Fractional
Differential
Calculus
Volume 4, Number 1 (2014), 45–53 doi:10.7153/fdc-04-03

BLOWING–UP SOLUTIONS AND GLOBAL SOLUTIONS TO A FRACTIONAL DIFFERENTIAL EQUATION

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Abstract. In this paper, we give a positive answer to a problem posed by Nakagawa, Sakamoto and Yamamoto concerning a nonlinear equation with a fractional derivative.

1. Introduction

In their overview paper concerning the mathematical analysis of fractional equations, Nakagawa, Sakamoto and Yamamoto [7] posed the problem concerning global solutions and blowing-up in a finite time of solutions to the equation

$$\begin{cases} {}^{C}D^{\alpha}_{0_+}u(t) = -u(t)(1-u(t)), \ t>0, \\ u(0) = u_0, \end{cases} \tag{1}$$

where ${}^{C}D^{\alpha}_{0_+}$ is the Caputo derivative defined for $g \in C^1[0,T]$ by

$$ {}^{C}D^{\alpha}_{0_+}g(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}g'(\tau)\,d\tau, $$

for $0<\alpha<1$.

Let us recall, in the case $\alpha = 1$, the results concerning solutions of (1):

- For $0<u(0)<1$, the solution exists globally. Moreover,

$$ |u(t)| \leqslant \frac{1}{e^t(1-u_0)} \longrightarrow 0, \text{ as } t \longrightarrow +\infty. $$

- For $u(0)>1$, the solution can not exist globally.

Here, we show that the same conclusions are valid for equation (1). Moreover we analyse:

1. The large time behavior of the global solution.
2. The blow-up time and profile of the blowing-up solutions.

Note that if we set $w = u-1$, then (1) reads

$$ {}^{C}D^{\alpha}_{0_+}w(t) = w(t)(1+w(t)), $$

which describes the evolution of a certain species; the reaction term $w(1+w)$ describes the law of increase of the species.

Mathematics subject classification (2010): 34A08, 34B10.
Keywords and phrases: Fractional differential equation, global existence, asymptotic behavior, blow-up time, blow-up profile.

Paper FDC-04-03

2. Preliminaries

In this section, we present some definitions and results concerning fractional calculus that will be used in the sequel. For more information see [1].

The Riemann-Liouville fractional integral of order $0<\alpha<1$ of the integrable function $f:\mathbb{R}^+\longrightarrow\mathbb{R}$ is

$$J_{0_+}^{\alpha}f(t):=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)\,d\tau,\quad t>0,$$

where $\Gamma(\alpha)$ is the Euler Gamma function.

The Riemann-Liouville fractional derivative of an absolutely continous function $f(t)$ of order $0<\alpha<1$ is

$$D_{0_+}^{\alpha}f(t):=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t(t-\tau)^{-\alpha}f(\tau)\,d\tau.$$

The Caputo fractional derivative of an absolutely continous function $f(t)$ of order $0<\alpha<1$ is defined by

$${}^{C}D_{0_+}^{\alpha}f(t):=J_{0_+}^{1-\alpha}f'(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-\tau)^{-\alpha}f'(\tau)\,d\tau.$$

Both derivatives present a drawback:

• The Riemann-Liouville derivative of a constant is different from zero,

$$D_{0_+}^{\alpha}C\neq 0,$$

while the Caputo derivative require $f'(t)$ to calculate ${}^{C}D_{0_+}^{\alpha}f(t)$, for $0<\alpha<1$.

• We know that the Riemann-Liouville derivative of the Weierstrass function exists for any $0<\alpha<1$, but not for $\alpha=1$.

But for regular function with $f(0)=0$, both definitions coincide.

Next, we recall a lemma that will be used hereafter.

LEMMA 2.1. (see [3]) *Let $a,\ b,\ K,\ \psi$ be non-negative continuous functions on the interval $I=(0,T)$, $(0<T\leqslant\infty)$, let $\omega:(0,\infty)\longrightarrow\mathbb{R}$ be a continuous, non-negative and non-decreasing function with $\omega(0)=0$ and $\omega(u)>0$ for $u>0$, and let $A(t)=\max_{0\leqslant s\leqslant t}a(s)$ and $B(t)=\max_{0\leqslant s\leqslant t}b(s)$. Assume that*

$$\psi(t)\leqslant a(t)+b(t)\int_0^t K(s)\omega(\psi(s))\,ds,\ \ t\in I.$$

Then

$$\psi(t)\leqslant H^{-1}\Big[H(A(t))+B(t)\int_0^t K(s)ds\Big],\ \ t\in(0,T_1),$$

where $H(v)=\int_{v_0}^{v}\frac{d\tau}{\omega(\tau)}$ $(v\geqslant v_0>0)$, H^{-1} is the inverse of H and $T_1>0$ is such that $H(A(t))+B(t)\int_0^t K(s)ds\in D(H^{-1})$ for all $t\in(0,T_1)$.

Here, we consider the problem

$$\begin{cases} {}^{C}D^{\alpha}_{0_+}u(t) = -u(t)(1-u(t)), \\ u(0) \quad = u_0, \end{cases} \tag{2}$$

for $0 < \alpha < 1$ and $u_0 > 0$.

3. Main results

The local existence of solutions to (2) is assured by the

THEOREM 3.1. (see [1]) *We consider the fractional differential equation of Caputo's type given by*

$$\begin{cases} {}^{C}D^{\alpha}_{0_+}u(t) = f(t,u(t)), \ t > 0, \\ u(0) \quad = u_0. \end{cases} \tag{3}$$

For $0 < \alpha < 1$, $u_0 \in \mathbb{R}$, $b > 0$ and $T > 0$ assume that

1. $f \in C(R_0,\mathbb{R})$ *where* $R_0 = \{(t,u), \ 0 \leqslant t \leqslant T, \ |u-u_0| \leqslant b\}$ *and* $|f(t,u)| \leqslant M$ *on* R_0*;*
2. $|f(t,u) - f(t,v)| \leqslant L|u-v|, \ L > 0, \ (t,u) \in R_0$.

Then there exists a unique solution $u \in C([0,h])$ for (3), where

$$h = \min\Big\{T, \Big(\frac{b\Gamma(\alpha+1)}{M}\Big)^{\frac{1}{\alpha}}\Big\}.$$

THEOREM 3.2. *Let u be the solution of problem (2). We have:*
• *If $0 < u_0 < 1$, the solution is global and it satisfies $0 < u < 1$. Moreover, u is given by*

$$u(t) = E_{\alpha}(-t^{\alpha})u_0 + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^{\alpha})\, u^2(s)ds,$$

and for some constants $c > 0$ and $c_1 > 0$, we have

$$0 < u(t) \leqslant \frac{1}{\frac{1}{cu_0} - \frac{c_1}{\alpha}t^{\alpha}}, \quad 0 < t < T_0 := \Big(\frac{\alpha}{c_1 c u_0}\Big)^{\frac{1}{\alpha}}.$$

In addition, we have for $0 < \varepsilon < 1$

$$u(t) \leqslant \frac{c}{1+\varepsilon(1-\varepsilon)t^{\alpha}}, \ t > 0$$

• *If $u_0 > 1$, the solution blows-up in a finite time T^*: $\lim_{t \longrightarrow T^*} u(t) = +\infty$.*
Moreover, we have the bilateral estimate:

$$\overline{w}(t) + 1 \leqslant u(t) \leqslant \widetilde{w}(t) + 1,$$

and

$$\left(\frac{\Gamma(\alpha+1)}{4(u_0-\frac{1}{2})}\right)^{\frac{1}{\alpha}} \leqslant T^* \leqslant \left(\frac{\Gamma(\alpha+1)}{u_0-1}\right)^{\frac{1}{\alpha}},$$

where

$$\widetilde{w}(t)+\frac{1}{2} \sim \frac{\Gamma(2\alpha)}{\Gamma(\alpha)}(T_{\widetilde{w}}-t)^{-\alpha}, \ \ as \ \ t \longrightarrow T_{\widetilde{w}},$$

$$\overline{w}(t) \sim \frac{\Gamma(2\alpha)}{\Gamma(\alpha)}(T_{\overline{w}}-t)^{-\alpha}, \ \ as \ \ t \longrightarrow T_{\overline{w}}.$$

Here, $T_{\widetilde{w}}$ is the blow-up time of $\widetilde{w}$, which satisfies

$$\left(\frac{\Gamma(\alpha+1)}{4(u_0-\frac{1}{2})}\right)^{\frac{1}{\alpha}} \leqslant T_{\widetilde{w}} \leqslant \left(\frac{\Gamma(\alpha+1)}{u_0-\frac{1}{2}}\right)^{\frac{1}{\alpha}},$$

and $T_{\overline{w}}$ is the blow-up time of $\overline{w}$, which satisfies

$$\left(\frac{\Gamma(\alpha+1)}{4(u_0-1)}\right)^{\frac{1}{\alpha}} \leqslant T_{\overline{w}} \leqslant \left(\frac{\Gamma(\alpha+1)}{u_0-1}\right)^{\frac{1}{\alpha}}.$$

Proof of Theorem 3.2.

Part 1. If $0<u_0<1$, then the solution is global. The solution to (2) is given by

$$u(t) = E_\alpha(-t^\alpha)u_0 + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^\alpha)\, u^2(s)ds. \tag{4}$$

Where the Mittag-Leffler functions $E_\alpha(-t^\alpha)$ and $E_{\alpha,\alpha}(-t^\alpha)$ are defined by:

$$E_\alpha(-t^\alpha) = \sum_{j=0}^{\infty} \frac{(-1)^j t^{\alpha j}}{\Gamma(\alpha j+1)},$$

$$E_{\alpha,\alpha}(-t^\alpha) = \sum_{j=0}^{\infty} \frac{(-1)^j t^{\alpha j}}{\Gamma(\alpha j+\alpha)}.$$

If $u_0>0$, then $u(t)>0$ as $E_\alpha(-t^\alpha)>0$ and $E_{\alpha,\alpha}(-t^\alpha)>0$.

Now, we set the function $\overline{u}(t)=1, \ \ t>0$.

As $0<u_0<1$, then $u_0<\overline{u}(0)$. In addition, we have

$${}^{C}D^{\alpha}_{0_+}\overline{u}(t) = 0 = -\overline{u}(t)(1-\overline{u}(t)).$$

Hence $\overline{u}$ is an upper solution of the equation (2), and we have $u(t)<\overline{u}(t)=1$, (see [6], Thm. 2.4.3, p. 32).

Now, we examine the large time behavior of the global solution $0<u<1$.

For, let us recall the estimates (see [5]):

- For $0<\alpha<1$ and $\mu>0$, there exists a constant $c>0$ such that,

$$0 < E_\alpha(-\mu t^\alpha) \leqslant \frac{c}{1+\mu t^\alpha} \leqslant c, \quad t>0. \tag{5}$$

- For $0 < \alpha < 1$, there exists a constant $c_1 > 0$ such that

$$0 < t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha}) \leqslant c_1 t^{\alpha-1}, \quad t > 0. \tag{6}$$

From (4) and using the inequalities (5) and (6), we obtain

$$u(t) \leqslant cu_0 + c_1 \int_0^t (t-s)^{\alpha-1} u^2(s)ds. \tag{7}$$

We apply Lemma 2.1 to (7) with $\omega(x) = x^2$, $K(s) = (t-s)^{\alpha-1}$, $A(t) = cu_0$, $B(t) = c_1$.

For $0 < t < T_0$, we have

$$H(cu_0) + \frac{c_1}{\alpha}t^{\alpha} \in D(H^{-1}),$$

where $H(v) = \frac{1}{v_0} - \frac{1}{v}$ and $H^{-1}(z) = \frac{1}{\frac{1}{v_0} - z}$, $z \neq \frac{1}{v_0}$.

So we obtain,

$$u(t) \leqslant H^{-1}\left[H(cu_0) + \frac{c_1}{\alpha}t^{\alpha}\right].$$

Therefore

$$u(t) \leqslant \frac{1}{\frac{1}{cu_0} - \frac{c_1}{\alpha}t^{\alpha}}, \quad 0 < t < T_0.$$

Let $0 < \varepsilon < 1$, we observe that for the function $\overline{u}(t) := E_{\alpha,1}(-\varepsilon(1-\varepsilon)t^{\alpha})$ we have

$$\begin{cases} {}^{C}D^{\alpha}_{0+}\overline{u}(t) &= -\varepsilon(1-\varepsilon)\overline{u}(t) \\ \overline{u}(0) &= 1 > u(0), \end{cases}$$

from the comparaison principle [6] and inequality (5) it follows that

$$u(t) \leqslant \frac{c}{1+\varepsilon(1-\varepsilon)t^{\alpha}}, \quad t > 0.$$

Part 2. If $u_0 > 1$, then the solution blows-up in a finite time.

- We show that $u > 1$. For, let us define the new unknown function $w = u - 1$. The function w satisfies

$$\begin{cases} {}^{C}D^{\alpha}_{0_+}w(t) &= w(t)(1+w(t)), \\ w(0) := w_0 &= u_0 - 1. \end{cases} \tag{8}$$

As $u_0 > 1$, then $w_0 > 0$. Moreover, we have ([1])

$$w(t) = E_{\alpha}(t^{\alpha})w_0 + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}((t-s)^{\alpha})\, w^2(s)ds.$$

Therefore, $w > 0$; hence $u > 1$.

- We prove that u blows-up in a finite time.

Since we have $w(t) = u(t) - 1$, it is seen that if $u(t) \longrightarrow \infty$ as $t \longrightarrow T^*$, then $w(t) \longrightarrow \infty$ as $t \longrightarrow T^*$ and vice versa. That is w and u will have the same blow-up time.

We now must examine the blow-up properties of w, the solution of problem (8). These are obtained by comparing $w(t)$ with the solutions of the following problems:

$$\begin{cases} {}^C D^{\alpha}_{0_+} \overline{w}(t) = \overline{w}^{\,2}(t), \\ \overline{w}(0) \quad = w_0, \end{cases} \tag{9}$$

and

$$\begin{cases} {}^C D^{\alpha}_{0_+} \widetilde{w}(t) = (\widetilde{w}(t) + \frac{1}{2})^2, \\ \widetilde{w}(0) \quad = w_0. \end{cases} \tag{10}$$

We see by comparaison ([6]) that

$$\overline{w}(t) \leqslant w(t) \leqslant \widetilde{w}(t), \;\; 0 \leqslant t < \min\{T_{\overline{w}}, T_{\widetilde{w}}\}.$$

Following the paper of Kirk, Olmstead and Roberts [4], we may assert that the solution $\overline{w}$ (resp. $\widetilde{w}$) blows-up in a finite time $T_{\overline{w}}$ (resp. $T_{\widetilde{w}}$), such that

$$\Big(\frac{\Gamma(\alpha+1)}{4w_0}\Big)^{\frac{1}{\alpha}} \leqslant T_{\overline{w}} \leqslant \Big(\frac{\Gamma(\alpha+1)}{w_0}\Big)^{\frac{1}{\alpha}},$$

and

$$\Big(\frac{\Gamma(\alpha+1)}{4(w_0+\frac{1}{2})}\Big)^{\frac{1}{\alpha}} \leqslant T_{\widetilde{w}} \leqslant \Big(\frac{\Gamma(\alpha+1)}{w_0+\frac{1}{2}}\Big)^{\frac{1}{\alpha}}.$$

So we have the following estimates

$$T_{\widetilde{w}} \leqslant T^* \leqslant T_{\overline{w}}.$$

Whereupon

$$\Big(\frac{\Gamma(\alpha+1)}{4(w_0+\frac{1}{2})}\Big)^{\frac{1}{\alpha}} \leqslant T^* \leqslant \Big(\frac{\Gamma(\alpha+1)}{w_0}\Big)^{\frac{1}{\alpha}}. \quad \square$$

4. Numerical implementation

In this section, we will approximate the solution u given by (4). For, we need a numerical approximation of the convolution integral; this can be obtained using the convolution quadrature method.

As it has been explained in [2], a convolution quadrature approximates the continuous convolution

$$\int_0^t K(t-s) f(s)\, ds, \;\; t > 0,$$

by a discrete convolution with a step size $h > 0$. Then

$$\int_0^{t_n} K(t_n - s) f(s)\, ds \sim \sum_{j=0}^{n} \omega_{n-j} f(t_j),$$

where $t_j = jh,\ j = 0,1,2,...,n$ and the convolution quadrature weights ω_j are determined from their generating power series as

$$\sum_{j=0}^{\infty} \omega_j \zeta^j = \mathscr{L}\Big\{K(t) : \frac{\delta(\zeta)}{h}\Big\}.$$

Here $\mathscr{L}\{K(t) : s\}$ is the Laplace transform of $K(t)$ and $\delta(\zeta)$ is the generating polynomial for a linear multistep method.

Let u_n be the approximation of $u(t_n)$ for $n \geqslant 0$. Using the convolution quadrature method we obtain

$$u_n = (1-\omega_0)^{-1}\Big[E_\alpha(-t^\alpha)u_0 + \sum_{j=0}^{n-1} \omega_{n-j}u_j\Big],\ \ n = 1,2,3....$$

Now, we introduce the following algorithm which gives the numerical approximation of solution to equation (2).

ALGORITHM.

Input: Give α, $0 < \alpha < 1$ and u_0, $u_0 > 1$.

Initializations: Discretize the time with a step size $h > 0$; $t_i = ih$, for all $i = 1,2,...,n$, $u^1_{appx} = u_0$, $u^1 = (u_0)^2$.

Step 1: Approximate the Mittag-Leffler function **GML**.

Step 2: Calculate convolution quadrature weights **W** using the fast Fourier transform (FFT).

Step 3: Calculate u^i_{appx}.

do
$u^i = \mathbf{GML} * u^1_{appx} + \mathbf{W} * u^{i-1}$.
$u^i_{appx} = (1-\mathbf{W}(1))^{-1} * u^i$.
$u^i = (u^i_{appx})^2$.
$i = i+1$.

until (u^i_{appx} blows up) or ($i > n$).

Output: Numerical approximation of u.

EXAMPLE 1. For *Figure*1, we set $\alpha = 0.5$; the initial conditions are respectively $u_0 = 5,\ u_0 = 3$ and $u_0 = 2$.

For *Figure*2, we take the initial condition $u_0 = 5$ and we plot the solutions; the dotted curve is the solution for $\alpha = 0.3$ and the solid curve corresponds to the solution for $\alpha = 0.5$.

As it has been proved, the solution blows up in a finite time which depends on u_0 and α.

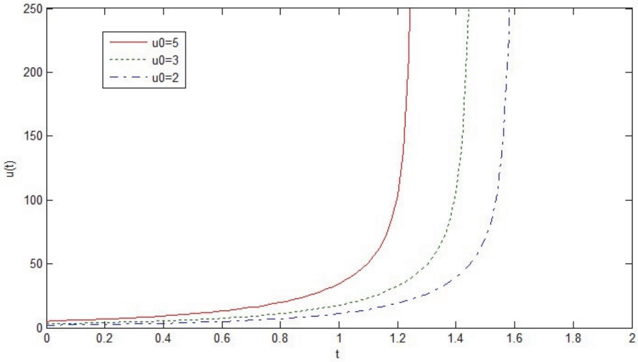

Figure 1: *Solutions for* $\alpha = 0.5$ *and* $u_0 = 5,\ 3,\ 2.$

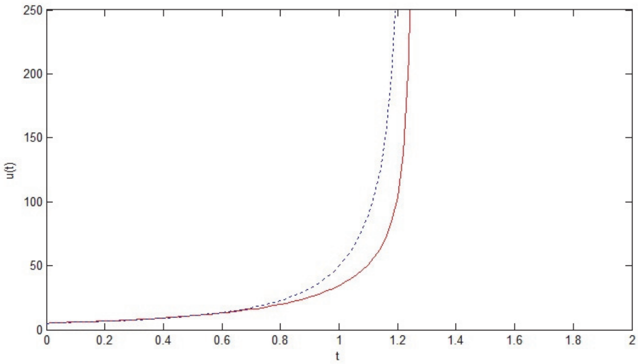

Figure 2: *Solutions for* $u_0 = 5$ *and* $\alpha = 0.3,\ 0.5.$

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(Received August 31, 2012)

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