. Identity*

$$D_a^\alpha f(x) = \frac{1}{\Gamma(\beta-\alpha)} \int_a^x (x-y)^{\beta-\alpha-1} D_a^\beta f(y)\, dy, \quad x \in [a,b]. \tag{4.2}$$

is valid if one of the following conditions holds:

(*i*) $f \in I_a^\beta(L(a,b))$.

(*ii*) $I_a^{n-\beta} f \in AC^n[a,b]$ *and* $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \ldots n$.

(*iii*) $D_a^{\beta-k} f \in C[a,b]$ *for* $k = 1, \ldots, n$, $D_a^{\beta-1} f \in AC[a,b]$ *and* $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \ldots n$.

(*iv*) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$, $\beta - \alpha \notin \mathbb{N}$, $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \ldots, n$ *and* $D_a^{\alpha-k} f(a) = 0$ *for* $k = 1, \ldots, m$.

(*v*) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$, $\beta - \alpha = l \in \mathbb{N}$, $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \ldots, l$.

(*vi*) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$ *and* $f(a) = f'(a) = \cdots = f^{(n-2)}(a) = 0$.

(*vii*) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$, $\beta \notin \mathbb{N}$ *and* $D_a^{\beta-1} f$ *is bounded in a neighborhood of* $t = a$.

THEOREM 4.6. *Let* $0 < p \leqslant q < \infty$, $s \geqslant 1$, $\beta - \alpha > 1 - \frac{p}{q}$, $D_a^\beta f(y) > \frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} D_a^\alpha f(x)$ *for* $y \in I \subset (a,b)$ *and let the assumptions in Lemma 4.5 be satisfied. Then for the non-negative functions* $D_a^\beta f$ *and* $D_a^\alpha f$ *the following inequality holds true:*

$$\pi_7(s) \leqslant \overline{H}_7(s) - B_7(s) \leqslant \overline{H}_7(s),$$

where

$$\begin{aligned}
\pi_7(s) = & \frac{(\beta-\alpha)^{\frac{q}{p}}}{(\beta-\alpha-1)\frac{q}{p}+1} \left(\int_a^b (b-y)^{\beta-\alpha-1+\frac{p}{q}} (D_a^\beta f(y))^s dy \right)^{\frac{q}{p}} \\
& - (\Gamma(\beta-\alpha+1))^{\frac{sq}{p}} \int_a^b (x-a)^{\frac{(\beta-\alpha)q(1-s)}{p}} (D_a^\alpha f(x))^{\frac{sq}{p}} dx - B_7(s),
\end{aligned}$$

$$\begin{aligned}
B_7(s) = & \frac{q(\beta-\alpha)(\Gamma(\beta-\alpha+1))^{s(\frac{q}{p}-1)}}{p} \left| \int_a^b \int_a^x sgn \left(D_a^\beta f(y) - \frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} D_a^\alpha f(x) \right) \right. \\
& \times (x-a)^{\frac{(\beta-\alpha)(q-p)(1-s)}{p}} (D_a^\alpha f(x))^{s(\frac{q}{p}-1)} (x-y)^{\beta-\alpha-1} \\
& \times \left[(D_a^\beta f(y))^s - \left(\frac{\Gamma(\beta-\alpha+1)}{(x-a)^\alpha} D_a^\alpha f(x) \right)^s \right. \\
& \left. \left. - s \left| \frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} D_a^\alpha f(x) \right|^{s-1} \left(D_a^\beta f(y) - \frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} D_a^\alpha f(x) \right) \right] dy dx \right|
\end{aligned}$$

and

$$\begin{aligned}
\overline{H}_7(s) = (b-a)^{(\beta-\alpha)\frac{q}{p}(1-s)} & \left[\frac{(\beta-\alpha)^{\frac{q}{p}} (b-a)^{\frac{q((\beta-\alpha)s-1)+p}{p}}}{(\beta-\alpha-1)\frac{q}{p}+1} \left(\int_a^b (D_a^\beta f(y))^s dy \right)^{\frac{q}{p}} \right. \\
& \left. - (\Gamma(\beta-\alpha+1))^{\frac{sq}{p}} \int_a^b (D_a^\alpha f(x))^{\frac{sq}{p}} dx \right].
\end{aligned}$$

Proof. Similar to the proof of Theorem 4.2. □

REMARK 4.7. For the case $p = q$ we can get the similar improvements of the inequality given in (2.4) for different fractional integrals and fractional derivative.

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