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AN OPIAL–TYPE INEQUALITY FOR FRACTIONAL DERIVATIVES OF TWO FUNCTIONS

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Abstract. This paper presents improvements of some Opial-type inequalities involving the Riemann-Liouville, Caputo and Canavati fractional derivatives, and presents some new Opial-type inequalities.

1. Introduction and preliminaries

The paper is motivated by the work of Agarwal, Pang and Alzer [1, 2, 3] and their study of Opial-type inequalities involving ordinary derivatives. We will present some fractional versions of known Opial-type inequalities and they will include three main types of fractional derivatives: Riemann-Liouville, Caputo and Canavati type.

First we survey some facts about fractional derivatives needed in this paper. For more details see the monographs [9, Chapter 2] and [10, Chapter 1].
By $C^n[a,b]$ we denote the space of all functions on $[a,b]$ which have continuous derivatives up to order n, and $AC[a,b]$ is the space of all absolutely continuous functions on $[a,b]$. By $AC^n[a,b]$ we denote the space of all functions $f \in C^{n-1}[a,b]$ with $f^{(n-1)} \in AC[a,b]$.

By $L_p[a,b]$, $1 \leqslant p < \infty$, we denote the space of all Lebesgue measurable functions f for which $|f^p|$ is Lebesgue integrable on $[a,b]$, and by $L_\infty[a,b]$ the set of all functions measurable and essentially bounded on $[a,b]$. Clearly, $L_\infty[a,b] \subset L_p[a,b]$ for all $p \geqslant 1$.

Let $x \in [a,b]$, $\alpha > 0$, $n = [\alpha]+1$ ($[\cdot]$ is the integral part) and Γ is the gamma function $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1}\,dt$. For $f \in L_1[a,b]$ the *Riemann-Liouville fractional integral* $J^\alpha f$ of order α is defined by

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt\,. \tag{1.1}$$

This is actually a definition for the left-sided Riemann-Liouville fractional integral. In the Remark 2 we give an explanation how to apply our results for the right-sided fractional integrals and derivatives.
For $f : [a,b] \to \mathbb{R}$ the *Riemann-Liouville fractional derivative* $D^\alpha f$ of order α is defined by

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t)\,dt = \frac{d^n}{dx^n} J^{n-\alpha} f(x)\,.$$

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In addition, we stipulate $J^0 f := f =: D^0 f$ and $J^{-\alpha} f := D^\alpha f$ if $\alpha > 0$.

Next, define n as

$$n = [\alpha] + 1, \text{ for } \alpha \notin \mathbb{N}_0; \quad n = \alpha, \text{ for } \alpha \in \mathbb{N}_0. \tag{1.2}$$

For n given by (1.2) and $f \in AC^n[a,b]$ the *Caputo fractional derivative* ${}^C D^\alpha f$ of order α is defined by

$$ {}^C D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\,dt = J^{n-\alpha} f^{(n)}(x).$$

A third fractional derivative, the *Canavati fractional derivative* ${}^{\bar{C}} D^\alpha f$ of order α, is defined for $f \in C^\alpha[a,b] = \left\{ f \in C^{n-1}[a,b] : J^{n-\alpha} f^{(n-1)} \in C^1[a,b] \right\}$ by

$$ {}^{\bar{C}} D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{n-\alpha-1} f^{(n-1)}(t)\,dt = \frac{d}{dx} J^{n-\alpha} f^{(n-1)}(x).$$

If $\alpha \in \mathbb{N}$ then $D^\alpha f = {}^C D^\alpha f = {}^{\bar{C}} D^\alpha f = f^{(\alpha)}$, the ordinary α-order derivatives.

Also, we will use composition identities for fractional derivatives, listed here for the Riemann-Liouville, Caputo and Canavati fractional derivatives, respectively:

THEOREM 1.1. [7, Theorem 4] *Let* $\alpha > \beta \geqslant 0$, $n = [\alpha] + 1$, $m = [\beta] + 1$ *and let* $f \in AC^n[a,b]$ *be such that* $D^\alpha f, D^\beta f \in L_1[a,b]$.

(*i*) *If* $\alpha - \beta \notin \mathbb{N}$ *and* f *is such that* $D^{\alpha-k} f(a) = 0$ *for* $k = 1, \ldots, n$ *and* $D^{\beta-k} f(a) = 0$ *for* $k = 1, \ldots, m$, *then*

$$D^\beta f(x) = \frac{1}{\Gamma(\alpha-\beta)} \int_a^x (x-t)^{\alpha-\beta-1} D^\alpha f(t)\,dt, \quad x \in [a,b]. \tag{1.3}$$

(*ii*) *If* $\alpha - \beta = l \in \mathbb{N}$ *and* f *is such that* $D^{\alpha-k} f(a) = 0$ *for* $k = 1, \ldots, l$, *then* (1.3) *holds.*

COROLLARY 1.2. [7, Corollary 1] *Let* $\alpha > \beta \geqslant 0$, $n = [\alpha] + 1$, $m = [\beta] + 1$. *Composition identity* (1.3) *is valid if one of the following conditions holds:*

(*i*) $f \in J^\alpha(L_1[a,b]) = \{ f : f = J^\alpha \varphi, \varphi \in L_1[a,b] \}$.

(*ii*) $J^{n-\alpha} f \in AC^n[a,b]$ *and* $D^{\alpha-k} f(a) = 0$ *for* $k = 1, \ldots n$.

(*iii*) $D^{\alpha-1} f \in AC[a,b]$, $D^{\alpha-k} f \in C[a,b]$ *and* $D^{\alpha-k} f(a) = 0$ *for* $k = 1, \ldots n$.

(*iv*) $f \in AC^n[a,b]$, $D^\alpha f, D^\beta f \in L_1[a,b]$, $\alpha - \beta \notin \mathbb{N}$, $D^{\alpha-k} f(a) = 0$ *for* $k = 1, \ldots, n$ *and* $D^{\beta-k} f(a) = 0$ *for* $k = 1, \ldots, m$.

(*v*) $f \in AC^n[a,b]$, $D^\alpha f, D^\beta f \in L_1[a,b]$, $\alpha - \beta = l \in \mathbb{N}$, $D^{\alpha-k} f(a) = 0$ *for* $k = 1, \ldots, l$.

(*vi*) $f \in AC^n[a,b]$, $D^\alpha f, D^\beta f \in L_1[a,b]$ *and* $f^{(k)}(a) = 0$ *for* $k = 0, \dots, n-2$.

(*vii*) $f \in AC^n[a,b]$, $D^\alpha f, D^\beta f \in L_1[a,b]$, $\alpha \notin \mathbb{N}$ *and* $D^{\alpha-1} f$ *is bounded in a neighborhood of* $t = a$.

THEOREM 1.3. [6, Theorem 2.1] *Let* $\alpha > \beta \geqslant 0$, n *and* m *given by* (1.2). *Let* $f \in AC^n[a,b]$ *be such that* $f^{(i)}(a) = 0$ *for* $i = m, m+1, \dots, n-1$. *Let* ${}^C D^\alpha f, {}^C D^\beta f \in L_1[a,b]$. *Then*

$$ {}^C D^\beta f(x) = \frac{1}{\Gamma(\alpha-\beta)} \int_a^x (x-t)^{\alpha-\beta-1}\, {}^C D^\alpha f(t)\, dt, \quad x \in [a,b]. \tag{1.4} $$

THEOREM 1.4. [5, Theorem 2.1] *Let* $\alpha > \beta > 0$, $n = [\alpha]+1$, $m = [\beta]+1$. *Let* $f \in C^\alpha[a,b]$ *be such that* $f^{(i)}(a) = 0$ *for* $i = m-1, m, \dots, n-2$. *Then* $f \in C^\beta[a,b]$ *and*

$$ {}^{\bar{C}} D^\beta f(x) = \frac{1}{\Gamma(\alpha-\beta)} \int_a^x (x-t)^{\alpha-\beta-1}\, {}^{\bar{C}} D^\alpha f(t)\, dt, \quad x \in [a,b]. \tag{1.5} $$

Our goal is to improve an Opial-type inequality involving fractional derivatives of two functions. For that we will need next Opial-type inequality involving ordinary derivatives that comes from [3].

TEOREM A. [3, Theorem 1] Let $p \geqslant 0$, $q > 0$, and $r > 1$ be real numbers with $r > q$, and let n and k be integers with $0 \leqslant k \leqslant n-1$. Let $\varphi > 0$ and $\omega \geqslant 0$ be measurable functions on $[a,b]$. Further, let $f, g \in AC^n[a,b]$ be such that $f^{(i)}(a) = g^{(i)}(a) = 0$ for $i = 0, \dots, n-1$, and let integrals $\int_a^b \varphi(t)|f^{(n)}(t)|^r dt$ and $\int_a^b \varphi(t)|g^{(n)}(t)|^r dt$ exist. Then we have

$$ \begin{aligned} &\int_a^b \omega(t) \left[|g^{(k)}(t)|^p |f^{(n)}(t)|^q + |f^{(k)}(t)|^p |g^{(n)}(t)|^q \right] dt \\ &\leqslant M \left(\int_a^b \varphi(t) \left[|f^{(n)}(t)|^r + |g^{(n)}(t)|^r \right] dt \right)^{\frac{p+q}{r}}, \end{aligned} $$

where

$$ M = \frac{2K}{[(n-k-1)!]^p} \left[\frac{q}{2(p+q)} \right]^{\frac{q}{r}} \left[\int_a^b [\omega(t)]^{\frac{r}{r-q}} [\varphi(t)]^{\frac{q}{q-r}} [P(t)]^{\frac{p(r-1)}{r-q}} dt \right]^{\frac{r-q}{r}}, $$

$$ P(t) = \int_a^t (t-\tau)^{\frac{r(n-k-1)}{r-1}} [\varphi(\tau)]^{\frac{1}{1-r}} d\tau $$

and

$$ K = \begin{cases} \left(1 - 2^{-\frac{p}{q}}\right)^{\frac{q}{r}}, & p \geqslant q, \\ 2^{-\frac{p}{r}}, & p \leqslant q. \end{cases} \tag{1.6} $$

It is Alzer's improvement of an Opial-type inequality involving higher-order derivatives of two functions which is due to Agarawal and Pang [2] (their monograph [1] is

an excellent survey on Opial inequalities). We will give it's fractional versions including the Riemann-Liouville, Caputo and Canavati fractional derivatives. They actually improve corresponding theorems from [4] (see Remark 1).
Also, we will give a new inequality, a counterpart of Theorem A for the case $r<0$. In the last section, we will present an application of the observed inequalities, a uniqueness of solution for a system of fractional differential equations.

2. Opial-type inequalities

Motivated by the Theorem A, we present it's Opial-type inequality which involves two functions, but with higher-order fractional derivatives. First we give a theorem involving the Riemann-Liouville fractional derivatives.

THEOREM 2.1. *Let $\alpha>\beta\geqslant 0$. Suppose that one of the conditions $(i)-(vii)$ in Corollary 1.2 holds for $\{\alpha,\beta,f\}$ and $\{\alpha,\beta,g\}$. Let $\varphi>0$ and $\omega\geqslant 0$ be measurable functions on $[a,x]$. Let $r>1$, $r>q>0$ and $p\geqslant 0$. Let $D^\alpha f, D^\alpha g\in L_r[a,b]$. Then*

$$\begin{aligned}&\int_a^x \omega(t)\Big[|D^\beta g(t)|^p\,|D^\alpha f(t)|^q+|D^\beta f(t)|^p\,|D^\alpha g(t)|^q\Big]dt\\&\leqslant K M_1\left(\int_a^x \varphi(t)\Big[|D^\alpha f(t)|^r+|D^\alpha g(t)|^r\Big]dt\right)^{\frac{p+q}{r}},\end{aligned}\tag{2.1}$$

where K is defined by (1.6) and

$$M_1=\frac{2}{[\Gamma(\alpha-\beta)]^p}\left[\frac{q}{2(p+q)}\right]^{\frac{q}{r}}\left[\int_a^x[\omega(t)]^{\frac{r}{r-q}}\,[\varphi(t)]^{\frac{q}{q-r}}\,[P_1(t)]^{\frac{p(r-1)}{r-q}}\,dt\right]^{\frac{r-q}{r}},\tag{2.2}$$

$$P_1(t)=\int_a^t (t-\tau)^{\frac{r(\alpha-\beta-1)}{r-1}}\,[\varphi(\tau)]^{\frac{1}{1-r}}\,d\tau.\tag{2.3}$$

Proof. Let $t\in[a,x]$. Using composition identity (1.3), the triangle inequality and Hölder's inequality for $\{\frac{r}{r-1},r\}$, we have

$$\begin{aligned}&|D^\beta g(t)|\\&\leqslant\frac{1}{\Gamma(\alpha-\beta)}\int_a^t(t-\tau)^{\alpha-\beta-1}\,[\varphi(\tau)]^{-\frac{1}{r}}\,[\varphi(\tau)]^{\frac{1}{r}}\,|D^\alpha g(\tau)|\,d\tau\\&\leqslant\frac{1}{\Gamma(\alpha-\beta)}\left(\int_a^t(t-\tau)^{\frac{r(\alpha-\beta-1)}{r-1}}\,[\varphi(\tau)]^{\frac{1}{1-r}}\,d\tau\right)^{\frac{r-1}{r}}\left(\int_a^t\varphi(\tau)\,|D^\alpha g(\tau)|^r\,d\tau\right)^{\frac{1}{r}}\\&=\frac{1}{\Gamma(\alpha-\beta)}\,[P_1(t)]^{\frac{r-1}{r}}\,[G(t)]^{\frac{1}{r}},\end{aligned}\tag{2.4}$$

where

$$G(t)=\int_a^t\varphi(\tau)\,|D^\alpha g(\tau)|^r\,d\tau.\tag{2.5}$$

Let

$$F(t)=\int_a^t \varphi(\tau)\,|D^{\alpha} f(\tau)|^r\,d\tau. \tag{2.6}$$

Then $F'(t)=\varphi(t)\,|D^{\alpha} f(t)|^r$, that is

$$|D^{\alpha} f(t)|^q=\left[F'(t)\right]^{\frac{q}{r}}[\varphi(t)]^{-\frac{q}{r}}. \tag{2.7}$$

Now (2.4) and (2.7) imply

$$\omega(t)\,|D^{\beta} g(t)|^p\,|D^{\alpha} f(t)|^q \leqslant h(t)\,[G(t)]^{\frac{p}{r}}\left[F'(t)\right]^{\frac{q}{r}}, \tag{2.8}$$

where

$$h(t)=\frac{1}{[\Gamma(\alpha-\beta)]^p}\,\omega(t)\,[\varphi(t)]^{-\frac{q}{r}}\,[P_1(t)]^{\frac{p(r-1)}{r}}. \tag{2.9}$$

Integrating (2.8) and applying Hölder's inequality for $\{\frac{r}{r-q},\frac{r}{q}\}$, we obtain

$$\begin{aligned}&\int_a^x \omega(t)\,|D^{\beta} g(t)|^p\,|D^{\alpha} f(t)|^q\,dt\\&\leqslant \int_a^x h(t)\,[G(t)]^{\frac{p}{r}}\left[F'(t)\right]^{\frac{q}{r}}dt\\&\leqslant \left(\int_a^x [h(t)]^{\frac{r}{r-q}}\,dt\right)^{\frac{r-q}{r}}\left(\int_a^x [G(t)]^{\frac{p}{q}}\,F'(t)\,dt\right)^{\frac{q}{r}}.\end{aligned} \tag{2.10}$$

Similarly we get

$$\begin{aligned}&\int_a^x \omega(t)\,|D^{\beta} f(t)|^p\,|D^{\alpha} g(t)|^q\,dt\\&\leqslant \left(\int_a^x [h(t)]^{\frac{r}{r-q}}\,dt\right)^{\frac{r-q}{r}}\left(\int_a^x [F(t)]^{\frac{p}{q}}\,G'(t)\,dt\right)^{\frac{q}{r}}.\end{aligned} \tag{2.11}$$

Now we need simple inequalities

$$c_{\varepsilon}(A+B)^{\varepsilon}\leqslant A^{\varepsilon}+B^{\varepsilon}\leqslant d_{\varepsilon}(A+B)^{\varepsilon},\quad (A,B\geqslant 0), \tag{2.12}$$

where

$$c_{\varepsilon}=\begin{cases}1,\ 0\leqslant\varepsilon\leqslant 1,\\ 2^{1-\varepsilon},\ \varepsilon\geqslant 1,\end{cases}\qquad d_{\varepsilon}=\begin{cases}2^{1-\varepsilon},\ 0\leqslant\varepsilon\leqslant 1,\\ 1,\ \varepsilon\geqslant 1.\end{cases}$$

Therefore, from (2.10), (2.11) and (2.12), with $r>q$, we conclude

$$\begin{aligned}&\int_a^x \omega(t)\Big[|D^{\beta} g(t)|^p\,|D^{\alpha} f(t)|^q+|D^{\beta} f(t)|^p\,|D^{\alpha} g(t)|^q\Big]\,dt\\&\leqslant \left(\int_a^x [h(t)]^{\frac{r}{r-q}}\,dt\right)^{\frac{r-q}{r}}\left[\left(\int_a^x [G(t)]^{\frac{p}{q}}\,F'(t)\,dt\right)^{\frac{q}{r}}+\left(\int_a^x [F(t)]^{\frac{p}{q}}\,G'(t)\,dt\right)^{\frac{q}{r}}\right]\\&\leqslant \left(\int_a^x [h(t)]^{\frac{r}{r-q}}\,dt\right)^{\frac{r-q}{r}}2^{1-\frac{q}{r}}\left(\int_a^x \Big[[G(t)]^{\frac{p}{q}}\,F'(t)+[F(t)]^{\frac{p}{q}}\,G'(t)\Big]\,dt\right)^{\frac{q}{r}}.\end{aligned} \tag{2.13}$$

Since $G(a) = F(a) = 0$, then with (2.12) follows

$$\begin{aligned}
&\int_a^x \left[[G(t)]^{\frac{p}{q}} F'(t) + [F(t)]^{\frac{p}{q}} G'(t) \right] dt \\
&= \int_a^x \left[[G(t)]^{\frac{p}{q}} + [F(t)]^{\frac{p}{q}} \right] \left[G'(t) + F'(t) \right] dt \\
&\qquad - \int_a^x \left[[G(t)]^{\frac{p}{q}} G'(t) + [F(t)]^{\frac{p}{q}} F'(t) \right] dt \\
&\leqslant d_{\frac{p}{q}} \int_a^x [G(t)+F(t)]^{\frac{p}{q}} [G(t)+F(t)]' dt - \frac{q}{p+q} \left[G(x)^{\frac{p}{q}+1} + F(x)^{\frac{p}{q}+1} \right] \\
&= \frac{q}{p+q} d_{\frac{p}{q}} [G(x)+F(x)]^{\frac{p}{q}+1} - \frac{q}{p+q} \left[G(x)^{\frac{p}{q}+1} + F(x)^{\frac{p}{q}+1} \right] \\
&\leqslant \frac{q}{p+q} \left(d_{\frac{p}{q}} - 2^{-\frac{p}{q}} \right) [G(x)+F(x)]^{\frac{p}{q}+1} .
\end{aligned} \tag{2.14}$$

Hence, from (2.13) and (2.14) we conclude

$$\begin{aligned}
&\int_a^x \omega(t) \left[|D^\beta g(t)|^p \, |D^\alpha f(t)|^q + |D^\beta f(t)|^p \, |D^\alpha g(t)|^q \right] dt \\
&\leqslant 2^{1-\frac{q}{r}} \left(d_{\frac{p}{q}} - 2^{-\frac{p}{q}} \right)^{\frac{q}{r}} \left(\frac{q}{p+q} \right)^{\frac{q}{r}} \left(\int_a^x [h(t)]^{\frac{r}{r-q}} \, dt \right)^{\frac{r-q}{r}} [G(x)+F(x)]^{\frac{p+q}{r}} ,
\end{aligned}$$

which is equivalent to inequality (2.1) $\square$

The following result deals with the extreme case of the preceding theorem when $r = \infty$.

THEOREM 2.2. *Let $\alpha > \beta_1, \beta_2 \geqslant 0$. Suppose that one of the conditions $(i) - (vii)$ in Corollary 1.2 holds for $\{\alpha, \beta_i, f\}$ and $\{\alpha, \beta_i, g\}$, $i = 1,2$. Let $w \geqslant 0$ be measurable function on $[a,x]$. Let $p, q_1, q_2 \geqslant 0$ and let $D^\alpha f, D^\alpha g \in L_\infty[a,b]$. Then*

$$\begin{aligned}
&\int_a^x w(t) \left[|D^{\beta_1} f(t)|^{q_1} \, |D^{\beta_2} g(t)|^{q_2} \, |D^\alpha f(t)|^p + |D^{\beta_2} f(t)|^{q_2} \, |D^{\beta_1} g(t)|^{q_1} \, |D^\alpha g(t)|^p \right] dt \\
&\qquad \leqslant M_2 \left[\|D^\alpha f\|_\infty^{2(q_1+p)} + \|D^\alpha f\|_\infty^{2q_2} + \|D^\alpha g\|_\infty^{2q_2} + \|D^\alpha g\|_\infty^{2(q_1+p)} \right],
\end{aligned} \tag{2.15}$$

where

$$M_2 = \frac{(x-a)^{q_1(\alpha-\beta_1)+q_2(\alpha-\beta_2)+1} \, \|w\|_\infty}{2 \, [\Gamma(\alpha-\beta_1+1)]^{q_1} \, [\Gamma(\alpha-\beta_2+1)]^{q_2} \, [q_1(\alpha-\beta_1)+q_2(\alpha-\beta_2)+1]} . \tag{2.16}$$

Proof. Let $t \in [a,x]$. Using identity (1.3), the triangle inequality and Hölder's inequality, for $i = 1,2$ we have

$$\begin{aligned}
|D^{\beta_i} f(t)|^{q_i} &\leqslant \frac{1}{[\Gamma(\alpha-\beta_i)]^{q_i}} \left(\int_a^t (t-\tau)^{\alpha-\beta_i-1} \, |D^\alpha f(\tau)| \, d\tau \right)^{q_i} \\
&\leqslant \frac{1}{[\Gamma(\alpha-\beta_i)]^{q_i}} \left(\int_a^t (t-\tau)^{\alpha-\beta_i-1} \, d\tau \right)^{q_i} \|D^\alpha f\|_\infty^{q_i} \\
&= \frac{(t-a)^{q_i(\alpha-\beta_i)}}{[\Gamma(\alpha-\beta_i+1)]^{q_i}} \|D^\alpha f\|_\infty^{q_i} .
\end{aligned}$$

By analogy, for $i=1,2$ we get

$$|D^{\beta_i} g(t)|^{q_i} \leqslant \frac{(t-a)^{q_i(\alpha-\beta_i)}}{[\Gamma(\alpha-\beta_i+1)]^{q_i}} \|D^\alpha g\|_\infty^{q_i}.$$

Also,

$$|D^\alpha f(t)|^p \leqslant \|D^\alpha f\|_\infty^p, \quad |D^\alpha g(t)|^p \leqslant \|D^\alpha g\|_\infty^p.$$

Hence

$$\begin{aligned} &|D^{\beta_1} f(t)|^{q_1} |D^{\beta_2} g(t)|^{q_2} |D^\alpha f(t)|^p \\ &\leqslant \frac{(t-a)^{q_1(\alpha-\beta_1)+q_2(\alpha-\beta_2)}}{[\Gamma(\alpha-\beta_1+1)]^{q_1} [\Gamma(\alpha-\beta_2+1)]^{q_2}} \|D^\alpha f\|_\infty^{q_1+p} \|D^\alpha g\|_\infty^{q_2}, \end{aligned} \tag{2.17}$$

$$\begin{aligned} &|D^{\beta_2} f(t)|^{q_2} |D^{\beta_1} g(t)|^{q_1} |D^\alpha g(t)|^p \\ &\leqslant \frac{(t-a)^{q_2(\alpha-\beta_2)+q_1(\alpha-\beta_1)}}{[\Gamma(\alpha-\beta_1+1)]^{q_1} [\Gamma(\alpha-\beta_2+1)]^{q_2}} \|D^\alpha f\|_\infty^{q_2} \|D^\alpha g\|_\infty^{q_1+p}. \end{aligned} \tag{2.18}$$

Form (2.17) and (2.18) follows

$$\begin{aligned} &\int_a^x w(t) \Big[|D^{\beta_1} f(t)|^{q_1} |D^{\beta_2} g(t)|^{q_2} |D^\alpha f(t)|^p + |D^{\beta_2} f(t)|^{q_2} |D^{\beta_1} g(t)|^{q_1} |D^\alpha g(t)|^p \Big] dt \\ &\leqslant \frac{1}{[\Gamma(\alpha-\beta_1+1)]^{q_1} [\Gamma(\alpha-\beta_2+1)]^{q_2}} \int_a^x w(t)\,(t-a)^{q_1(\alpha-\beta_1)+q_2(\alpha-\beta_2)}\,dt \\ &\quad \cdot \Big[\|D^\alpha f\|_\infty^{q_1+p} \|D^\alpha g\|_\infty^{q_2} + \|D^\alpha f\|_\infty^{q_2} \|D^\alpha g\|_\infty^{q_1+p} \Big] \\ &\leqslant \frac{\|w\|_\infty}{[\Gamma(\alpha-\beta_1+1)]^{q_1} [\Gamma(\alpha-\beta_2+1)]^{q_2}} \int_a^x (t-a)^{q_1(\alpha-\beta_1)+q_2(\alpha-\beta_2)}\,dt \\ &\quad \cdot \frac{1}{2} \Big[\|D^\alpha f\|_\infty^{2(q_1+p)} + \|D^\alpha f\|_\infty^{2q_2} + \|D^\alpha g\|_\infty^{2q_2} + \|D^\alpha g\|_\infty^{2(q_1+p)} \Big]. \end{aligned}$$

□

Now we present a counterpart of the Theorem 2.1 for the case $r<0$. Conditions on r and q allow us to apply reverse Hölder's inequalities, first with parameters $\{\frac{r}{r-1} \in (0,1), r<0\}$, then with $\{\frac{r}{r-q} \in (0,1), \frac{r}{q}<0\}$. Apart from using inequalities (2.12), we have to require similar inequalities for negative power, that is (2.25). Hence, instead of constant factor K we get L. We sketch a proof for the reader's convenience.

THEOREM 2.3. *Let $\alpha > \beta \geqslant 0$. Suppose that one of the conditions $(i)-(vii)$ in Corollary 1.2 holds for $\{\alpha,\beta,f\}$ and $\{\alpha,\beta,g\}$. Let $\varphi>0$ and $\omega \geqslant 0$ be measurable functions on $[a,x]$. Let $r<0$, $q>0$ and $p \geqslant 0$. Let $D^\alpha f, D^\alpha g \in L_r[a,b]$, each of which is of fixed sign a.e. on $[a,b]$, with $1/D^\alpha f, 1/D^\alpha g \in L_r[a,b]$. Then*

$$\begin{aligned} &\int_a^x \omega(t) \Big[|D^\beta g(t)|^p |D^\alpha f(t)|^q + |D^\beta f(t)|^p |D^\alpha g(t)|^q \Big] dt \\ &\geqslant L M_1 \left(\int_a^x \varphi(t) \Big[|D^\alpha f(t)|^r + |D^\alpha g(t)|^r \Big] dt \right)^{\frac{p+q}{r}}, \end{aligned} \tag{2.19}$$

where M_1 is defined with (2.2) *and*

$$L = \begin{cases} 2^{-\frac{p}{r}}, \; p \geqslant q, \\ \left(1-2^{-\frac{p}{q}}\right)^{\frac{q}{r}}, \; p \leqslant q. \end{cases} \tag{2.20}$$

Proof. Let $t \in [a,x]$. Using identity (1.3), fixed sign of $D^\alpha g$ on $[a,b]$ and reverse Hölder's inequality for $\{\frac{r}{r-1}, r\}$, we have

$$\begin{aligned} &|D^\beta g(t)| \\ &= \frac{1}{\Gamma(\alpha-\beta)} \int_a^t (t-\tau)^{\alpha-\beta-1} [\varphi(\tau)]^{-\frac{1}{r}} [\varphi(\tau)]^{\frac{1}{r}} |D^\alpha g(\tau)| \, d\tau \\ &\geqslant \frac{1}{\Gamma(\alpha-\beta)} \left(\int_a^t (t-\tau)^{\frac{r(\alpha-\beta-1)}{r-1}} [\varphi(\tau)]^{\frac{1}{1-r}} d\tau \right)^{\frac{r-1}{r}} \left(\int_a^t \varphi(\tau) |D^\alpha g(\tau)|^r d\tau \right)^{\frac{1}{r}} \\ &= \frac{1}{\Gamma(\alpha-\beta)} [P_1(t)]^{\frac{r-1}{r}} [G(t)]^{\frac{1}{r}}, \end{aligned} \tag{2.21}$$

where G is defined by (2.5). Let F be defined with (2.6). Then (2.7) holds, and by (2.21) and (2.7) follows

$$\omega(t) |D^\beta g(t)|^p |D^\alpha f(t)|^q \geqslant h(t) [G(t)]^{\frac{p}{r}} \left[F'(t)\right]^{\frac{q}{r}}, \tag{2.22}$$

where h is defined by (2.9). Integrating (2.22) and applying reverse Hölder's inequality for $\{\frac{r}{r-q}, \frac{r}{q}\}$, follows

$$\begin{aligned} &\int_a^x \omega(t) |D^\beta g(t)|^p |D^\alpha f(t)|^q dt \\ &\geqslant \left(\int_a^x [h(t)]^{\frac{r}{r-q}} dt \right)^{\frac{r-q}{r}} \left(\int_a^x [G(t)]^{\frac{p}{q}} F'(t) \, dt \right)^{\frac{q}{r}} \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} &\int_a^x \omega(t) |D^\beta f(t)|^p |D^\alpha g(t)|^q dt \\ &\geqslant \left(\int_a^x [h(t)]^{\frac{r}{r-q}} dt \right)^{\frac{r-q}{r}} \left(\int_a^x [F(t)]^{\frac{p}{q}} G'(t) \, dt \right)^{\frac{q}{r}}. \end{aligned} \tag{2.24}$$

For negative power we use inequality

$$A^\delta + B^\delta \geqslant 2^{1-\delta} (A+B)^\delta, \quad (\delta < 0; A, B > 0), \tag{2.25}$$

since x^δ is convex function on $(0,\infty)$ for $\delta < 0$. Using (2.25) for $\frac{q}{r} < 0$, (2.23) and (2.24), we conclude

$$\begin{aligned} &\int_a^x \omega(t) \left[|D^\beta g(t)|^p |D^\alpha f(t)|^q + |D^\beta f(t)|^p |D^\alpha g(t)|^q \right] dt \\ &\geqslant \left(\int_a^x [h(t)]^{\frac{r}{r-q}} dt \right)^{\frac{r-q}{r}} 2^{1-\frac{q}{r}} \left(\int_a^x \left[[G(t)]^{\frac{p}{q}} F'(t) + [F(t)]^{\frac{p}{q}} G'(t) \right] dt \right)^{\frac{q}{r}}. \end{aligned} \tag{2.26}$$

For $\frac{p}{q}>0$ we use (2.12), and with $G(a)=F(a)=0$ we get

$$\int_a^x \left[[G(t)]^{\frac{p}{q}}\, F'(t)+[F(t)]^{\frac{p}{q}}\, G'(t)\right] dt \geqslant \frac{q}{p+q}\left(c_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)[G(x)+F(x)]^{\frac{p}{q}+1}. \tag{2.27}$$

Now from (2.26) and (2.27) follows

$$\begin{aligned}&\int_a^x \omega(t)\left[|D^\beta g(t)|^p\,|D^\alpha f(t)|^q+|D^\beta f(t)|^p\,|D^\alpha g(t)|^q\right] dt\\ &\geqslant 2^{1-\frac{q}{r}}\left(c_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{r}}\left(\frac{q}{p+q}\right)^{\frac{q}{r}}\left(\int_a^x [h(t)]^{\frac{r}{r-q}}\, dt\right)^{\frac{r-q}{r}}[G(x)+F(x)]^{\frac{p+q}{r}}.\end{aligned}$$

□

The same results, under different assumptions, follows for the Caputo fractional derivatives. The proofs are similar to the previous ones, using composition identity (1.4), and are omitted.

THEOREM 2.4. *Let $\alpha>\beta\geqslant 0$ with n and m given by (1.2). Let $f,g\in AC^n[a,b]$ be such that $f^{(i)}(a)=g^{(i)}(a)=0$ for $i=m,\dots,n-1$. Let $\varphi>0$ and $\omega\geqslant 0$ be measurable functions on $[a,x]$. Let $r>1$, $r>q>0$ and $p\geqslant 0$. Let ${}^CD^\alpha f, {}^CD^\alpha g\in L_r[a,b]$. Then*

$$\begin{aligned}&\int_a^x \omega(t)\left[\left|{}^CD^\beta g(t)\right|^p\left|{}^CD^\alpha f(t)\right|^q+\left|{}^CD^\beta f(t)\right|^p\left|{}^CD^\alpha g(t)\right|^q\right] dt\\ &\leqslant KM_1\left(\int_a^x \varphi(t)\left[\left|{}^CD^\alpha f(t)\right|^r+\left|{}^CD^\alpha g(t)\right|^r\right] dt\right)^{\frac{p+q}{r}},\end{aligned} \tag{2.28}$$

where K and M_1 are defined by (1.6) and (2.2), respectively.

THEOREM 2.5. *Let $\alpha>\beta_1,\beta_2\geqslant 0$ with n, m_1 and m_2 given by (1.2). Let $m=\min\{m_1,m_2\}$ and $f,g\in AC^n[a,b]$ be such that $f^{(i)}(a)=g^{(i)}(a)=0$ for $i=m,\dots,n-1$. Let $w\geqslant 0$ be measurable function on $[a,x]$. Let $p,q_1,q_2\geqslant 0$ and let ${}^CD^\alpha f, {}^CD^\alpha g\in L_\infty[a,b]$. Then*

$$\begin{aligned}&\int_a^x w(t)\Big[\left|{}^CD^{\beta_1} f(t)\right|^{q_1}\left|{}^CD^{\beta_2} g(t)\right|^{q_2}\left|{}^CD^\alpha f(t)\right|^p\\ &\qquad+\left|{}^CD^{\beta_2} f(t)\right|^{q_2}\left|{}^CD^{\beta_1} g(t)\right|^{q_1}\left|{}^CD^\alpha g(t)\right|^p\Big] dt\\ &\leqslant M_2\left[\left\|{}^CD^\alpha f\right\|_\infty^{2(q_1+p)}+\left\|{}^CD^\alpha f\right\|_\infty^{2q_2}+\left\|{}^CD^\alpha g\right\|_\infty^{2q_2}+\left\|{}^CD^\alpha g\right\|_\infty^{2(q_1+p)}\right],\end{aligned} \tag{2.29}$$

where M_2 is defined by (2.16).

THEOREM 2.6. *Let $\alpha>\beta\geqslant 0$ with n and m given by (1.2). Let $f,g\in AC^n[a,b]$ be such that $f^{(i)}(a)=g^{(i)}(a)=0$ for $i=m,\dots,n-1$. Let $\varphi>0$ and $\omega\geqslant 0$ be* *measurable functions on* $[a,x]$. *Let* $r<0$, $q>0$ *i* $p\geqslant 0$. *Let* ${}^{C}D^{\alpha}f, {}^{C}D^{\alpha}g \in L_r[a,b]$, *each of which is of fixed sign a.e. on* $[a,b]$ *with* $1/{}^{C}D^{\alpha}f, 1/{}^{C}D^{\alpha}g \in L_r[a,b]$. *Then*

$$\int_a^x \omega(t)\left[\left|{}^{C}D^{\beta}g(t)\right|^p \left|{}^{C}D^{\alpha}f(t)\right|^q + \left|{}^{C}D^{\beta}f(t)\right|^p \left|{}^{C}D^{\alpha}g(t)\right|^q\right] dt$$

$$\geqslant LM_1\left(\int_a^x \varphi(t)\left[\left|{}^{C}D^{\alpha}f(t)\right|^r + \left|{}^{C}D^{\alpha}g(t)\right|^r\right] dt\right)^{\frac{p+q}{r}}, \tag{2.30}$$

where L *and* M_1 *are defined by* (2.20) *and* (2.2), *respectively.*

Finally, we give corresponding theorems involving the Canavati fractional derivatives and composition identity (1.5).

THEOREM 2.7. *Let* $\alpha>\beta\geqslant 0$, $n=[\alpha]+1$ *and* $m=[\beta]+1$. *Let* $f,g\in C^{\alpha}[a,b]$ *be such that* $f^{(i)}(a)=g^{(i)}(a)=0$ *for* $i=m-1,\dots,n-2$. *Let* $\varphi>0$ *and* $\omega\geqslant 0$ *be measurable functions on* $[a,x]$. *Let* $r>1$, $r>q>0$ *and* $p\geqslant 0$. *Let* ${}^{\bar{C}}D^{\alpha}f, {}^{\bar{C}}D^{\alpha}g \in L_r[a,b]$. *Then*

$$\int_a^x \omega(t)\left[\left|{}^{\bar{C}}D^{\beta}g(t)\right|^p \left|{}^{\bar{C}}D^{\alpha}f(t)\right|^q + \left|{}^{\bar{C}}D^{\beta}f(t)\right|^p \left|{}^{\bar{C}}D^{\alpha}g(t)\right|^q\right] dt$$

$$\leqslant KM_1\left(\int_a^x \varphi(t)\left[\left|{}^{\bar{C}}D^{\alpha}f(t)\right|^r + \left|{}^{\bar{C}}D^{\alpha}g(t)\right|^r\right] dt\right)^{\frac{p+q}{r}}, \tag{2.31}$$

where K *and* M_1 *are defined by* (1.6) *and* (2.2), *respectively.*

THEOREM 2.8. *Let* $\alpha>\beta_1,\beta_2\geqslant 0$, $n=[\alpha]+1$ *and* $m=\min\{[\beta_1]+1,[\beta_2]+1\}$. *Let* $f,g\in C^{\alpha}[a,b]$ *be such that* $f^{(i)}(a)=g^{(i)}(a)=0$ *for* $i=m-1,\dots,n-2$. *Let* $w\geqslant 0$ *be measurable function on* $[a,x]$. *Let* $p,q_1,q_2\geqslant 0$ *and let* ${}^{\bar{C}}D^{\alpha}f, {}^{\bar{C}}D^{\alpha}g \in L_{\infty}[a,b]$. *Then*

$$\begin{aligned}\int_a^x w(t)\Big[&\left|{}^{\bar{C}}D^{\beta_1}f(t)\right|^{q_1}\left|{}^{\bar{C}}D^{\beta_2}g(t)\right|^{q_2}\left|{}^{\bar{C}}D^{\alpha}f(t)\right|^{p}\\ &+\left|{}^{\bar{C}}D^{\beta_2}f(t)\right|^{q_2}\left|{}^{\bar{C}}D^{\beta_1}g(t)\right|^{q_1}\left|{}^{\bar{C}}D^{\alpha}g(t)\right|^{p}\Big]dt\\ \leqslant M_2\Big[&\left\|{}^{\bar{C}}D^{\alpha}f\right\|_{\infty}^{2(q_1+p)}+\left\|{}^{\bar{C}}D^{\alpha}f\right\|_{\infty}^{2q_2}+\left\|{}^{\bar{C}}D^{\alpha}g\right\|_{\infty}^{2q_2}+\left\|{}^{\bar{C}}D^{\alpha}g\right\|_{\infty}^{2(q_1+p)}\Big],\end{aligned} \tag{2.32}$$

where M_2 *is defined by* (2.16).

THEOREM 2.9. *Let* $\alpha>\beta\geqslant 0$, $n=[\alpha]+1$ *and* $m=[\beta]+1$. *Let* $f,g\in C^{\alpha}[a,b]$ *be such that* $f^{(i)}(a)=g^{(i)}(a)=0$ *for* $i=m-1,\dots,n-2$. *Let* $\varphi>0$ *and* $\omega\geqslant 0$ *be measurable functions on* $[a,x]$. *Let* $r<0$, $q>0$ *and* $p\geqslant 0$. *Let* ${}^{\bar{C}}D^{\alpha}f, {}^{\bar{C}}D^{\alpha}g \in L_r[a,b]$, *each of which is of fixed sign a.e. on* $[a,b]$ *with* $1/{}^{\bar{C}}D^{\alpha}f, 1/{}^{\bar{C}}D^{\alpha}g \in L_r[a,b]$. *Then*

$$\int_a^x \omega(t)\left[\left|{}^{\bar{C}}D^{\beta}g(t)\right|^p \left|{}^{\bar{C}}D^{\alpha}f(t)\right|^q + \left|{}^{\bar{C}}D^{\beta}f(t)\right|^p \left|{}^{\bar{C}}D^{\alpha}g(t)\right|^q\right] dt$$

$$\geqslant LM_1\left(\int_a^x \varphi(t)\left[\left|{}^{\bar{C}}D^{\alpha}f(t)\right|^r + \left|{}^{\bar{C}}D^{\alpha}g(t)\right|^r\right] dt\right)^{\frac{p+q}{r}}, \tag{2.33}$$

where L and M_1 are defined by (2.20) *and* (2.2), *respectively.*

REMARK 1. Comparing these theorems with ones from [4] we conclude:

With relaxed restrictions and smaller constant K, defined by (1.6), Theorem 2.1 improves [4, Theorem 7.5], Theorem 2.4 improves [4, Theorem 16.31] and Theorem 2.7 improves [4, Theorem 6.6]. In theorems from [4] the role of constant K has

$$\delta_3^{\frac{q}{r}} = \begin{cases} \left(2^{\frac{p}{q}} - 1\right)^{\frac{q}{r}}, & p \geqslant q, \\ 1, & p \leqslant q. \end{cases}$$

Obviously, $\delta_3^{q/r} \geqslant 1$, while $K \leqslant 1$. Since $\lim\limits_{p\to\infty} \delta_3^{q/r} = \infty$, for all sufficiently large p we obtain a substantial improvement of inequality.

Further, with relaxed restrictions Theorem 2.2 improves [4, Theorem 7.18], Theorem 2.5 improves [4, Theorem 16.38] and Theorem 2.8 improves [4, Theorem 6.18].

Theorems 2.3, 2.6 and 2.9 are newly presented.

REMARK 2. In this paper we consider left-sided fractional integrals and derivatives. A common notation for the left-sided Riemann-Lioville fractional integral is $J_{a+}^{\alpha} f$, defined by (1.1). For the right-sided we have

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt\,.$$

A connection between left-sided and right-sided Riemann-Liouville fractional integrals is given by a simple relation

$$QJ_{a+}^{\alpha} = J_{b-}^{\alpha} Q, \quad QJ_{b-}^{\alpha} = J_{a+}^{\alpha} Q,$$

where Q is the "reflection operator": $(Q\varphi)(x) = \varphi(a+b-x)$.
For the Riemann-Liouville, Caputo and Canavati fractional derivatives we have analogous relations

$$\begin{aligned} QD_{a+}^{\alpha} &= D_{b-}^{\alpha} Q, \quad QD_{b-}^{\alpha} = D_{a+}^{\alpha} Q, \\ Q\,{}^{C}D_{a+}^{\alpha} &= {}^{C}D_{b-}^{\alpha} Q, \quad Q\,{}^{C}D_{b-}^{\alpha} = {}^{C}D_{a+}^{\alpha} Q, \\ Q\,{}^{\bar{C}}D_{a+}^{\alpha} &= {}^{\bar{C}}D_{b-}^{\alpha} Q, \quad Q\,{}^{\bar{C}}D_{b-}^{\alpha} = {}^{\bar{C}}D_{a+}^{\alpha} Q. \end{aligned}$$

Using this operator, it's easy to prove composition identity for the right-sided fractional derivatives, e.g. for the Riemann-Liouville fractional derivatives

$$D_{b-}^{\beta} f(x) = \frac{1}{\Gamma(\alpha-\beta)} \int_x^b (t-x)^{\alpha-\beta-1} D_{b-}^{\alpha} f(t)\, dt = J_{b-}^{\alpha-\beta} D_{b-}^{\alpha} f(x),$$

follows

$$\begin{aligned} D_{b-}^{\beta} f &= Q\left(QD_{b-}^{\beta} f\right) = Q\left(D_{a+}^{\beta} Qf\right) = Q\left(J_{a+}^{\alpha-\beta} D_{a+}^{\alpha} Qf\right) \\ &= J_{b-}^{\alpha-\beta} Q\left(D_{a+}^{\alpha} Qf\right) = J_{b-}^{\alpha-\beta} D_{b-}^{\alpha} Q(Qf) = J_{b-}^{\alpha-\beta} D_{b-}^{\alpha} f\,. \end{aligned}$$

Now we have all we need for Opial-type inequalities involving right-sided fractional integral and derivatives, and right-sided versions of our theorems could be analogously done.

3. Applications

Opial's inequality and its several generalizations, extensions and discretizations, play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations. As an example for fractional calculus, we present a uniqueness of solution for a system of fractional differential equations involving Riemann-Liouville fractional derivatives. With relax conditions, it actually improve Theorem 7.26 form [4]. We sketch a proof for the reader's convenience.

THEOREM 3.1. *Let $\alpha > \beta_i \geqslant 0$, $i = 1,\dots,r \in \mathbb{N}$. Suppose that one of the conditions $(i)-(vii)$ in Corollary 1.2 holds for $\{\alpha,\beta_i,f_1\}$ and $\{\alpha,\beta_i,f_2\}$, $i=1,\dots,r$. Let $D^\alpha f_1, D^\alpha f_2 \in L_2[a,x]$. For $j=1,2$, let*

$$D^\alpha f_j(s) = F_j\left(s, \{D^{\beta_i} f_1(s)\}_{i=1}^r, \{D^{\beta_i} f_2(s)\}_{i=1}^r\right), \quad s\in[a,x], \tag{3.1}$$

where $F_j : [a,x]\times\mathbb{R}^r\times\mathbb{R}^r \to \mathbb{R}$ are continuous, bounded for $s\in[a,x]$, and satisfy the Lipschitz condition

$$\begin{aligned}&\left|F_j(s,z_1,\dots,z_r,y_1,\dots,y_r) - F_j(s,z_1',\dots,z_r',y_1',\dots,y_r')\right| \\ &\leqslant \sum_{i=1}^r \left[q_{1,i,j}(s)|z_i - z_i'| + q_{2,i,j}(s)|y_i - y_i'|\right],\end{aligned} \tag{3.2}$$

$j=1,2$, with $q_{1,i,j}(s), q_{2,i,j}(s) \geqslant 0$ bounded on $[a,x]$, $1\leqslant i\leqslant r$.
Further, assume that

$$\phi^*(x) := \sum_{i=1}^r \left(\frac{M_{1,i}}{2} + \frac{M_{2,i}}{\sqrt{2}}\right)\left(\frac{x^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i)\sqrt{\alpha-\beta_i}\sqrt{2\alpha-2\beta_i-1}}\right) < 1, \tag{3.3}$$

where

$$M_{1,i} = \max\left(\|q_{1,i,1}\|_\infty, \|q_{2,i,2}\|_\infty\right),\; M_{2,i} = \max\left(\|q_{2,i,1}\|_\infty, \|q_{1,i,2}\|_\infty\right).$$

Then the system (3.1) has at most one solution on $[a,x]$.

Proof. Assume there are two pairs of solutions (f_1,f_2), (f_1^*,f_2^*) to system (3.1). Set $g_j = f_j - f_j^*, j=1,2$. Then

$$D^{\alpha-k} g_j(a) = 0,\; k=1,\dots[\alpha]+1;\; j=1,2. \tag{3.4}$$

It holds

$$\begin{aligned}D^\alpha g_j(s) &= F_j\left(s,\{D^{\beta_i}f_1(s)\}_{i=1}^r, \{D^{\beta_i}f_2(s)\}_{i=1}^r\right) \\ &- F_j\left(s,\{D^{\beta_i}f_1^*(s)\}_{i=1}^r, \{D^{\beta_i}f_2^*(s)\}_{i=1}^r\right).\end{aligned}$$

By (3.2) we have

$$|D^{\alpha}g_j(s)| \leqslant \sum_{i=1}^{r}\left[q_{1,i,j}(s)|D^{\beta_i}g_1(s)|+q_{2,i,j}(s)|D^{\beta_i}g_2(s)|\right].$$

Therefore,

$$(D^{\alpha}g_j(s))^2 \leqslant \sum_{i=1}^{r}\left[\|q_{1,i,j}\|_{\infty}|D^{\beta_i}g_1(s)|\,|D^{\alpha}g_j(s)|+\|q_{2,i,j}\|_{\infty}|D^{\beta_i}g_2(s)|\,|D^{\alpha}g_j(s)|\right].$$

Now follows

$$\begin{aligned}
I &:= \int_a^x \left((D^{\alpha}g_1(s))^2+(D^{\alpha}g_2(s))^2\right)ds \\
&\leqslant \sum_{i=1}^{r} M_{1,i}\left(\int_a^x \left[|D^{\beta_i}g_1(s)|\,|D^{\alpha}g_1(s)|+|D^{\beta_i}g_2(s)|\,|D^{\alpha}g_2(s)|\right]ds\right) \\
&\quad +\sum_{i=1}^{r} M_{2,i}\left(\int_a^x \left[|D^{\beta_i}g_2(s)|\,|D^{\alpha}g_1(s)|+|D^{\beta_i}g_1(s)|\,|D^{\alpha}g_2(s)|\right)ds\right) \\
&\leqslant \sum_{i=1}^{r} M_{1,i}\left(\frac{x^{\alpha-\beta_i}I}{2\Gamma(\alpha-\beta_i)\sqrt{\alpha-\beta_i}\sqrt{2\alpha-2\beta_i-1}}\right) && (3.5)\\
&\quad +\sum_{i=1}^{r} M_{2,i}\left(\frac{x^{\alpha-\beta_i}I}{\sqrt{2}\Gamma(\alpha-\beta_i)\sqrt{\alpha-\beta_i}\sqrt{2\alpha-2\beta_i-1}}\right) && (3.6)\\
&= \phi^*(x)I,
\end{aligned}$$

where (3.6) follows by Theorem 2.1 for $\varphi=\omega\equiv 1$, $p=q=1$ and $r=2$, while (3.5) is obtain similarly. We have established that

$$I \leqslant \phi^*(x)I.$$

If $I\neq 0$ then $\phi^*(x)\geqslant 1$, a contradiction by the assumption (3.3) that $\phi^*(x)<1$. Therefore $I=0$, implying that

$$(D^{\alpha}g_1(s))^2+(D^{\alpha}g_2(s))^2=0,\ a.e. \text{ in } [a,x].$$

That is,

$$D^{\alpha}g_1(s)=0,\ D^{\alpha}g_2(s)=0,\ a.e. \text{ in } [a,x].$$

By (3.4) and Theorem 1.1 (applying (1.3) for $\beta=0$), we find $g_1(s)\equiv g_2(s)\equiv 0$ over $[a,x]$. This implies $f_j=f_j^*, j=1,2$, over $[a,x]$, thus proving the uniqueness of the solution to the initial value problem of this theorem. □

For more applications, such as upper bounds on $D^{\alpha}f_j$ and solutions f_j included in a system of fractional differential equations involving Riemann-Liouville fractional derivatives see Section 7.4 in [4]. Also, similar applications in fractional differential equations involving Canavati fractional derivatives can be find in [4, Section 6.4], and for Caputo fractional derivatives in [4, Section 16.6].

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