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HERMITE–HADAMARD TYPE INEQUALITIES FOR RIEMANN–LIOUVILLE FRACTIONAL INTEGRALS OF (α,m)–CONVEX FUNCTIONS

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Abstract. In the paper, the authors establish some new Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals of functions whose derivatives in absolute value are of (α,m)-convexity.

1. Introduction

The following definition is well known in the literature.

DEFINITION 1.1. A function $f : I \subseteq \mathbb{R} = (-\infty,\infty) \to \mathbb{R}$ is said to be convex if

$$f(\lambda x+(1-\lambda)y) \leqslant \lambda f(x)+(1-\lambda)f(y) \tag{1.1}$$

holds for all $x,y \in I$ and $\lambda \in [0,1]$. If the inequality (1.1) reverses, then f is said to be concave on I.

Hermite-Hadamard inequality asserts that for every convex function $f : I \subseteq \mathbb{R} \to \mathbb{R}$, we have

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}, \tag{1.2}$$

where $a,b \in I$ with $a<b$. Both inequalities hold in reversed direction if f is concave.

DEFINITION 1.2. ([13]) A function $f : [0,b] \to \mathbb{R}$ is said to be m-convex if

$$f(tx+m(1-t)y) \leqslant tf(x)+m(1-t)f(y) \tag{1.3}$$

holds for all $x,y \in [0,b]$ and $t \in [0,1]$ and for some $m \in (0,1]$.

DEFINITION 1.3. ([9]) Let $f : [0,b] \to \mathbb{R}$ and $(\alpha,m) \in (0,1]\times(0,1]$. If

$$f(tx+m(1-t)y) \leqslant t^{\alpha} f(x)+m(1-t^{\alpha})f(y) \tag{1.4}$$

is valid for all $x,y \in [0,b]$ and $t \in [0,1]$, then we say that $f(x)$ is a (α,m)-convex function on $[0,b]$.

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In recent decades, a lot of inequalities of Hermite-Hadamard type for various kinds of (α,m)-convex functions have been established. Some of them may be recited as follows.

THEOREM 1.1. ([5, Theorem 2] and [6]) *Let* $f:\mathbb{R}_0=[0,\infty)\to\mathbb{R}$ *be* m*-convex and* $m\in(0,1]$. *If* $f\in L([a,b])$ *for* $0\leqslant a<b<\infty$, *then*

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\leqslant\min\left\{\frac{f(a)+mf(b/m)}{2},\frac{mf(a/m)+f(b)}{2}\right\}. \tag{1.5}$$

THEOREM 1.2. ([3, Theorem 3.1]) *Let* $I\supseteq\mathbb{R}_0$ *be an open real interval and let* $f:I\to\mathbb{R}$ *be a differentiable function on* I *such that* $f'\in L([a,b])$ *for* $0\leqslant a<b<\infty$. *If* $|f'|^q$ *is* (α,m)*-convex on* $[a,b]$ *for some given numbers* $\alpha,m\in(0,1]$ *and* $q\geqslant1$, *then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\leqslant\frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q}$$
$$\times\min\left\{\left[v_1|f'(a)|^q+v_2m\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q},\left[v_2m\left|f'\left(\frac{a}{m}\right)\right|^q+v_1|f'(b)|^q\right]^{1/q}\right\}, \tag{1.6}$$

where

$$v_1=\frac{1}{(\alpha+1)(\alpha+2)}\left(\alpha+\frac{1}{2^\alpha}\right)\quad\textit{and}\quad v_2=\frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{\alpha^2+\alpha+2}{2}-\frac{1}{2^\alpha}\right).$$

For more information on Hermite-Hadamard type inequalities for (α,m)-convex functions, please refer to [1, 2, 4, 8, 10, 11, 12, 14, 15] and closely related references therein.

DEFINITION 1.4. ([7]) Let $f\in L([a,b])$ and $a\geqslant0$. Riemann-Liouville integrals $J_{a+}^{\alpha}f(x)$ and $J_{b-}^{\alpha}f(x)$ of order $\alpha>0$ are defined by

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)\,\mathrm{d}t,\quad x>a \tag{1.7}$$

and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)\,\mathrm{d}t,\quad x<b \tag{1.8}$$

respectively, where Γ is the classical Euler gamma function which may be defined for $\Re(z)>0$ by

$$\Gamma(z)=\int_0^\infty u^{z-1}e^{-u}\,\mathrm{d}u. \tag{1.9}$$

Moreover, assume that $J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)$.

In this paper, motivated by the above mentioned results, we will establish a Riemann-Liouville fractional integral identity including a differentiable mapping and then find some new Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals of functions whose derivatives in absolute value are of (α,m)-convexity.

2. A Riemann-Liouville fractional integral identity

Let $f:[a,b]\subseteq\mathbb{R}\to\mathbb{R}$ and $\alpha>0$ and $Q_\alpha(a,b)$ be defined by

$$Q_\alpha(a,b)=\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{a+}f\left(\frac{3a+b}{4}\right)\right.$$
$$\left.+J^\alpha_{[(3a+b)/4]+}f\left(\frac{a+b}{2}\right)+J^\alpha_{[(a+b)/2]+}f\left(\frac{a+3b}{4}\right)+J^\alpha_{[(a+3b)/4]+}f(b)\right]. \quad (2.1)$$

It is easy to see that

$$Q_1(a,b)=\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x. \quad (2.2)$$

LEMMA 2.1. *Let $f:[a,b]\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on (a,b) such that $f'\in L([a,b])$. Then*

$$Q_\alpha(a,b)=\frac{b-a}{16}\left\{\int_0^1(1-t^\alpha)f'\left(\frac{3a+b}{4}t+(1-t)\frac{a+b}{2}\right)\mathrm{d}t\right.$$
$$-\int_0^1 t^\alpha f'\left(at+(1-t)\frac{3a+b}{4}\right)\mathrm{d}t+\int_0^1(1-t^\alpha)f'\left(\frac{a+3b}{4}t+(1-t)b\right)\mathrm{d}t$$
$$\left.-\int_0^1 t^\alpha f'\left(\frac{a+b}{2}t+(1-t)\frac{a+3b}{4}\right)\mathrm{d}t\right\}. \quad (2.3)$$

Proof. Letting $u=at+(1-t)\frac{3a+b}{4}$ and integrating by parts yield

$$\begin{aligned}I_1&\triangleq-\frac{b-a}{16}\int_0^1 t^\alpha f'\left(at+(1-t)\frac{3a+b}{4}\right)\mathrm{d}t\\&=\frac{1}{4}\left[f(a)-\alpha\int_0^1 f\left(at+(1-t)\frac{3a+b}{4}\right)t^{\alpha-1}\,\mathrm{d}t\right]\\&=\frac{1}{4}f(a)+\frac{\alpha 4^{\alpha-1}}{(b-a)^\alpha}\int_{(3a+b)/4}^a f(u)\left(\frac{3a+b}{4}-u\right)^{\alpha-1}\mathrm{d}u\\&=\frac{1}{4}f(a)-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}J^\alpha_{a+}f\left(\frac{3a+b}{4}\right).\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}I_2&\triangleq\frac{b-a}{16}\int_0^1(1-t^\alpha)f'\left(\frac{3a+b}{4}t+(1-t)\frac{a+b}{2}\right)\mathrm{d}t\\&=\frac{1}{4}f\left(\frac{a+b}{2}\right)-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}J^\alpha_{[(3a+b)/4]+}f\left(\frac{a+b}{2}\right),\\I_3&\triangleq-\frac{b-a}{16}\int_0^1 t^\alpha f'\left(\frac{a+b}{2}t+(1-t)\frac{a+3b}{4}\right)\mathrm{d}t\end{aligned}$$

$$= \frac{1}{4} f\left(\frac{a+b}{2}\right) - \frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} J^{\alpha}_{[(a+b)/2]+} f\left(\frac{a+3b}{4}\right),$$

$$I_4 \triangleq \frac{b-a}{16} \int_0^1 (1-t^{\alpha}) f'\left(\frac{a+3b}{4}t + (1-t)b\right) \mathrm{d}t$$

$$= \frac{1}{4} f(b) - \frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} J^{\alpha}_{[(a+3b)/4]+} f(b).$$

Adding the above quantities leads to the identity (2.3). The proof of Lemma 2.1 is complete. □

REMARK 2.1. Under conditions of Lemma 2.1, if $\alpha = 1$, then

$$\begin{aligned} Q_1(a,b) = \frac{b-a}{16} \bigg\{ & \int_0^1 (1-t) f'\left(\frac{3a+b}{4}t + (1-t)\frac{a+b}{2}\right) \mathrm{d}t \\ & - \int_0^1 t f'\left(at + (1-t)\frac{3a+b}{4}\right) \mathrm{d}t + \int_0^1 (1-t) f'\left(\frac{a+3b}{4}t + (1-t)b\right) \mathrm{d}t \\ & - \int_0^1 t f'\left(\frac{a+b}{2}t + (1-t)\frac{a+3b}{4}\right) \mathrm{d}t \bigg\}. \end{aligned} \tag{2.4}$$

3. Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals

Now we start out to establish Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals of functions whose derivatives in absolute value are of (α,m)-convexity.

THEOREM 3.1. *Let $f : \mathbb{R}_0 \to \mathbb{R}$ be differentiable on $\mathbb{R}_0$ and $f' \in L([a,b])$ for $0 \leqslant a < b$ and $\alpha > 0$. If $|f'|^q$ is (α_1,m)-convex function on $\left[0, \frac{b}{m}\right]$ for some $(\alpha_1,m) \in (0,1] \times (0,1]$ and $q \geqslant 1$, then*

$$\begin{aligned} |Q_{\alpha}(a,b)| \leqslant \frac{b-a}{16(\alpha+1)} & \left[\frac{1}{(\alpha_1+1)(\alpha+\alpha_1+1)}\right]^{1/q} \\ & \times \left[\left((\alpha+1)(\alpha_1+1)|f'(a)|^q + m\alpha_1(\alpha_1+1)\left|f'\left(\frac{3a+b}{4m}\right)\right|^q\right)^{1/q}\right. \\ & + \alpha\left((\alpha+1)\left|f'\left(\frac{3a+b}{4}\right)\right|^q + m\alpha_1(\alpha_1+\alpha+2)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{1/q} \\ & + \left((\alpha+1)(\alpha_1+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q + m\alpha_1(\alpha_1+1)\left|f'\left(\frac{a+3b}{4m}\right)\right|^q\right)^{1/q} \\ & \left. + \alpha\left((\alpha+1)\left|f'\left(\frac{a+3b}{4}\right)\right|^q + m\alpha_1(\alpha_1+\alpha+2)\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{1/q}\right]. \end{aligned}$$

Proof. From Lemma 2.1 and by the power mean inequality and the (α_1,m)-convexity of $|f'|^q$ on $\left[0,\frac{b}{m}\right]$, we obtain

$$\begin{aligned}
|Q_\alpha(a,b)| \leqslant \frac{b-a}{16}\bigg\{&\int_0^1 t^\alpha\left|f'\left(at+(1-t)\frac{3a+b}{4}\right)\right|\mathrm{d}t \\
&+\int_0^1 (1-t^\alpha)\left|f'\left(\frac{3a+b}{4}t+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}t \\
&+\int_0^1 t^\alpha\left|f'\left(\frac{a+b}{2}t+(1-t)\frac{a+3b}{4}\right)\right|\mathrm{d}t \\
&+\int_0^1 (1-t^\alpha)\left|f'\left(\frac{a+3b}{4}t+(1-t)b\right)\right|\mathrm{d}t\bigg\} \\
\leqslant \frac{b-a}{16}\bigg\{&\left(\int_0^1 t^\alpha\,\mathrm{d}t\right)^{1-1/q}\left[\int_0^1 t^\alpha\left(t^{\alpha_1}|f'(a)|^q+m(1-t^{\alpha_1})\right.\right. \\
&\times\left.\left.\left|f'\left(\frac{3a+b}{4m}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}+\left[\int_0^1(1-t^\alpha)\,\mathrm{d}t\right]^{1-1/q}\left[\int_0^1(1-t^\alpha)\right. \\
&\times\left.\left(t^{\alpha_1}\left|f'\left(\frac{3a+b}{4}\right)\right|^q+m(1-t^{\alpha_1})\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)\mathrm{d}t\right]^{1/q} \\
&+\left(\int_0^1 t^\alpha\,\mathrm{d}t\right)^{1-1/q}\left[\int_0^1 t^\alpha\left(t^{\alpha_1}\left|f'\left(\frac{a+b}{2}\right)\right|^q+m(1-t^{\alpha_1})\right.\right. \\
&\times\left.\left.\left|f'\left(\frac{a+3b}{4m}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}+\left[\int_0^1(1-t^\alpha)\,\mathrm{d}t\right]^{1-1/q}\left[\int_0^1(1-t^\alpha)\right. \\
&\times\left.\left(t^{\alpha_1}\left|f'\left(\frac{a+3b}{4}\right)\right|^q+m(1-t^{\alpha_1})\left|f'\left(\frac{b}{m}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}\bigg\}.
\end{aligned}$$

Substituting $\int_0^1 t^\alpha\,\mathrm{d}t=\frac{1}{\alpha+1}$ and

$$\begin{aligned}
&\int_0^1 t^\alpha\left(t^{\alpha_1}|f'(a)|^q+m(1-t^{\alpha_1})\left|f'\left(\frac{3a+b}{4m}\right)\right|^q\right)\mathrm{d}t \\
&=\frac{1}{(\alpha+1)(\alpha+\alpha_1+1)}\left((\alpha+1)|f'(a)|^q+\alpha_1 m\left|f'\left(\frac{3a+b}{4m}\right)\right|^q\right), \\
&\int_0^1(1-t^\alpha)\left(t^{\alpha_1}\left|f'\left(\frac{3a+b}{4}\right)\right|^q+m(1-t^{\alpha_1})\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)\mathrm{d}t \\
&=\frac{\alpha}{(\alpha+1)(\alpha_1+1)(\alpha+\alpha_1+1)}\left((\alpha+1)\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right. \\
&\quad\left.+m\alpha_1(\alpha+\alpha_1+2)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right), \\
&\int_0^1 t^\alpha\left(t^{\alpha_1}\left|f'\left(\frac{a+b}{2}\right)\right|^q+m(1-t^{\alpha_1})\left|f'\left(\frac{a+3b}{4m}\right)\right|^q\right)\mathrm{d}t \\
&=\frac{1}{(\alpha+1)(\alpha+\alpha_1+1)}\left((\alpha+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\alpha_1\left|f'\left(\frac{a+3b}{4m}\right)\right|^q\right),
\end{aligned}$$

$$
\begin{aligned}
&\int_0^1 (1-t^{\alpha})\left(t^{\alpha_1}\left|f'\left(\frac{a+3b}{4}\right)\right|^q + m(1-t^{\alpha_1})\left|f'\left(\frac{b}{m}\right)\right|^q\right)\mathrm{d}t \\
&= \frac{\alpha}{(\alpha+1)(\alpha_1+1)(\alpha+\alpha_1+1)}\left((\alpha+1)\left|f'\left(\frac{a+3b}{4}\right)\right|^q \right.\\
&\quad \left. + m\alpha_1(\alpha+\alpha_1+2)\left|f'\left(\frac{b}{m}\right)\right|^q\right)
\end{aligned}
$$

into the above inequality and simplifying result in the required inequality. The proof of Theorem 3.1 is complete. □

COROLLARY 3.1. *Under conditions of Theorem 3.1,*

1. *if $q=1$, then*

$$
\begin{aligned}
|Q_\alpha(a,b)| \leqslant{}& \frac{b-a}{16(\alpha+1)(\alpha_1+1)(\alpha+\alpha_1+1)}\left[(\alpha+1)(\alpha_1+1)|f'(a)|\right.\\
&+ m\alpha_1(\alpha_1+1)\left|f'\left(\frac{3a+b}{4m}\right)\right| + \alpha(\alpha+1)\left|f'\left(\frac{3a+b}{4}\right)\right|\\
&+ m\alpha_1(\alpha_1+\alpha+2)\left|f'\left(\frac{a+b}{2m}\right)\right| + (\alpha+1)(\alpha_1+1)\left|f'\left(\frac{a+b}{2}\right)\right|\\
&+ m\alpha_1(\alpha_1+1)\left|f'\left(\frac{a+3b}{4m}\right)\right| + \alpha(\alpha+1)\left|f'\left(\frac{a+3b}{4}\right)\right|\\
&\left.+ m\alpha_1(\alpha_1+\alpha+2)\left|f'\left(\frac{b}{m}\right)\right|\right];
\end{aligned}
$$

2. *if $m=1$, then*

$$
\begin{aligned}
|Q_\alpha(a,b)| \leqslant{}& \frac{b-a}{16(\alpha+1)}\left[\frac{1}{(\alpha_1+1)(\alpha+\alpha_1+1)}\right]^{1/q}\\
&\times\left[\left((\alpha+1)(\alpha_1+1)|f'(a)|^q + \alpha_1(\alpha_1+1)\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right)^{1/q}\right.\\
&+ \alpha\left((\alpha+1)\left|f'\left(\frac{3a+b}{4}\right)\right|^q + \alpha_1(\alpha_1+\alpha+2)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}\\
&+ \left((\alpha+1)(\alpha_1+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q + \alpha_1(\alpha_1+1)\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right)^{1/q}\\
&\left.+ \alpha\left((\alpha+1)\left|f'\left(\frac{a+3b}{4}\right)\right|^q + \alpha_1(\alpha_1+\alpha+2)|f'(b)|^q\right)^{1/q}\right];
\end{aligned}
$$

3. *if $m=\alpha=\alpha_1=1$, then*

$$
|Q_1(a,b)| \leqslant \frac{b-a}{32}\left(\frac{1}{3}\right)^{1/q}\left[\left(2|f'(a)|^q + \left|f'\left(\frac{3a+b}{4}\right)\right|^q\right)^{1/q}\right.
$$

$$+\left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q+2\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}+\left(2\left|f'\left(\frac{a+b}{2}\right)\right|^q\right.$$

$$\left.+\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right)^{1/q}+\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q+2\left|f'(b)\right|^q\right)^{1/q}\Bigg]; \quad (3.1)$$

4. *if* $m=\alpha=\alpha_1=q=1$, *then*

$$|Q_1(a,b)|\leqslant\frac{b-a}{48}\left[|f'(a)|+\left|f'\left(\frac{3a+b}{4}\right)\right|\right.$$

$$\left.+2\left|f'\left(\frac{a+b}{2}\right)\right|+\left|f'\left(\frac{a+3b}{4}\right)\right|+|f'(b)|\right]. \quad (3.2)$$

THEOREM 3.2. *Let* $f:\mathbb{R}_0\to\mathbb{R}$ *be differentiable on* $\mathbb{R}_0$ *and* $f'\in L([a,b])$ *for* $0\leqslant a<b$ *and* $\alpha>0$. *If* $|f'|^q$ *is* (α_1,m)*-convex function on* $\left[0,\frac{b}{m}\right]$ *for some* $(\alpha_1,m)\in(0,1]\times(0,1]$ *and for* $q>1$ *and* $q\geqslant r\geqslant 0$, *then*

$$|Q_\alpha(a,b)|\leqslant\frac{b-a}{16}\left\{\left(\frac{q-1}{\alpha(q-r)+q-1}\right)^{1-1/q}\left[\frac{1}{\alpha r+\alpha_1+1}|f'(a)|^q\right.\right.$$

$$\left.+\frac{m\alpha_1}{(\alpha r+1)(\alpha r+\alpha_1+1)}\left|f'\left(\frac{3a+b}{4m}\right)\right|^q\right]^{1/q}$$

$$+\frac{1}{\alpha}\left(B\left(\frac{2q-r-1}{q-1},\frac{1}{\alpha}\right)\right)^{1-1/q}\left[B\left(r+1,\frac{\alpha_1+1}{\alpha}\right)\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right.$$

$$\left.+m\left(B\left(r+1,\frac{1}{\alpha}\right)-B\left(r+1,\frac{\alpha_1+1}{\alpha}\right)\right)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}$$

$$+\left(\frac{q-1}{\alpha(q-r)+q-1}\right)^{1-1/q}\left[\frac{1}{\alpha r+\alpha_1+1}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right.$$

$$\left.+\frac{m\alpha_1}{(\alpha r+1)(\alpha r+\alpha_1+1)}\left|f'\left(\frac{a+3b}{4m}\right)\right|^q\right]^{1/q}$$

$$+\frac{1}{\alpha}\left(B\left(\frac{2q-r-1}{q-1},\frac{1}{\alpha}\right)\right)^{1-1/q}\left[B\left(r+1,\frac{\alpha_1+1}{\alpha}\right)\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right.$$

$$\left.\left.+m\left(B\left(r+1,\frac{1}{\alpha}\right)-B\left(r+1,\frac{\alpha_1+1}{\alpha}\right)\right)\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\right\},$$

where $B(u,v)$ *denotes the well known Beta function which may be defined by*

$$B(u,v)=\int_0^1 t^{u-1}(1-t)^{v-1}\,\mathrm{d}t,\quad u,v>0. \quad (3.3)$$

Proof. Using Lemma 2.1, Hölder inequality, and the (α_1,m)-convexity of $|f'|^q$ on $\left[0,\frac{b}{m}\right]$ gives

$$|Q_\alpha(a,b)|\leqslant\frac{b-a}{16}\left\{\int_0^1 t^\alpha\left|f'\left(at+(1-t)\frac{3a+b}{4}\right)\right|\mathrm{d}t+\int_0^1(1-t^\alpha)\left|f'\left(\frac{3a+b}{4}t\right.\right.\right.$$

$$
\begin{aligned}
&+(1-t)\frac{a+b}{2}\bigg)\bigg|\,\mathrm{d}t+\int_0^1 t^\alpha\bigg|f'\bigg(\frac{a+b}{2}t+(1-t)\frac{a+3b}{4}\bigg)\bigg|\,\mathrm{d}t\\
&+\int_0^1(1-t^\alpha)\bigg|f'\bigg(\frac{a+3b}{4}t+(1-t)b\bigg)\bigg|\,\mathrm{d}t\bigg\}\\
\leqslant{}&\frac{b-a}{16}\bigg\{\bigg(\int_0^1 t^{\alpha(q-r)/(q-1)}\,\mathrm{d}t\bigg)^{1-1/q}\bigg[\int_0^1 t^{\alpha r}\bigg(t^{\alpha_1}|f'(a)|^q\\
&+m(1-t^{\alpha_1})\bigg|f'\bigg(\frac{3a+b}{4m}\bigg)\bigg|^q\bigg)\,\mathrm{d}t\bigg]^{1/q}+\bigg[\int_0^1(1-t^\alpha)^{(q-r)/(q-1)}\,\mathrm{d}t\bigg]^{1-1/q}\\
&\times\bigg[\int_0^1(1-t^\alpha)^r\bigg(t^{\alpha_1}\bigg|f'\bigg(\frac{3a+b}{4}\bigg)\bigg|^q+m(1-t^{\alpha_1})\bigg|f'\bigg(\frac{a+b}{2m}\bigg)\bigg|^q\bigg)\,\mathrm{d}t\bigg]^{1/q}\\
&+\bigg(\int_0^1 t^{\alpha(q-r)/(q-1)}\,\mathrm{d}t\bigg)^{1-1/q}\bigg[\int_0^1 t^{\alpha r}\bigg(t^{\alpha_1}\bigg|f'\bigg(\frac{a+b}{2}\bigg)\bigg|^q\\
&+m(1-t^{\alpha_1})\bigg|f'\bigg(\frac{a+3b}{4m}\bigg)\bigg|^q\bigg)\,\mathrm{d}t\bigg]^{1/q}+\bigg(\int_0^1(1-t^\alpha)^{(q-r)/(q-1)}\,\mathrm{d}t\bigg)^{1-1/q}\\
&\times\bigg[\int_0^1(1-t^\alpha)^r\bigg(t^{\alpha_1}\bigg|f'\bigg(\frac{a+3b}{4}\bigg)\bigg|^q+m(1-t^{\alpha_1})\bigg|f'\bigg(\frac{b}{m}\bigg)\bigg|^q\bigg)\,\mathrm{d}t\bigg]^{1/q}\bigg\}.
\end{aligned}
$$

Substituting

$$
\begin{gathered}
\int_0^1 t^{\alpha(q-r)/(q-1)}\,\mathrm{d}t=\frac{q-1}{\alpha(q-r)+q-1},\\
\int_0^1 t^{\alpha r+\alpha_1}\,\mathrm{d}t=\frac{1}{\alpha r+\alpha_1+1},\\
\int_0^1 t^{\alpha r}(1-t^{\alpha_1})\,\mathrm{d}t=\frac{\alpha_1}{(\alpha r+1)(\alpha r+\alpha_1+1)},\\
\int_0^1(1-t^\alpha)^{(q-r)/(q-1)}\,\mathrm{d}t=\frac{1}{\alpha}B\bigg(\frac{2q-r-1}{q-1},\frac{1}{\alpha}\bigg),\\
\int_0^1(1-t^\alpha)^r t^{\alpha_1}\,\mathrm{d}t=\frac{1}{\alpha}B\bigg(r+1,\frac{\alpha_1+1}{\alpha}\bigg),
\end{gathered}
$$

and

$$
\int_0^1(1-t^\alpha)^r(1-t^{\alpha_1})\,\mathrm{d}t=\frac{1}{\alpha}B\bigg(r+1,\frac{1}{\alpha}\bigg)-\frac{1}{\alpha}B\bigg(r+1,\frac{\alpha_1+1}{\alpha}\bigg).
$$

into the above inequality and simplifying lead to the required inequality. The proof of Theorem 3.2 is complete. □

COROLLARY 3.2. *With assumptions in Theorem 3.2,*

1. if $r=0$, then

$$
|Q_\alpha(a,b)|\leqslant\frac{b-a}{16}\bigg(\frac{1}{\alpha_1+1}\bigg)^{1/q}\bigg\{\bigg(\frac{q-1}{\alpha q+q-1}\bigg)^{1-1/q}\bigg[|f'(a)|^q
$$

$$+m\alpha_1\left|f'\left(\frac{3a+b}{4m}\right)\right|^q\Bigg]^{1/q}+\left(\frac{1}{\alpha}B\left(\frac{2q-1}{q-1},\frac{1}{\alpha}\right)\right)^{1-1/q}$$
$$\times\left[\left|f'\left(\frac{3a+b}{4}\right)\right|^q+m\alpha_1\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}$$
$$+\left(\frac{q-1}{\alpha q+q-1}\right)^{1-1/q}\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\alpha_1\left|f'\left(\frac{a+3b}{4m}\right)\right|^q\right]^{1/q}$$
$$+\left(\frac{1}{\alpha}B\left(\frac{2q-1}{q-1},\frac{1}{\alpha}\right)\right)^{1-1/q}\left[\left|f'\left(\frac{a+3b}{4}\right)\right|^q+m\alpha_1\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\Bigg\};$$

2. *if* $r=q$, *then*

$$|Q_\alpha(a,b)|\leqslant\frac{b-a}{16}\Bigg\{\Bigg[\frac{1}{\alpha q+\alpha_1+1}|f'(a)|^q+\frac{m\alpha_1}{(\alpha q+1)(\alpha q+\alpha_1+1)}$$
$$\times\left|f'\left(\frac{3a+b}{4m}\right)\right|^q\Bigg]^{1/q}+\Bigg[B\left(q+1,\frac{\alpha_1+1}{\alpha}\right)\left|f'\left(\frac{3a+b}{4}\right)\right|^q$$
$$+m\left(B\left(q+1,\frac{1}{\alpha}\right)-B\left(q+1,\frac{\alpha_1+1}{\alpha}\right)\right)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\Bigg]^{1/q}$$
$$+\Bigg[\frac{1}{\alpha q+\alpha_1+1}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{m\alpha_1}{(\alpha q+1)(\alpha q+\alpha_1+1)}$$
$$\times\left|f'\left(\frac{a+3b}{4m}\right)\right|^q\Bigg]^{1/q}+\Bigg[B\left(q+1,\frac{\alpha_1+1}{\alpha}\right)\left|f'\left(\frac{a+3b}{4}\right)\right|^q$$
$$+m\left(B\left(q+1,\frac{1}{\alpha}\right)-B\left(q+1,\frac{\alpha_1+1}{\alpha}\right)\right)\left|f'\left(\frac{b}{m}\right)\right|^q\Bigg]^{1/q}\Bigg\};$$

3. *if* $\alpha=1$, *then*

$$|Q_1(a,b)|\leqslant\frac{b-a}{16}\left(\frac{q-1}{2q-r-1}\right)^{1-1/q}\Bigg\{\Bigg[\frac{1}{r+\alpha_1+1}|f'(a)|^q$$
$$+\frac{m\alpha_1}{(r+1)(r+\alpha_1+1)}\left|f'\left(\frac{3a+b}{4m}\right)\right|^q\Bigg]^{1/q}+\Bigg[B(r+1,\alpha_1+1)$$
$$\times\left|f'\left(\frac{3a+b}{4}\right)\right|^q+m\left(\frac{1}{r+1}-B(r+1,\alpha_1+1)\right)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\Bigg]^{1/q}$$
$$+\Bigg[\frac{1}{r+\alpha_1+1}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{m\alpha_1}{(r+1)(r+\alpha_1+1)}\left|f'\left(\frac{a+3b}{4m}\right)\right|^q\Bigg]^{1/q}$$
$$+\Bigg[B(r+1,\alpha_1+1)\left|f'\left(\frac{a+3b}{4}\right)\right|^q$$
$$+m\left(\frac{1}{r+1}-B(r+1,\alpha_1+1)\right)\left|f'\left(\frac{b}{m}\right)\right|^q\Bigg]^{1/q}\Bigg\};$$

4. *if* $\alpha = \alpha_1 = m = 1$, *we have*

$$\begin{aligned}|Q_1(a,b)| \leqslant \frac{b-a}{16}\left(\frac{q-1}{2q-r-1}\right)^{1-1/q}\left[\frac{1}{(r+1)(r+2)}\right]^{1/q}\Bigg\{\bigg[(r+1)|f'(a)|^q \\ +\left|f'\left(\frac{3a+b}{4}\right)\right|^q\bigg]^{1/q}+\left[\left|f'\left(\frac{3a+b}{4}\right)\right|^q+(r+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q} \\ +\left[(r+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right]^{1/q} \\ +\left[\left|f'\left(\frac{a+3b}{4}\right)\right|^q+(r+1)\left|f'(b)\right|^q\right]^{1/q}\Bigg\}.\end{aligned} \tag{3.4}$$

4. Applications

For two positive numbers $a>0$ and $b>0$, let $A(a,b)=\frac{a+b}{2}$, $H(a,b)=\frac{2ab}{a+b}$, and

$$L_p(a,b)=\begin{cases}\left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & p\neq 0,-1 \text{ and } a\neq b,\\ \dfrac{b-a}{\ln b-\ln a}, & p=-1 \text{ and } a\neq b,\\ I(a,b), & p=0 \text{ and } a\neq b,\\ a, & a=b.\end{cases}$$

It is well known that A, H, $L=L_{-1}$ and L_p are respectively called the arithmetic, harmonic, logarithmic, and generalized logarithmic means of two positive number a and b.

THEOREM 4.1. *Let* $b>a>0$, $q\geqslant 1$, *and* $p\in\mathbb{R}$.

1. *If* $p>1$ *and* $(p-1)q\geqslant 1$, *or* $p<0$ *and* $p\neq -1$, *then*

$$\begin{aligned}\left|\frac{A(a^p,b^p)+[A(a,b)]^p}{2}-[L_p(a,b)]^p\right| \leqslant \frac{b-a}{32}\left(\frac{|p|}{3}\right)^{1/q} \\ \times\Big\{\big(2a^{(p-1)q}+[A(a,A(a,b))]^{(p-1)q}\big)^{1/q}+\big([A(a,A(a,b))]^{(p-1)q} \\ +2[A(a,b)]^{(p-1)q}\big)^{1/q}+\big(2[A(a,b)]^{(p-1)q}+[A(A(a,b),b)]^{(p-1)q}\big)^{1/q} \\ +\big([A(A(a,b),b)]^{(p-1)q}+2b^{(p-1)q}\big)^{1/q}\Big\}.\end{aligned} \tag{4.1}$$

2. *If* $p=-1$, *then*

$$\left|\frac{1}{2}\left[\frac{1}{H(a,b)}+\frac{1}{A(a,b)}\right]-\frac{1}{L(a,b)}\right| \leqslant \frac{b-a}{32}\left(\frac{1}{3}\right)^{1/q}\Bigg\{\left(\frac{1}{2a^{2q}}+\frac{1}{[A(a,A(a,b))]^{2q}}\right)^{1/q}$$

$$+\left(\frac{1}{[A(a,A(a,b))]^{2q}}+\frac{2}{[A(a,b)]^{2q}}\right)^{1/q}+\left(\frac{2}{[A(a,b)]^{2q}}+\frac{1}{[A(A(a,b),b)]^{2q}}\right)^{1/q}+\left(\frac{1}{[A(A(a,b),b)]^{2q}}+\frac{2}{b^{2q}}\right)^{1/q}\Bigg\}. \tag{4.2}$$

3. If $q=1$ and $p\geqslant 2$, or $q=1$ and $-1\neq p<0$, then

$$\left|\frac{A(a^p,b^p)+[A(a,b)]^p}{2}-[L_p(a,b)]^p\right|\leqslant\frac{(b-a)|p|}{48}\left\{a^{p-1}+[A(a,A(a,b))]^{p-1}+2[A(a,b)]^{p-1}+[A(A(a,b),b)]^{p-1}+b^{p-1}\right\}. \tag{4.3}$$

4. If $p=-1$ and $q=1$, then

$$\left|\frac{1}{2}\left[\frac{1}{H(a,b)}+\frac{1}{A(a,b)}\right]-\frac{1}{L(a,b)}\right|\leqslant\frac{b-a}{48}\left\{\frac{1}{a^2}+\frac{1}{[A(a,A(a,b))]^2}+\frac{2}{[A(a,b)]^2}+\frac{1}{[A(A(a,b),b)]^2}+\frac{1}{b^2}\right\}. \tag{4.4}$$

Proof. Let $f(x)=x^p$ for $x>0$ and $p\neq 0,1$. Then $f'(x)=px^{p-1}$, $|f'(x)|^q=|p|x^{(p-1)q}$, and $\left(|f'(x)|^q\right)''=|p|^q(p-1)q[(p-1)q-1]x^{(p-1)q-2}$. If $p>1$ and $(p-1)q\geqslant 1$, or $p<0$, the function $|f'(x)|^q=|p|^q x^{(p-1)q}$ is convex on $[a,b]$. By (3.1), we obtain (4.1) and (4.2). The proof of Theorem 4.1 is complete. $\square$

THEOREM 4.2. *Let $b>a>0$, $q>1$, $q\geqslant r\geqslant 0$, and $p\in\mathbb{R}$.*

1. If $p>1$ and $(p-1)q\geqslant 1$, or $p<0$ and $p\neq -1$, then

$$\left|\frac{A(a^p,b^p)+[A(a,b)]^p}{2}-[L_p(a,b)]^p\right|\leqslant\frac{b-a}{16}\left(\frac{q-1}{2q-r-1}\right)^{1-1/q}\times\left[\frac{|p|}{(r+1)(r+2)}\right]^{1/q}\Big\{\left[(r+1)a^{(p-1)q}+[A(a,A(a,b))]^{(p-1)q}\right]^{1/q}+\left([A(a,A(a,b))]^{(p-1)q}+(r+1)[A(a,b)]^{(p-1)q}\right)^{1/q}+\left[(r+1)[A(a,b)]^{(p-1)q}+[A(A(a,b),b)]^{(p-1)q}\right]^{1/q}+\left([A(A(a,b),b)]^{(p-1)q}+(r+1)b^{(p-1)q}\right)^{1/q}\Big\}.$$

2. If $p=-1$, then

$$\left|\frac{1}{2}\left[\frac{1}{H(a,b)}+\frac{1}{A(a,b)}\right]-\frac{1}{L(a,b)}\right|\leqslant\frac{b-a}{16}\left(\frac{q-1}{2q-r-1}\right)^{1-1/q}\times\left[\frac{1}{(r+1)(r+2)}\right]^{1/q}\left\{\left[\frac{r+1}{a^{2q}}+\frac{1}{[A(a,A(a,b))]^{2q}}\right]^{1/q}\right.$$

$$+\left[\frac{1}{[A(a,A(a,b))]^{2q}}+\frac{r+1}{[A(a,b)]^{2q}}\right]^{1/q}$$
$$+\left[\frac{r+1}{[A(a,b)]^{2q}}+\frac{1}{[A(A(a,b),b)]^{2q}}\right]^{1/q}+\left[\frac{1}{[A(A(a,b),b)]^{2q}}+\frac{r+1}{b^{2q}}\right]^{1/q}\Bigg\}.$$

Proof. This follows from putting $f(x)=x^p$ for $x>0$ and $p\neq 0,1$ in (3.4). □

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