BLOWING–UP SOLUTIONS AND GLOBAL SOLUTIONS TO A FRACTIONAL DIFFERENTIAL EQUATION

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Abstract. In this paper, we give a positive answer to a problem posed by Nakagawa, Sakamoto and Yamamoto concerning a nonlinear equation with a fractional derivative.

1. Introduction

In their overview paper concerning the mathematical analysis of fractional equations, Nakagawa, Sakamoto and Yamamoto [7] posed the problem concerning global solutions and blowing-up in a finite time of solutions to the equation

\(\begin{align*}
\frac{CD_0^\alpha}{u(t)} = -u(t)(1-u(t)), & \quad t > 0, \\
\frac{u(0)}{} = u_0,
\end{align*}\)

(1)

where \(\frac{CD_0^\alpha}{g(t)}\) is the Caputo derivative defined for \(g \in C^1[0, T]\) by

\[
\frac{CD_0^\alpha}{g(t)} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} g'(\tau) d\tau,
\]

for \(0 < \alpha < 1\).

Let us recall, in the case \(\alpha = 1\), the results concerning solutions of (1):

- For \(0 < u(0) < 1\), the solution exists globally. Moreover,
  \[|u(t)| \leq \frac{1}{e^t(1-u_0)} \to 0, \quad \text{as} \quad t \to +\infty.\]

- For \(u(0) > 1\), the solution can not exist globally.

Here, we show that the same conclusions are valid for equation (1). Moreover we analyse:

1. The large time behavior of the global solution.
2. The blow-up time and profile of the blowing-up solutions.

Note that if we set \(w = u - 1\), then (1) reads

\[
\frac{CD_0^\alpha}{w(t)} = w(t)(1+w(t)),
\]

which describes the evolution of a certain species; the reaction term \(w(1+w)\) describes the law of increase of the species.


Keywords and phrases: Fractional differential equation, global existence, asymptotic behavior, blow-up time, blow-up profile.
2. Preliminaries

In this section, we present some definitions and results concerning fractional calculus that will be used in the sequel. For more information see [1].

The Riemann-Liouville fractional integral of order \(0 < \alpha < 1\) of the integrable function \(f : \mathbb{R}^+ \to \mathbb{R}\) is

\[
J_0^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad t > 0,
\]

where \(\Gamma(\alpha)\) is the Euler Gamma function.

The Riemann-Liouville fractional derivative of an absolutely continuous function \(f(t)\) of order \(0 < \alpha < 1\) is

\[
D_0^\alpha f(t) := \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t (t - \tau)^{-\alpha} f(\tau) d\tau.
\]

The Caputo fractional derivative of an absolutely continuous function \(f(t)\) of order \(0 < \alpha < 1\) is defined by

\[
CD_0^\alpha f(t) := J_0^{1-\alpha} f'(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - \tau)^{-\alpha} f'(\tau) d\tau.
\]

Both derivatives present a drawback:
- The Riemann-Liouville derivative of a constant is different from zero,
  \(D_0^\alpha C \neq 0\),
- We know that the Riemann-Liouville derivative of the Weierstrass function exists for any \(0 < \alpha < 1\), but not for \(\alpha = 1\).
  But for regular function with \(f(0) = 0\), both definitions coincide.

Next, we recall a lemma that will be used hereafter.

**Lemma 2.1.** (see [3]) Let \(a, b, K, \psi\) be non-negative continuous functions on the interval \(I = (0, T)\), \((0 < T \leq \infty)\), let \(\omega : (0, \infty) \to \mathbb{R}\) be a continuous, non-negative and non-decreasing function with \(\omega(0) = 0\) and \(\omega(u) > 0\) for \(u > 0\), and let \(A(t) = \max_{0 \leq s \leq t} a(s)\) and \(B(t) = \max_{0 \leq s \leq t} b(s)\). Assume that

\[
\psi(t) \leq a(t) + b(t) \int_0^t K(s)\omega(\psi(s)) ds, \quad t \in I.
\]

Then

\[
\psi(t) \leq H^{-1} \left[ H(A(t)) + B(t) \int_0^t K(s)ds \right], \quad t \in (0, T_1),
\]

where \(H(v) = \int_{v_0}^v \frac{d\tau}{\omega(\tau)}\) (\(v \geq v_0 > 0\)), \(H^{-1}\) is the inverse of \(H\) and \(T_1 > 0\) is such that \(H(A(t)) + B(t) \int_0^t K(s)ds \in D(H^{-1})\) for all \(t \in (0, T_1)\).
Here, we consider the problem
\[
\begin{cases}
C^{\alpha}_{D_0^+} u(t) = -u(t)(1-u(t)), \\
u(0) = u_0,
\end{cases}
\]
for \(0 < \alpha < 1\) and \(u_0 > 0\).

3. Main results

The local existence of solutions to (2) is assured by the

THEOREM 3.1. (see [1]) We consider the fractional differential equation of Caputo’s type given by
\[
\begin{cases}
C^{\alpha}_{D_0^+} u(t) = f(t,u(t)), \\
u(0) = u_0.
\end{cases}
\]
For \(0 < \alpha < 1\), \(u_0 \in \mathbb{R}\), \(b > 0\) and \(T > 0\) assume that
1. \(f \in C(R_0, \mathbb{R})\) where \(R_0 = \{(t,u), \ t \leq T, \ |u-u_0| \leq b\}\) and \(|f(t,u)| \leq M\) on \(R_0\);
2. \(|f(t,u) - f(t,v)| \leq L|u-v|, \ L > 0, \ (t,u) \in R_0\).

Then there exists a unique solution \(u \in C([0,h])\) for (3), where
\[
h = \min \left\{ T, \left( \frac{b\Gamma(\alpha+1)}{M} \right)^{\frac{1}{\alpha}} \right\}.
\]

THEOREM 3.2. Let \(u\) be the solution of problem (2). We have:
- If \(0 < u_0 < 1\), the solution is global and it satisfies \(0 < u < 1\). Moreover, \(u\) is given by
  \[u(t) = E^\alpha_{\alpha}(-t^\alpha)u_0 + \int_0^t (t-s)^{\alpha-1} E^\alpha_{\alpha}(-(t-s)^\alpha) u^2(s)ds,\]
  and for some constants \(c > 0\) and \(c_1 > 0\), we have
  \[0 < u(t) \leq \frac{1}{c u_0 c_1^{1/\alpha}}, \ 0 < t < T_0 := \left( \frac{\alpha}{c_1 c u_0} \right)^{\frac{1}{\alpha}}.\]
In addition, we have for \(0 < \varepsilon < 1\)
\[u(t) \leq \frac{c}{1+\varepsilon(1-\varepsilon)t^{\alpha}}, \ t > 0\]
- If \(u_0 > 1\), the solution blows-up in a finite time \(T^* : \lim_{t \to T^*} u(t) = +\infty.\)
Moreover, we have the bilateral estimate:
\[\underline{w}(t) + 1 \leq u(t) \leq \overline{w}(t) + 1,\]
and
\[
\left( \frac{\Gamma(\alpha + 1)}{4(u_0 - \frac{1}{2})} \right)^\frac{1}{\alpha} \leq T^* \leq \left( \frac{\Gamma(\alpha + 1)}{u_0 - \frac{1}{2}} \right)^\frac{1}{\alpha},
\]
where
\[
\tilde{w}(t) + \frac{1}{2} \sim \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} (T_{\tilde{w}} - t)^{-\alpha}, \quad \text{as } t \to T_{\tilde{w}},
\]
\[
\bar{w}(t) \sim \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} (T_{\bar{w}} - t)^{-\alpha}, \quad \text{as } t \to T_{\bar{w}}.
\]

Here, \(T_{\tilde{w}}\) is the blow-up time of \(\tilde{w}\), which satisfies
\[
\left( \frac{\Gamma(\alpha + 1)}{4(u_0 - \frac{1}{2})} \right)^\frac{1}{\alpha} \leq T_{\tilde{w}} \leq \left( \frac{\Gamma(\alpha + 1)}{u_0 - \frac{1}{2}} \right)^\frac{1}{\alpha},
\]
and \(T_{\bar{w}}\) is the blow-up time of \(\bar{w}\), which satisfies
\[
\left( \frac{\Gamma(\alpha + 1)}{4(u_0 - \frac{1}{2})} \right)^\frac{1}{\alpha} \leq T_{\bar{w}} \leq \left( \frac{\Gamma(\alpha + 1)}{u_0 - \frac{1}{2}} \right)^\frac{1}{\alpha}.
\]

**Proof of Theorem 3.2.**

**Part I.** If \(0 < u_0 < 1\), then the solution is global. The solution to (2) is given by
\[
u(t) = E_\alpha(-t^\alpha)u_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha) u^2(s)ds.
\] (4)

Where the Mittag-Leffler functions \(E_\alpha(-t^\alpha)\) and \(E_{\alpha,\alpha}(-t^\alpha)\) are defined by:
\[
E_\alpha(-t^\alpha) = \sum_{j=0}^{\infty} \frac{(-1)^j t^{\alpha j}}{\Gamma(\alpha j + 1)},
\]
\[
E_{\alpha,\alpha}(-t^\alpha) = \sum_{j=0}^{\infty} \frac{(-1)^j t^{\alpha j \alpha}}{\Gamma(\alpha j + \alpha)}.
\]

If \(u_0 > 0\), then \(u(t) > 0\) as \(E_\alpha(-t^\alpha) > 0\) and \(E_{\alpha,\alpha}(-t^\alpha) > 0\).

Now, we set the function \(\pi(t) = 1, \quad t > 0\).

As \(0 < u_0 < 1\), then \(u_0 < \pi(0)\). In addition, we have
\[
^CD^{\alpha}_{0+} \pi(t) = 0 = -\pi(t)(1 - \pi(t)).
\]

Hence \(\pi\) is an upper solution of the equation (2), and we have \(u(t) < \pi(t) = 1\), (see [6], Thm. 2.4.3, p. 32).

Now, we examine the large time behavior of the global solution \(0 < u < 1\).

For, let us recall the estimates (see [5]):

- For \(0 < \alpha < 1\) and \(\mu > 0\), there exists a constant \(c > 0\) such that,
\[
0 < E_\alpha(-\mu t^\alpha) \leq \frac{c}{1 + \mu t^\alpha} \leq c, \quad t > 0.
\] (5)
• For $0 < \alpha < 1$, there exists a constant $c_1 > 0$ such that
\[
0 < t^{\alpha-1} E_{\alpha,1}(-t^\alpha) \leq c_1 t^{\alpha-1}, \quad t > 0.
\]  
(6)

From (4) and using the inequalities (5) and (6), we obtain
\[
u(t) \leq cu_0 + c_1 \int_0^t (t-s)^{\alpha-1} u^2(s) ds.
\]  
(7)

We apply Lemma 2.1 to (7) with $\omega(x) = x^2$, $K(s) = (t-s)^{\alpha-1}$, $A(t) = cu_0$, $B(t) = c_1$.

For $0 < t < T_0$, we have
\[
H(cu_0) + \frac{c_1 t^\alpha}{\alpha} \in D(H^{-1}),
\]
where $H(v) = \frac{1}{v_0} - \frac{1}{v}$ and $H^{-1}(z) = \frac{1}{v_0 - z}$, $z \neq \frac{1}{v_0}$.

So we obtain,
\[
u(t) \leq H^{-1}\left[H(cu_0) + \frac{c_1 t^\alpha}{\alpha}\right].
\]

Therefore
\[
u(t) \leq \frac{1}{cu_0 - \frac{c_1 t^\alpha}{\alpha}}, \quad 0 < t < T_0.
\]

Let $0 < \varepsilon < 1$, we observe that for the function $\overline{u}(t) := E_{\alpha,1}(-\varepsilon(1-\varepsilon)t^\alpha)$ we have
\[
\left\{
\begin{array}{l}
\mathcal{CD}_{0+}^\alpha \overline{u}(t) = -\varepsilon(1-\varepsilon)\overline{u}(t) \\
\overline{u}(0) = 1 > u(0),
\end{array}
\right.
\]

from the comparison principle [6] and inequality (5) it follows that
\[
u(t) \leq \frac{c}{1 + \varepsilon(1-\varepsilon)t^\alpha}, \quad t > 0.
\]

Part 2. If $u_0 > 1$, then the solution blows-up in a finite time.

• We show that $u > 1$. For, let us define the new unknown function $w = u - 1$.

The function $w$ satisfies
\[
\left\{
\begin{array}{l}
\mathcal{CD}_{0+}^\alpha w(t) = w(t)(1+w(t)), \\
w(0) := w_0 = u_0 - 1.
\end{array}
\right.
\]  
(8)

As $u_0 > 1$, then $w_0 > 0$. Moreover, we have ([1])
\[
w(t) = E_{\alpha}(t^\alpha)w_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha) w^2(s) ds.
\]

Therefore, $w > 0$; hence $u > 1$.

• We prove that $u$ blows-up in a finite time.
Since we have \( w(t) = u(t) - 1 \), it is seen that if \( u(t) \to \infty \) as \( t \to T^* \), then \( w(t) \to \infty \) as \( t \to T^* \) and vice versa. That is \( w \) and \( u \) will have the same blow-up time.

We now must examine the blow-up properties of \( w \), the solution of problem (8). These are obtained by comparing \( w(t) \) with the solutions of the following problems:

\[
\begin{cases}
C D^{\alpha}_0 w(t) = \overline{w}^2(t), \\
\overline{w}(0) = w_0,
\end{cases}
\]

(9)

and

\[
\begin{cases}
C D^{\alpha}_0 \tilde{w}(t) = (\tilde{w}(t) + \frac{1}{2})^2, \\
\tilde{w}(0) = w_0.
\end{cases}
\]

(10)

We see by comparison ([6]) that

\[
\overline{w}(t) \leq w(t) \leq \tilde{w}(t), \quad 0 \leq t < \min\{T_{\overline{w}}, T_{\tilde{w}}\}.
\]

Following the paper of Kirk, Olmstead and Roberts [4], we may assert that the solution \( \overline{w} \) (resp. \( \tilde{w} \)) blows-up in a finite time \( T_{\overline{w}} \) (resp. \( T_{\tilde{w}} \)), such that

\[
\left( \frac{\Gamma(\alpha + 1)}{4w_0} \right)^\frac{1}{\alpha} \leq T_{\overline{w}} \leq \left( \frac{\Gamma(\alpha + 1)}{w_0} \right)^\frac{1}{\alpha},
\]

and

\[
\left( \frac{\Gamma(\alpha + 1)}{4(w_0 + \frac{1}{2})} \right)^\frac{1}{\alpha} \leq T_{\tilde{w}} \leq \left( \frac{\Gamma(\alpha + 1)}{w_0 + \frac{1}{2}} \right)^\frac{1}{\alpha}.
\]

So we have the following estimates

\[ T_{\overline{w}} \leq T^* \leq T_{\tilde{w}}. \]

Whereupon

\[
\left( \frac{\Gamma(\alpha + 1)}{4(w_0 + \frac{1}{2})} \right)^\frac{1}{\alpha} \leq T^* \leq \left( \frac{\Gamma(\alpha + 1)}{w_0} \right)^\frac{1}{\alpha}. \quad \square
\]

4. Numerical implementation

In this section, we will approximate the solution \( u \) given by (4). For, we need a numerical approximation of the convolution integral; this can be obtained using the convolution quadrature method.

As it has been explained in [2], a convolution quadrature approximates the continuous convolution

\[
\int_0^t K(t-s)f(s)ds, \quad t > 0,
\]

by a discrete convolution with a step size \( h > 0 \). Then

\[
\int_0^t K(t_n-s)f(s)ds \sim \sum_{j=0}^n \omega_{n-j}f(t_j),
\]
where \( t_j = jh, \ j = 0, 1, 2, \ldots, n \) and the convolution quadrature weights \( \omega_j \) are determined from their generating power series as

\[
\sum_{j=0}^{\infty} \omega_j \zeta^j = \mathcal{L}\{K(t) : \frac{\delta(\zeta)}{h}\}.
\]

Here \( \mathcal{L}\{K(t) : s\} \) is the Laplace transform of \( K(t) \) and \( \delta(\zeta) \) is the generating polynomial for a linear multistep method.

Let \( u_n \) be the approximation of \( u(t_n) \) for \( n \geq 0 \). Using the convolution quadrature method we obtain

\[
u_n = (1 - \omega_0)^{-1} \left[ E_\alpha(-t^\alpha)u_0 + \sum_{j=0}^{n-1} \omega_{n-j} u_j \right], \quad n = 1, 2, 3, \ldots
\]

Now, we introduce the following algorithm which gives the numerical approximation of solution to equation (2).

**ALGORITHM.**

**Input:** Give \( \alpha, \ 0 < \alpha < 1 \) and \( u_0, \ u_0 > 1 \).

**Initializations:** Discretize the time with a step size \( h > 0; \ t_i = ih, \) for all \( i = 1, 2, \ldots, n, \ u_{i\text{appx}}^1 = u_0, \ u^1 = (u_0)^2.\)

**Step 1:** Approximate the Mittag-Leffler function GML.

**Step 2:** Calculate convolution quadrature weights \( W \) using the fast Fourier transform (FFT).

**Step 3:** Calculate \( u_{i\text{appx}}^i. \)

\[
u_i = \text{GML} \ast u_{i\text{appx}}^1 + W \ast u^{i-1},
\]

\[
u_{i\text{appx}}^i = (1 - W(1))^{-1} \ast u^i.
\]

**do**

\[
u_i = (u_{i\text{appx}}^i)^2.
\]

\[i = i + 1.
\]

**until** \( (u_{i\text{appx}}^i \text{ blows up}) \) or \( (i > n) \).

**Output:** Numerical approximation of \( u \).

**EXAMPLE 1.** For Figure 1, we set \( \alpha = 0.5 \); the initial conditions are respectively \( u_0 = 5, \ u_0 = 3 \) and \( u_0 = 2 \).

For Figure 2, we take the initial condition \( u_0 = 5 \) and we plot the solutions; the dotted curve is the solution for \( \alpha = 0.3 \) and the solid curve corresponds to the solution for \( \alpha = 0.5 \).

As it has been proved, the solution blows up in a finite time which depends on \( u_0 \) and \( \alpha \).
Figure 1: Solutions for $\alpha = 0.5$ and $u_0 = 5, 3, 2$.

Figure 2: Solutions for $u_0 = 5$ and $\alpha = 0.3, 0.5$.

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(Received August 31, 2012)

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