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EXISTENCE OF MILD SOLUTIONS FOR NONLOCAL SEMILINEAR FRACTIONAL EVOLUTION EQUATIONS

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Abstract. In this paper, we investigate a class of semilinear fractional evolution equations with nonlocal initial conditions given by

$$(1) \qquad \begin{cases} \dfrac{\mathrm{d}^q u(t)}{\mathrm{d}t^q} = Au(t) + (Fu)(t), \quad t \in I, \\ u(0) + g(u) = u_0, \end{cases}$$

where $0 < q < 1$, I is a compact interval. Sufficient conditions for the existence of mild solutions for the equation (1) are derived. The main tools include Laplace transform, Arzela-Ascoli's Theorem, Schauder's fixed point theorem and Operator theorem.

1. Introduction

In this paper, we are concerned with the existence of mild solutions for the following semilinear fractional evolution equations with nonlocal initial conditions

$$\begin{cases} \dfrac{\mathrm{d}^q u(t)}{\mathrm{d}t^q} = Au(t) + (Fu)(t), \quad t \in I, \\ u(0) + g(u) = u_0, \end{cases} \tag{1.1}$$

where $\mathrm{d}^q u(t)/\mathrm{d}t^q$ is the standard Riemann-Liouville fractional derivative, $0 < q < 1$, $I = [0,T]$ is a compact interval, $A : D(A) \subseteq X \to X$ is a closed bounded linear operator on a Banach space X, $g : C(I,X) \to X$ is a given X-valued function and $F : C(I,X) \to L^p(I,X)$ is a given nonlinear operator.

The fractional derivative is understood in the Riemann-Liouville sense. The origin of fractional calculus goes back to Newton and Leibnitz in the seventieth century. One observes that fractional order can be very complex in viewpoint of mathematics and they have recently proved to be valuable in various fields of science and engineering. In fact, one can find numerous applications in electrochemistry, electromagnetism, viscoelasticity, biology and hydrogeology. For example space-fractional diffusion equations have been used in groundwater hydrology to model the transport of passive tracers carried by fluid flow in a porous medium [1, 2] or to model activator-inhibitor dynamics with anomalous diffusion [3]. For details, see [4–7] and the references therein.

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Differential equations of fractional order have appeared in many branches of physics and technical sciences [8, 9]. It has seen considerable development in the last decade, see [3–25] and the references therein. Recently, the existence and uniqueness problem for various fractional differential equations were considered by Ahmad [10], Bhaskar [11], Zhou [12–14], Lakshmikantham and Leela [15] et al. The nonlocal Cauchy problem was considered by Anguraj, Karthikeyan and N'Guèrèkata [16], and the importance of nonlocal initial conditions in different fields has been discussed in [6, 7] and the references therein.

The equation (1.1) can be viewed as a generalization of the following equation

$$\begin{cases} \dfrac{\mathrm{d}^q u(t)}{\mathrm{d}t^q} = Au(t) + f(t,u(t)), \quad t \in I, \\ u(0) + g(u) = u_0, \end{cases} \tag{1.2}$$

where $f : I \times X \to X$ is a given X-valued function. Indeed, under suitable conditions, the operator $(Fu)(t) = f(t,u(t))$ maps $C(I,X)$ into $L^p(I,X)$. Thus the equation (1.2) reads equation (1.1).

The nonlocal problem (1.2) was motivated by physical problems. Indeed, the nonlocal initial condition $u(0)+g(u) = u_0$ can be applied in physics with better effect than the classical initial condition $u(0) = u_0$. For this reason, the equation (1.2) has gotten considerable attention in recent years, see [26, 27, 28] and the references therein. See also [29, 30, 31] and the references therein for recent generalizations of equation (1.2) to various kinds of differential equations.

In this paper, we are interested in the case that A generates a compact C_0-semigroup. In [32] the Leray-Schauder Alternative was used to study the existence of solutions for the equation (1.2) in which $q = 1$. However, as is shown in [33], the proof of the main results in [32] does not work because the most important place at $t = 0$ was neglected when checking the compactness of the solution operator. To fill this gap, some authors added conditions on the compactness of g, see e.g. [34, 35, 33, 36, 37] and the references therein. However, in application to physics, these conditions is too strong. In fact, in many references on nonlocal Cauchy problems (see e.g. [38, 39, 40, 41]), the mapping g is given by

$$g(u) = \sum_{i=1}^{p} c_i u(t_i), \tag{1.3}$$

where $0 < t_1 < t_2 < \cdots < t_p \leqslant T$, $c_1, c_2, \cdots, c_p$ are given constants. Obviously, compactness condition is not valid for g.

Without assumptions on the compactness of g, Liang, Liu and Xiao [40] developed a method to deal with the case that f is Lipschitz continuous in the second argument. The main assumptions in [40] are that the function g depends only on u in the interval $[\delta, T]$ for some constant $\delta > 0$. Clearly, these assumptions cover (1.3). The authors defined a solution operator by different methods. In this way, the point $t = 0$ is not needed to be considered. By Schauder's Fixed Point Theorem, the authors obtained the existence of mild solutions for the equation (1.2) in which $q = 1$.

The present paper is motivated by the following facts. Firstly, to the best of our knowledge, the more general and more valuable fractional differential equations were considered seldom in this case. Secondly, the approach used in [40] relies on the assumption that the function f is Lipschitz continuous with respect to the second variable. Thirdly, as is mentioned in the previous paragraph, the classes of equations covered in [40] do not include the general case where $g(u)$ depends on u on the whole interval $[0,T]$. For example, the following nonlocal problem

$$\begin{cases} \dfrac{\mathrm{d}^q u(t)}{\mathrm{d}t^q} = Au(t) + f(t,u(t)), \ \ t \in I, \\ u(0) + \displaystyle\int_0^T h(s,u(s))ds = u_0. \end{cases} \tag{1.4}$$

Neither the compactness condition nor the assumptions given in [40] are satisfied for g.

In this work, we continue discussing the existence of mild solutions for the nonlocal problem (1.2) under more general hypotheses. We consider the more general nonlocal Cauchy problem (1.1). Firstly, we prove an existence result for equation (1.1) under the assumptions that g and F depend only on u in the interval $[\delta,T]$ for some constant $\delta > 0$. Subsequently, we construct a family of nonlocal Cauchy problems $\{(Q_{\delta_n}) : n \in \mathbb{N}\}$ such that the above conditions are satisfied. For every $n \in \mathbb{N}$, we obtain one mild solution of the problem (Q_{δ_n}), say, u_n by the above result. Finally, a mild solution of the original equation (1.1) was obtained by a diagonal argument.

2. Some Lemmas

We begin this section by giving some notations. $\mathbb{N}$ and $\mathbb{R}_+$ stand for the set of natural and nonnegative numbers, respectively. Denote by X a Banach space with norm $\|\cdot\|$. Let A be the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geqslant 0}$ on X. Denote by $C(I,X)$ the Banach space of all continuous functions from I to X with the norm

$$\|u\|_\infty = \max\{\|u(t)\| : t \in I\},$$

and by $L^p(I,X)$ the Banach space of all X-valued functions defined on I with the norm

$$\|u\|_p = \left(\int_I \|u(t)\|^p \mathrm{d}t\right)^{\frac{1}{p}}.$$

For any positive number r, let

$$\bar{B}_r := \{x \in X : \|x\| \leqslant r\}, \ \ Y_r := \{\phi \in C(I,X) : \|\phi\|_\infty \leqslant r\}.$$

Following Gelfand and Shilov, define the fractional integral of order $q > 0$ as

$$I_a^q f(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s) \mathrm{d}s,$$

also, the Riemann-Liouville fractional derivative of function f of order $0 < q < 1$ as

$${}_aD_t^q f(t) = \frac{1}{\Gamma(1-q)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t (t-s)^{-q} f(s) \mathrm{d}s,$$

where f is an abstract continuous function on $[a,b]$ and Γ is the Gamma function [42].

According to the definitions of fractional integral and Riemann-Liouville fractional derivative, it is suitable to rewrite the nonlocal Cauchy problem (1.2) in the equivalent integral equation

$$u(t) = u_0 - g(u) + \frac{1}{\Gamma(q)} \int_0^t (t-\sigma)^{q-1}[Au(\sigma) + f(\sigma, u(\sigma))]\mathrm{d}\sigma, \tag{2.1}$$

provided that the integral in (2.1) exists.

Before giving the definition of mild solutions of the equation (1.1) and (1.2), we first give the following lemma.

LEMMA 2.1. ([43]) *If (2.1) holds, then one has*

$$u(t) = \int_0^\infty \phi_q(\theta)T(t^q\theta)[u_0 - g(u)]d\theta + q\int_0^t \int_0^\infty \theta(t-\sigma)^{q-1}\phi_q(\theta)T(t^q\theta)f(\sigma,u(\sigma))d\theta d\sigma,$$

where ϕ_q is a probability density function defined on $(0,+\infty)$, that is

$$\phi_q(\theta) \geqslant 0,\ \forall\, \theta \in (0,+\infty) \ \ and \ \ \int_0^\infty \phi_q(\theta)d\theta = 1.$$

For any $u \in X$, define operators $\{S_q(t)\}_{t\geqslant 0}$ and $\{T_q(t)\}_{t\geqslant 0}$ by

$$S_q(t)u = \int_0^\infty \phi_q(\theta)T(t^q\theta)u\mathrm{d}\theta,\ \ T_q(t)u = q\int_0^\infty \theta\phi_q(\theta)T(t^q\theta)u\mathrm{d}\theta.$$

Thus due to Lemma 2.1, we give the following definition of the mild solutions of the equation (1.1) and (1.2).

DEFINITION 2.1. A function $u \in C(I,X)$ is said to be a mild solution of the equation (1.2) if for every $t \in I$,

$$u(t) = S_q(t)(u_0 - g(u)) + \int_0^t (t-s)^{q-1}T_q(t-s)f(s,u(s))\mathrm{d}s.$$

By a mild solution of the equation (1.1) we understand a function $u \in C(I,X)$ which satisfies

$$u(t) = S_q(t)(u_0 - g(u)) + \int_0^t (t-s)^{q-1}T_q(t-s)(Fu)(s)\mathrm{d}s,$$

for any $t \in I$.

Let $r > 0$, we introduce the following assumptions.

(H_1) The operator A generates a compact C_0-semigroup $\{T(t)\}_{t\geqslant 0}$ on X, i.e., for any $t > 0$, the operator $T(t)$ is compact.

(H_2) The operator $F : C(I,X) \to L^p(I,X)$ is continuous. There exists a positive function $\alpha : I \to \mathbb{R}_+$ such that for all $u \in Y_r$ and a.e. $t \in I$, $\|(Fu)(t)\| \leqslant \alpha(t)$ with the function $s \to \alpha(s)$ belongs to $L^p(I,\mathbb{R}_+)$,

$$\gamma(t) := \left(\int_0^t \alpha^p(s)\mathrm{d}s\right)^{\frac{1}{p}} \leqslant M_T < \infty.$$

(H_3) The function $g : C(I,X) \to X$ is continuous, and maps Y_r into a bounded set.

(H_4) There exists a constant $\delta \in (0,T)$, such that

$$F(u) = F(v), \quad g(u) = g(v),$$

for any $u, v \in Y_r$ with $u(s) = v(s)$, $s \in [\delta, T]$.

(H_5) $M(\|u_0\| + \sup_{v \in Y_r} \|g(v)\|) + \frac{qMM_T}{\Gamma(q+1)} \left(\frac{p-1}{pq-1}\right)^{\frac{p-1}{p}} T^{q-\frac{1}{p}} \leqslant r.$

It follows from (H_1) and [44, Theorem 2.3.2] that $\{T(t)\}_{t \geqslant 0}$ is continuous in the uniform operator topology for all $t > 0$, i.e.,

$$\lim_{\eta \to 0} \|T(t+\eta) - T(t)\| = 0, \ \forall\, t > 0.$$

Furthermore, $\|T(t)\|$ is bounded on the interval I, i.e.,

$$M := \sup\{\|T(t)\| : t \in I\} < \infty. \tag{2.2}$$

We give the following lemmas relative to operators $\{S_q(t)\}_{t \geqslant 0}$ and $\{T_q(t)\}_{t \geqslant 0}$ before we proceed further.

LEMMA 2.2. ([45]) *For any fixed $t \geqslant 0$, $\{S_q(t)\}_{t \geqslant 0}$ and $\{T_q(t)\}_{t \geqslant 0}$ are linear and bounded operators, i.e.,*

$$\|S_q(t)u\| = \left\| \int_0^\infty \phi_q(\theta) T(t^q \theta) x d\theta \right\| \leqslant M\|u\|,$$

$$\|T_q(t)u\| = \left\| q \int_0^\infty \theta \phi_q(\theta) T(t^q \theta) x d\theta \right\| \leqslant \frac{qM}{\Gamma(1+q)} \|u\|.$$

LEMMA 2.3. ([45]) *The operators $\{S_q(t)\}_{t \geqslant 0}$ and $\{T_q(t)\}_{t \geqslant 0}$ are strongly continuous, which means that for $\forall u \in X$ and $0 < t' < t''$, one has*

$$\|S_q(t'')u - S_q(t')u\| \to 0, \ \|T_q(t'')u - T_q(t')u\| \to 0 \ \text{ as } \ t' \to t''.$$

LEMMA 2.4. ([45]) *If $T(t)$ is a compact operator for every $t > 0$, then $\{S_q(t)\}_{t \geqslant 0}$ and $\{T_q(t)\}_{t \geqslant 0}$ are also compact operators for every $t > 0$.*

Now we give other two lemmas.

LEMMA 2.5. *Assume that there exists a constant $r > 0$ such that the conditions (H_1)–(H_5) are satisfied. Then the nonlocal problem (1.1) has at least one mild solution in Y_r.*

Proof. Set

$$Y_r(\delta) := \{u \in C([\delta, T], X) : \|u\|_\infty \leqslant r, \forall\, t \in [\delta, T]\}.$$

For any $u \in Y_r(\delta)$, one can find a function $v \in Y_r$ with $u(t) = v(t)$, $t \in [\delta, T]$. Let

$$(\tilde{F}u)(t) := (Fv)(t), \ \forall\, t \in [0,T], \ \tilde{g}(u) := g(v).$$

By conditions (H_2)–(H_4), the mappings $\tilde{F} : Y_r(\delta) \to L^p(I, X)$ and $\tilde{g} : Y_r(\delta) \to X$ are well defined and continuous. Moreover,

$$\|(\tilde{F}u)(t)\| \leqslant \alpha(t), \ \text{a.e. } t \in I, \ \forall\, u \in Y_r(\delta), \tag{2.3}$$

$$\sup_{u\in Y_r(\delta)}\|\tilde{g}(u)\| = \sup_{v\in Y_r}\|g(v)\| < \infty. \tag{2.4}$$

Now, define a mapping Ψ on $Y_r(\delta)$ as follows

$$(\Psi u)(t) := S_q(t)(u_0-\tilde{g}(u)) + \int_0^t (t-s)^{q-1}T_q(t-s)(\tilde{F}u)(s)\mathrm{d}s,\ t\in\ [\delta,T]. \tag{2.5}$$

It follows from Lemma 2.2, (2.2)–(2.5) and (H_5) that Ψ maps $Y_r(\delta)$ into itself. Indeed

$$\begin{aligned}
\|(\Psi u)(t)\| &= \left\|S_q(t)(u_0-\tilde{g}(u)) + \int_0^t (t-s)^{q-1}T_q(t-s)(\tilde{F}u)(s)\mathrm{d}s\right\| \\
&\leqslant \|S_q(t)(u_0-\tilde{g}(u))\| + \left\|\int_0^t (t-s)^{q-1}T_q(t-s)(\tilde{F}u)(s)\mathrm{d}s\right\| \\
&\leqslant M(\|u_0\|+\|\tilde{g}(u))\|) + \int_0^t (t-s)^{q-1}\|T(t-s)(\tilde{F}u)(s)\|\mathrm{d}s \\
&\leqslant M(\|u_0\|+\|\tilde{g}(u))\|) + \frac{qM}{\Gamma(q+1)}\int_0^t (t-s)^{q-1}\alpha(s)\mathrm{d}s \\
&\leqslant M(\|u_0\|+\|\tilde{g}(u))\|) + \frac{qM}{\Gamma(q+1)}\left(\int_0^t (t-s)^{\frac{(q-1)p}{p-1}}\mathrm{d}s\right)^{\frac{p-1}{p}}\left(\int_0^t \alpha^p(s)\mathrm{d}s\right)^{\frac{1}{p}} \\
&\leqslant M(\|u_0\|+\|\tilde{g}(u))\|) + \frac{qMM_T}{\Gamma(q+1)}\left(\frac{p-1}{pq-1}\right)^{\frac{p-1}{p}} T^{q-\frac{1}{p}} \leqslant r.
\end{aligned}$$

Next we prove Ψ has a fixed point in $Y_r(\delta)$. From the continuity of $\tilde{F}$ and $\tilde{g}$, one obtains that Ψ is continuous. Hence, one only needs to prove the set $\{\Psi u : u\in\ Y_r(\delta)\}$ is relatively compact in $C([\delta,T],X)$. Then the result follows from Schauder's Fixed Point Theorem.

It follows from (H_1) and Lemma 2.4 that $\{S_q(t)\}_{t\geqslant 0}$ and $\{T_q(t)\}_{t\geqslant 0}$ are compact for $t>0$. Thus, by (2.3) and (2.4), one can further deduce that for any $t\in[\delta,T]$, the set $\{(\Psi u)(t) : u\in Y_r(\delta)\}$ is relatively compact in X. The norm continuity of $\{S_q(t)\}_{t\geqslant 0}$ (from Lemma 2.3), together with (2.4), yields that the family of functions on $[\delta,T]$, $\{S_q(\cdot)(u_0-\tilde{g}(u)) : u\in Y_r(\delta)\}$ is equicontinuous.

On the other hand,

$$\begin{aligned}
&\left\|\int_0^{t+h}(t+h-s)^{q-1}T_q(t+h-s)(\tilde{F}u)(s)\mathrm{d}s - \int_0^t (t-s)^{q-1}T_q(t-s)(\tilde{F}u)(s)\mathrm{d}s\right\| \\
\leqslant &\left\|\int_0^t \left[(t+h-s)^{q-1}-(t-s)^{q-1}\right]T_q(t+h-s)(\tilde{F}u)(s)\mathrm{d}s\right\| \\
&+\left\|\int_t^{t+h}(t+h-s)^{q-1}T_q(t+h-s)(\tilde{F}u)(s)\right\|\mathrm{d}s \\
&+\left\|\int_0^t (t-s)^{q-1}\left[T_q(t+h-s)-T_q(t-s)\right](\tilde{F}u)(s)\right\|\mathrm{d}s \\
=\ &I+II+III,
\end{aligned} \tag{2.6}$$

where

$$
\begin{aligned}
I &= \left\| \int_0^t \left[(t+h-s)^{q-1} - (t-s)^{q-1} \right] T_q(t+h-s)(\tilde{F}u)(s)\mathrm{d}s \right\|, \\
II &= \left\| \int_t^{t+h} (t+h-s)^{q-1} T_q(t+h-s)(\tilde{F}u)(s) \right\| \mathrm{d}s, \\
III &= \left\| \int_0^t (t-s)^{q-1} \left[T_q(t+h-s) - T_q(t-s) \right] (\tilde{F}u)(s) \right\| \mathrm{d}s.
\end{aligned}
$$

According to Lemma 2.2, estimating the terms on the right-hand side of (2.6) yields

$$
\begin{aligned}
I &= \left\| \int_0^t \left[(t+h-s)^{q-1} - (t-s)^{q-1} \right] T_q(t+h-s)(\tilde{F}u)(s)\mathrm{d}s \right\| \\
&\leqslant \int_0^t |(t+h-s)^{q-1} - (t-s)^{q-1}| \|T_q(t-s)(\tilde{F}u)(s)\| \mathrm{d}s \\
&\leqslant \frac{qM}{\Gamma(q+1)} \int_0^t |(t+h-s)^{q-1} - (t-s)^{q-1}| \alpha(s)\mathrm{d}s \\
&\leqslant \frac{qM}{\Gamma(q+1)} \left[\int_0^{t-\varepsilon} [(t-s)^{q-1} - (t+h-s)^{q-1}]\alpha(s)\mathrm{d}s + \int_{t-\varepsilon}^t (t-s)^{q-1}\alpha(s)\mathrm{d}s \right] \\
&= I' + II',
\end{aligned}
\tag{2.7}
$$

where

$$
\begin{aligned}
I' &= \frac{qM}{\Gamma(q+1)} \left[\int_0^{t-\varepsilon} [(t-s)^{q-1} - (t+h-s)^{q-1}]\alpha(s)\mathrm{d}s \right], \\
II' &= \frac{qM}{\Gamma(q+1)} \left[\int_{t-\varepsilon}^t (t-s)^{q-1}\alpha(s)\mathrm{d}s \right].
\end{aligned}
$$

It follows from the assumption of $\alpha(s)$ that I' tends to 0 as $h \to 0$.

For II', using Hölder inequality, one can see that II' tends to 0 as $\varepsilon \to 0$.

For II, according to Lemma 2.2 again, one has

$$
\begin{aligned}
II &= \left\| \int_t^{t+h} (t+h-s)^{q-1} T_q(t+h-s)(\tilde{F}u)(s) \right\| \mathrm{d}s \\
&\leqslant \int_t^{t+h} (t+h-s)^{q-1} \|T_q(t+h-s)(\tilde{F}u)(s)\| \mathrm{d}s \\
&\leqslant \frac{qM}{\Gamma(q+1)} \int_t^{t+h} (t+h-s)^{q-1} \alpha(s)\mathrm{d}s.
\end{aligned}
\tag{2.8}
$$

Thus, one can see that II tends to 0 as $h \to 0$.

As to III, from Lemma 2.2, one gets

$$
\begin{aligned}
III &= \left\| \int_0^t (t-s)^{q-1} \left[T_q(t+h-s) - T_q(t-s)\right] (\tilde{F}u)(s) \right\| \mathrm{d}s \\
&\leqslant \left\| \int_0^{t-\varepsilon} (t-s)^{q-1} \left[T_q(t+h-s) - T_q(t-s)\right] (\tilde{F}u)(s) \right\| \mathrm{d}s \\
&\quad + \left\| \int_{t-\varepsilon}^t (t-s)^{q-1} \left[T_q(t+h-s) - T_q(t-s)\right] (\tilde{F}u)(s) \right\| \mathrm{d}s \\
&\leqslant \int_0^{t-\varepsilon} (t-s)^{q-1} \left\| T_q(t+h-s) - T_q(t-s)(\tilde{F}u)(s) \right\| \mathrm{d}s \\
&\quad + \frac{2qM}{\Gamma(q+1)} \int_{t-\varepsilon}^t (t-s)^{q-1} \alpha(s) \mathrm{d}s \\
&\leqslant \int_0^{t-\varepsilon} (t-s)^{q-1} \left\| T_q(t+h-s) - T_q(t-s) \right\| \alpha(s) \mathrm{d}s \\
&\quad + \frac{2qM}{\Gamma(q+1)} \int_{t-\varepsilon}^t (t-s)^{q-1} \alpha(s) \mathrm{d}s \\
&= I'' + II'',
\end{aligned}
\tag{2.9}
$$

where

$$
\begin{aligned}
I'' &= \int_0^{t-\varepsilon} (t-s)^{q-1} \left\| T_q(t+h-s) - T_q(t-s) \right\| \alpha(s) \mathrm{d}s, \\
II'' &= \frac{2qM}{\Gamma(q+1)} \int_{t-\varepsilon}^t (t-s)^{q-1} \alpha(s) \mathrm{d}s.
\end{aligned}
$$

Since A is a bounded operator, using the compactness of $T_q(t)$ in X again implies the continuity of $t \to \|T_q(t)\|$ for $t \in I$, integrating with $s \to \alpha(s) \in L^p(I, \mathbb{R}^+)$, we see that I'' tends to 0 as $h \to 0$.

For II'', from the assumption of $\alpha(s)$ and Hölder inequality, it is easy to see that II'' tends to 0 as $\varepsilon \to 0$.

Therefore, the family of functions on $[\delta, T]$,

$$
\left\{ \int_0^{\cdot} T_q(\cdot - s)(\tilde{F}u)(s) ds : u \in Y_r(\delta) \right\}
$$

is equicontinuous and so is $\{\Psi u : u \in Y_r(\delta)\}$. Applying Arzela-Ascoli's Theorem, one obtains that $\{\Psi u : u \in Y_r(\delta)\}$ is relatively compact in $C([\delta, T], X)$. Thus there is a function $\phi \in Y_r(\delta)$ such that $\Psi\phi = \phi$, i.e.,

$$
\phi(t) = S_q(t)(u_0 - \tilde{g}(\phi)) + \int_0^t (t-s)^{q-1} T_q(t-s)(\tilde{F}\phi)(s) \mathrm{d}s, \ \forall\, t \in [\delta, T].
$$

Set

$$
\psi(t) = S_q(t)(u_0 - \tilde{g}(\phi)) + \int_0^t (t-s)^{q-1} T_q(t-s)(\tilde{F}\phi)(s) \mathrm{d}s, \ \forall\, t \in [0, T].
$$

Then $\phi(t) = \psi(t), t \in [\delta, T]$. By (2.2)–(2.4) and (H_5) one has $\psi \in Y_r(\delta)$. Therefore, it follows from the definitions of $\tilde{F}$ and $\tilde{g}$ that

$$
\psi(t) = S_q(t)(u_0 - g(\psi)) + \int_0^t (t-s)^{q-1} T_q(t-s)(F\psi)(s) \mathrm{d}s, \ \forall\, t \in [0, T],
$$

i.e., ψ is a mild solution of the equation (1.1). □

In the following, we construct a family of nonlocal Cauchy problems. For each $\delta \in (0,T)$, define an operator $\mathscr{B}_\delta$ on $C(I,X)$ as follows: for every $u \in C(I,X)$,

$$[\mathscr{B}_\delta u](t) := \begin{cases} u(\delta),\ 0 \leqslant t \leqslant \delta, \\ u(t),\ \delta \leqslant t \leqslant T. \end{cases} \tag{2.10}$$

It is easy to check that $\mathscr{B}_\delta$ is a bounded linear operator on $C(I,X)$ and $\|\mathscr{B}_\delta\| = 1$. Therefore, $\mathscr{B}_\delta Y_r \subset Y_r$, $\forall\, r > 0$. Now one can define a function $g_\delta : C(I,X) \to X$ and an operator $F_\delta : C(I,X) \to L^p(I,X)$ by

$$g_\delta(u) = g(\mathscr{B}_\delta u), \quad F_\delta u = F\mathscr{B}_\delta u, \quad u \in C(I,X).$$

Consider the nonlocal Cauchy problem

$$(Q_\delta) \qquad \begin{cases} \dfrac{\mathrm{d}^q u(t)}{\mathrm{d}t^q} = Au(t) + (F_\delta u)(t),\ t \in I, \\ u(0) + g_\delta(u) = u_0. \end{cases}$$

One has the following result.

LEMMA 2.6. *Suppose that there exists a constant $r > 0$ such that (H_1)–(H_3) and (H_5) are satisfied. Then for any $\delta \in (0,T)$, the problem (Q_δ) has at least one mild solution in Y_r.*

Proof. For every $\delta \in (0,T)$, by the definitions of $\mathscr{B}_\delta, F_\delta$ and g_δ, one sees that the conditions of Lemma 2.5 are satisfied with F and g replaced by F_δ and g_δ respectively. Applying Lemma 2.5 one obtains the result. □

3. Main results

Since $C(I,X)$ is a subset of $L^p(I,X)$, one can regard g as a function from $L^p(I,X)$ to X, although in the hypothesis (H_3) the function g is only needed to be defined in $C(I,X)$. Indeed, in many practical examples g is also well-defined and continuous in $L^p(I,X)$. Thus we introduce the following condition.

(H_3') The function $g : L^p(I,X) \to X$ is continuous, and maps Y_r into a bounded set.

Now we state and prove our first main result. In this theorem, in order to drop the condition (H_4), we replace the assumption (H_3) by (H_3').

THEOREM 3.1. *Suppose that there exists a constant $r > 0$ such that (H_1), (H_2), (H_3') and (H_5) are satisfied. Then the nonlocal problem (1.1) has at least one mild solution in Y_r.*

Proof. The proof is divided into the following four steps.

Step 1. Let $\{\delta_n : n \in \mathbb{N}\}$ be a decreasing sequence in $(0,T)$ such that $\lim_{n\to\infty} \delta_n = 0$. Then for any n, by Lemma 2.6, the nonlocal problem (Q_{δ_n}) has a mild solution $u_n \in Y_r$, i.e.,

$$u_n(t) = S_q(t)(u_0 - g_{\delta_n}(u_n)) + \int_0^t (t-s)^{q-1} T_q(t-s)(F_{\delta_n} u_n)(s) \mathrm{d}s, \ t \in I. \tag{3.1}$$

Define a function $v_n \in C(I,X)$ by

$$v_n(t) = \begin{cases} u_n(\delta_n), \ 0 \leqslant t \leqslant \delta_n, \\ u_n(t), \ \delta_n \leqslant t \leqslant T. \end{cases} \tag{3.2}$$

Then $v_n \in Y_r$. By (3.1) and the definitions of g_{δ_n} and F_{δ_n}, one has

$$u_n(t) = S_q(t)(u_0 - g(v_n)) + \int_0^t (t-s)^{q-1} T_q(t-s)(F v_n)(s) \mathrm{d}s, \ t \in I. \tag{3.3}$$

Let

$$\phi_n(t) = S_q(t)(u_0 - g(v_n)), \ \ \psi_n(t) = \int_0^t (t-s)^{q-1} T_q(t-s)(F v_n)(s) \mathrm{d}s, \ t \in I.$$

Then $\phi_n, \psi_n \in C(I,X)$ and $u_n = \phi_n + \psi_n$. In the following two steps we will prove that the sequence $\{u_n\}$ is relatively compact in $C(I,X)$ by showing that both $\{\psi_n : n \in \mathbb{N}\}$ and $\{\phi_n : n \in \mathbb{N}\}$ are relatively compact.

Step 2. Proving that $\{\psi_n : n \in \mathbb{N}\}$ is relatively compact in $C(I,X)$.

For any $n \in \mathbb{N}$, noticing that $v_n \in Y_r$, by (H_2) one has

$$\|(F v_n)(t)\| \leqslant \alpha(t), \ t \in I.$$

This, together with the compactness of $\{S_q(t)\}_{t>0}$, implies that for all $t \in I$, the set $\{\psi_n(t) : n \in \mathbb{N}\}$ is relatively compact in X. This follows from Theorem 2 in [46].

On the other hand, similarly to the proof of Lemma 2.5 (see (2.6)–(2.9)), one obtains that $\{\psi_n : n \in \mathbb{N}\}$ is equicontinuous. It follows from Arzela-Ascoli's Theorem that $\{\psi_n : n \in \mathbb{N}\}$ is relatively compact.

Step 3. Claiming that $\{\phi_n : n \in \mathbb{N}\}$ is relatively compact in $C(I,X)$.

It is sufficient to prove that the sequence $\{u_0 - g(v_n) : n \in \mathbb{N}\}$ is relatively compact in X. Since the function $g : L^p(I,X) \to X$ is continuous, one only needs to prove that $\{v_n : n \in \mathbb{N}\}$ is relatively compact in $L^p(I,X)$. Noticing that $u_n, v_n \in Y_r$, by (3.2) one has

$$\|u_n - v_n\|_p \leqslant 2^{1+\frac{1}{p}} r \delta_n^{\frac{1}{p}}, \ n \in \mathbb{N}. \tag{3.4}$$

We will show that for any subsequence of $\{u_n : n \in \mathbb{N}\}$ denoted by $\{\tilde{u}_n : n \in \mathbb{N}\}$, there exist a subsequence $\{z_n : n \in \mathbb{N}\} \subset \{\tilde{u}_n : n \in \mathbb{N}\}$ and a function $z_\infty \in L^p(I,X)$ such that

$$\lim_{n\to\infty} \|z_n - z_\infty\|_p = 0. \tag{3.5}$$

Then the claim follows from (3.4).

Let $\{\varepsilon_n : n \in \mathbb{N}\}$ be a decreasing sequence in $(0,T)$ such that $\lim_{n\to\infty} \varepsilon_n = 0$. Define a family of functions $w_n : [\varepsilon_1, T] \to X$ by $w_n(t) = \tilde{u}_n(t), \ t \in [\varepsilon_1, T]$.

Note that $\{v_n : n \in \mathbb{N}\} \subset Y_r$, by assumption (H_5) one has

$$\|u_0 - g(v_n)\| \leqslant r/M, \ n \in \mathbb{N}.$$

Then it follows from the compactness and norm continuity of $\{S_q(t)\}_{t>0}$ that for any $t \in [\varepsilon_1, T]$, the set

$$\{S_q(t)(u_0 - g(v_n)) : n \in \mathbb{N}\}$$

is relatively compact in X and that the family of functions on $[\varepsilon_1, T]$,

$$\{S_q(\cdot)(u_0 - g(v_n)) : n \in \mathbb{N}\}$$

is equicontinuous. Applying Arzela-Ascoli's Theorem again one obtains that

$$\{S_q(\cdot)(u_0 - g(v_n)) : n \in \mathbb{N}\}$$

is relatively compact in $C([\varepsilon_1, T], X)$.

On the other hand, as is proved in step 2, $\{\psi_n : n \in \mathbb{N}\}$ is relatively compact in $C(I, X)$. Thus $\{w_n : n \in \mathbb{N}\}$ is relatively compact in $C([\varepsilon_1, T], X)$. Therefore, there exists a subsequence of $\{\tilde{u}_n : n \in \mathbb{N}\}$ denoted by $\{\tilde{u}_n^1 : n \in \mathbb{N}\}$ which is a Cauchy sequence in $C([\varepsilon_1, T], X)$.

Similarly, we can select a subsequence of $\{\tilde{u}_n^1 : n \in \mathbb{N}\}$ denoted by $\{\tilde{u}_n^2 : n \in \mathbb{N}\}$ which is a Cauchy sequence in $C([\varepsilon_2, T], X)$. Repeating the above approach and using a diagonal argument, we get a subsequence of $\{\tilde{u}_n : n \in \mathbb{N}\}$ denoted by $\{z_n : n \in \mathbb{N}\}$, such that for every $t \in (0, T]$, $\{z_n(t) : n \in \mathbb{N}\}$ is a Cauchy sequence in X. Define the function z_∞ by

$$z_\infty(t) = \begin{cases} 0, & t = 0, \\ \lim_{n\to\infty} z_n(t), & 0 < t \leqslant T. \end{cases} \tag{3.6}$$

Then z_∞ is strongly measurable and

$$\left(\int_0^T \|z_\infty(t)\| \mathrm{d}t \right)^{\frac{1}{p}} \leqslant T^{\frac{1}{p}} r < \infty.$$

This shows that $z_\infty \in L^p(I, X)$. Now by Lebesgue's dominated convergence theorem and (3.6) one gets (3.5).

Step 4. In steps 2 and 3, we proved that both $\{\phi_n : n \in \mathbb{N}\}$ and $\{\psi_n : n \in \mathbb{N}\}$ are relatively compact in $C(I, X)$. This fact implies the relatively compactness of $\{u_n : n \in \mathbb{N}\}$ in $C(I, X)$. Therefore, there exist a subsequence of $\{u_n : n \in \mathbb{N}\}$ denoted again by $\{u_n : n \in \mathbb{N}\}$ and a function $u_\infty \in C(I, X)$ such that

$$\lim_{n\to\infty} \|u_n(t) - u_\infty(t)\|_\infty = 0. \tag{3.7}$$

Obviously, $u_\infty \in Y_r$.

We claim that u_∞ is a mild solution of the equation (1.1). In fact, by (3.2) one has

$$\begin{aligned} \|v_n - u_\infty\|_\infty &= \max_{t\in I} \|v_n(t) - u_\infty(t)\| \\ &\leqslant \max_{t\in[\delta_n, T]} \|u_n(t) - u_\infty(t)\| + \max_{t\in[0,\delta_n]} \|u_n(\delta_n) - u_\infty(t)\| \\ &\leqslant \|u_n - u_\infty\|_\infty + \|u_n(\delta_n) - u_\infty(\delta_n)\| + \max_{t\in[0,\delta_n]} \|u_\infty(\delta_n) - u_\infty(t)\| \\ &\leqslant 2\|u_n - u_\infty\|_\infty + \max_{t\in[0,\delta_n]} \|u_\infty(\delta_n) - u_\infty(t)\|. \end{aligned} \tag{3.8}$$

Now it follows from (3.7), (3.8) and the uniformly continuity of u_∞ on the interval I that

$$\lim_{n\to\infty} \|v_n - u_\infty\|_\infty = 0.$$

Therefore, taking limits in (3.3) one has

$$u_\infty(t) = S_q(t)(u_0 - g(u_\infty)) + \int_0^t (t-s)^{q-1} T_q(t-s)(Fu_\infty)(s)\mathrm{d}s,\ t\in I,$$

i.e., u_∞ is a mild solution of the equation (1.2). □

Next, we consider the case that the function g is defined on $C(I,X)$ rather than $L^p(I,X)$. We make the following assumption.

(H_6) The following assertion holds uniformly for all $\phi \in Y_r$:

$$\lim_{\varepsilon\to 0} \|g(\phi) - g(\phi^\varepsilon)\| = 0, \tag{3.9}$$

where

$$\phi^\varepsilon(t) = \begin{cases} \phi(\varepsilon),\ 0 \leqslant t \leqslant \varepsilon, \\ \phi(t),\ \varepsilon \leqslant t \leqslant T. \end{cases}$$

This assumption is reasonable and natural. (3.9) means that the function g depends mainly on the value of ϕ on the interval $[\varepsilon, T]$ when ε is small enough. Obviously, it is satisfied for the function g in (1.3). And under suitable conditions (for example, the condition (3.18) in Corollary 3.2 below), it holds true for the function g in (1.4). We also remark that the condition (H_3') does not imply (H_6).

The following theorem is our second main result.

THEOREM 3.2. *Suppose that there exists a constant $r > 0$ such that (H_1)–(H_3), (H_5) and (H_6) are satisfied. Then the nonlocal problem (1.1) has at least one mild solution in Y_r.*

Proof. We divide the proof into 4 steps. Steps 1 and 2 are the same as in the proof of Theorem 3.1. So we begin with step 3.

Step 3. Proving that $\{\phi_n : n \in \mathbb{N}\}$ is relatively compact in $C(I,X)$.

It suffices to show that the sequence $\{g(v_n) : n \in \mathbb{N}\}$ is relatively compact in X.

Similarly to step 3 in the proof of Theorem 3.1, for any subsequence $\{\tilde{u}_n : n \in \mathbb{N}\} \subset \{u_n : n \in \mathbb{N}\}$, there exist a subsequence $\{z_n : n \in \mathbb{N}\} \subset \{\tilde{u}_n : n \in \mathbb{N}\}$ and a continuous function $z_\infty : (0,T] \to X$ such that for every ε_k,

$$\lim_{n\to\infty} \max_{t\in[\varepsilon_k,T]} \|z_n(t) - z_\infty(t)\| = 0. \tag{3.10}$$

Now we prove that $\{g(z_n) : n \in \mathbb{N}\}$ is a Cauchy sequence in X. In fact, by assumption (H_6), for any $\varepsilon > 0$, there exists a constant $\delta > 0$, such that

$$\|g(\phi) - g(\psi)\| < \varepsilon/4, \tag{3.11}$$

for any $\phi, \psi \in C(I,X)$ with $\phi(t) = \psi(t)$, $\delta \leqslant t \leqslant T$. Let

$$\varphi(t) = \begin{cases} z_\infty(\delta),\ 0 \leqslant t \leqslant \delta, \\ z_\infty(t),\ \delta < t \leqslant T. \end{cases} \tag{3.12}$$

Then $\varphi \in C(I,X)$ and by (3.10) one has

$$\lim_{n\to\infty} \max_{t\in[\delta,T]} \|z_n(t) - \varphi(t)\| = 0. \tag{3.13}$$

This implies that

$$\lim_{n\to\infty} \|\mathscr{B}_\delta z_n - \varphi\|_\infty = 0. \tag{3.14}$$

By assumption (H_3) there exists a natural number N such that

$$\|g(\mathscr{B}_\delta z_n) - g(\varphi)\| < \varepsilon/4, \ \forall n > N. \tag{3.15}$$

Therefore, for any $m,n > N$,

$$\begin{aligned}\|g(z_m) - g(z_n)\| \leqslant &\|g(z_m) - g(\mathscr{B}_\delta z_m)\| + \|g(\mathscr{B}_\delta z_m) - g(\varphi)\| \\ &+ \|g(\varphi) - g(\mathscr{B}_\delta z_n)\| + \|g(\mathscr{B}_\delta z_n) - g(z_n)\| < \varepsilon.\end{aligned}$$

This shows that $\{g(z_n) : n \in \mathbb{N}\}$ is a Cauchy sequence in X. Thus the sequence $\{g(u_n) : n \in \mathbb{N}\}$ is relatively compact in X. On the other hand, by (3.2) and assumption (H_6) one has

$$\lim_{n\to\infty} \|g(u_n) - g(v_n)\| = 0. \tag{3.16}$$

Therefore, the sequence $\{g(v_n) : n \in \mathbb{N}\}$ is relatively compact in X.

Step 4. This step is the same as in the proof of Theorem 3.1. The proof is completed. □

Now we consider the nonlocal Cauchy problem (1.2). Firstly, we introduce the following condition.

(H_2') The function $f : I \times X \to X$ satisfies the Carathéodory condition, i.e., $f(\cdot,x)$ is measurable for each $x \in X$ and $f(t,\cdot)$ is continuous for any $t \in I$. Moreover, there exists a function $\alpha \in L^p(I,\mathbb{R}_+)$ such that for all $x \in \bar{B}_r$ and a.e. $t \in I$.

$$\|f(t,x)\| \leqslant \alpha(t).$$

As a consequence of Theorems 3.1 and 3.2, one has the following result for the equation (1.2).

COROLLARY 3.1. *Assume that there exists a constant $r > 0$ such that (H_1), (H_2'), (H_3), (H_5) and (H_6) hold or (H_1), (H_2'), (H_3') and (H_5) are satisfied. Then the nonlocal problem (P) has at least one mild solution in Y_r.*

Proof. Consider the operator

$$(Fu)(t) = f(t,u(t)).$$

One sees that condition (H_2') implies (H_2). Therefore, by Theorems 3.1 and 3.2 one obtains the result. □

Corollary 3.1 implies the following corollaries immediately.

COROLLARY 3.2. *Consider the nonlocal Cauchy problem (1.4). Assume (H_1) holds. Suppose that the functions f and h satisfy the Carathéodory condition and there exist a function $\alpha \in L^p(I,\mathbb{R}_+)$ and a nondecreasing function $\Upsilon : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|f(t,u)\| \leqslant \alpha(t)\Upsilon(\|u\|), \ a.e.\ t \in I,\ \forall\, x \in X, \tag{3.17}$$

$$\|h(t,u)\| \leqslant \alpha(t)\Upsilon(\|u\|), \ a.e.\ t \in I,\ \forall\, x \in X, \tag{3.18}$$

$$\Omega = \varliminf_{r\to\infty} \frac{\Upsilon(r)}{r} < \frac{1}{\frac{qM}{\Gamma(q+1)}\left(\frac{p-1}{pq-1}\right)^{\frac{p-1}{p}} T^{q-\frac{1}{p}} \left(\int_0^T \alpha^p(t)ds\right)^{\frac{1}{p}}}. \tag{3.19}$$

Then (1.4) has at least one mild solution in $C(I,X)$.

COROLLARY 3.3. *Consider the following nonlocal Cauchy problem*

$$\begin{cases} \dfrac{d^q u(t)}{dt^q} = Au(t) + f(t,u(t)),\ t \in I, \\ u(0) + \displaystyle\sum_{i=1}^{p} c_i u(t_i) = u_0, \end{cases} \tag{3.20}$$

where $0 < t_1 < t_2 < \cdots < t_p \leqslant T$, $c_1, c_2, \cdots, c_p$ are given constants.

Assume (H_1) holds. Suppose that the function f satisfies the Carathéodory condition and there exist a function $\alpha \in L^p(I,\mathbb{R}_+)$ and a nondecreasing function $\Upsilon : \mathbb{R}_+ \to \mathbb{R}_+$ such that (3.17) and (3.19) hold. Moreover,

$$C := \sum_{i=1}^{p} |c_i| < 1/M.$$

Then (3.20) has at least one mild solution in $C(I,X)$.

4. Applications

In this section, we give three examples to illustrate the practical usefulness of the results that we establish in the paper.

Consider the following nonlocal problem of fractional differential equation

$$\begin{aligned} &\partial_t^q u(t,x) = \partial_x^2 u(t,x) + (Fu)(t),\ \ t \in [0,1], x \in [0,\pi], \\ &u(t,0) = u(t,\pi) = 0, \quad t \in [0,1], \\ &u(0,x) + g(u) = u_0(x) \end{aligned} \tag{4.1}$$

where ∂_t^q is a Riemann-Liouville fractional partial derivative of order $0 < q < 1$.

Let $u(s,x) = \varphi(s,x)$, $\varphi(\cdot,x) \in C([0,1],\mathbb{R})$, $\varphi(s,\cdot) \in L^2([0,\pi],\mathbb{R})$, $s \in [0,1]$, $x \in [0,\pi]$. Denote $X = L^2([0,\pi],\mathbb{R})$ and define $A : D(A) \subset X \to X$ given by $A = \frac{\partial^2}{\partial x^2}$ with the domain

$$D(A) = \Big\{u(\cdot) \in X : u'' \in X, u' \in X \text{ is absolutely continuous on } [0,\pi], u(0) = u(\pi) = 0\Big\}.$$

It is well known that A is self-adjoint with compact resolvent and is the infinitesimal generator of an analytic, strongly continuous, compact and self-adjoint semigroup $\{T(t)\}_{t\geqslant 0}$ satisfying

$$\|T(t)\| \leqslant e^{-t}, \text{ for } t \geqslant 0.$$

Furthermore, A has a discrete spectrum with eigenvalues of the form $-n^2$, $n \in \mathbb{N}$. In fact, let $u \in D(A)$ and $\lambda \in \mathbb{R}$, such that $Au = -u'' = \lambda u$, that is,

$$u'' + \lambda u = 0. \tag{4.2}$$

Thus one has $\langle Au, u\rangle = \langle \lambda u, u\rangle$, that is $\langle -u'', u\rangle = \|u'\|^2_{L^2([0,\pi],\mathbb{R})} = \lambda \|u\|^2_{L^2([0,\pi],\mathbb{R})}$. The solutions of (4.2) have the form

$$u(x) = C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x).$$

From $u(0) = u(\pi)$, it follows that $C = 0$ and $\sqrt{\lambda} = n$, $n \in N$. Put $\lambda_n = n^2$, the solutions of equation (4.2) are

$$u_n(x) = D\sin(\sqrt{\lambda_n}x),\ n \in N.$$

According to $\langle u_n, u_m\rangle = 0$, for $n \neq m$ and $\langle u_n, u_n\rangle = 1$, one has $D = \sqrt{2}$ and

$$u_n(x) = \sqrt{2}\sin(\sqrt{\lambda_n}x).$$

For $u \in D(A)$, there exists a sequence of reals (α_n) such that

$$u(x) = \sum_{n\in N} \alpha_n u_n(x),$$

$$\sum_{n\in N} \alpha_n^2 < +\infty, \quad \sum_{n\in N} \lambda_n^2 \alpha_n^2 < +\infty.$$

In addition, $\{u_n : n \in \mathbb{N}\}$ is an orthogonal basis for X,

$$T(t)u = \Sigma_{n=1}^{\infty} e^{-n^2 t}\langle u, u_n\rangle u_n, \text{ for all } u \in X \text{ and every } t > 0.$$

From these expressions it follows that $\{T(t)\}_{t\geqslant 0}$ is uniformly bounded compact semigroup, so that $R(\lambda, A) = (\lambda - A)^{-1}$ is compact operator for all $\lambda \in \rho(A)$.

EXAMPLE 4.1. Let $0 < t_1 < t_2 < \cdots < t_n \leqslant 1$ and c_0, c_i $(i = 1, 2, \cdots, n)$ be constants satisfying $\sum\limits_{i=1}^{n} |c_i| \leqslant 1$. Define

$$(Fu)(t) = c_0 \sin(u(t,x)), \text{ and } g(u(t,x)) = \sum_{i=1}^{n} c_i u(t_i, x).$$

for each $t \in [0,1]$ and $u \in C([0,1],\mathbb{R})$. Let $q = \frac{1}{2}$, note that $(Fu)(t)$ satisfy (H_2) and (H_2') with $\alpha(t) = c_0$ and $M_T = c_0$, $g(u)$ satisfy (H_3), (H_3') and (H_6). Let

$$r = \frac{\|u_0\| + \frac{c_0}{\sqrt{3\pi}}}{1 - \sum\limits_{i=1}^{n} |c_i|}.$$

Then from Theorem 3.1 and Theorem 3.2, it follows that Eq. (4.1) has at least one mild solution in Y_r.

EXAMPLE 4.2. Let $\mu(t) \in C([0,1],\mathbb{R})$ satisfying $\eta = \max_{t\in[0,1]} \mu(t)$. Define

$$f(t,u(t,x)) = \beta e^{-t}\sin(\sqrt[3]{u(t,x)}), \text{ and } g(u(t,x)) = \int_0^1 \mu(s)\log(1+\sqrt[3]{|u(s,x)|})\mathrm{d}s.$$

for each $t \in [0,1]$ and $u \in C([0,1],\mathbb{R})$. Let $q = \frac{1}{2}$, note that $f(t,u(t,x))$ satisfy the Carathéodory condition and (3.17), (3.18) and (3.19) with

$$\alpha(t) = \max\{\beta,\gamma\}, \ \Upsilon(\|u\|) = 1+\sqrt[3]{\|u\|}.$$

Then from Corollary 3.2, it follows that Eq. (4.1) with nonlocal initial condition (1.4) has at least one mild solution in $C([0,1],X)$.

EXAMPLE 4.3. Let $0 < t_1 < t_2 < \cdots < t_n \leqslant 1$ and c_i, $i = 1,2,\cdots,n$ be constants satisfying $\sum_{i=1}^{n} |c_i| < 1$. Define

$$f(t,u(t,x)) = \beta e^{-t}\sin(\sqrt[3]{u(t,x)}), \text{ and } g(u(t,x)) = \sum_{i=1}^{n} c_i u(t_{t_i},x).$$

for each $t \in [0,1]$ and $u \in C([0,1],\mathbb{R})$. Let $q = \frac{1}{2}$. Note that $f(t,u(t,x))$ satisfy (H_2) and (H_2') with $\alpha(t) = \beta e^{-t}$ and $M_T = \beta$, $g(u)$ satisfy (H_3), (H_3') and (H_6). Moreover f satisfies the Carathéodory condition and (3.17), (3.19) with

$$\alpha(t) = \beta e^{-t}, \ \Upsilon(\|u\|) = \sqrt[3]{\|u\|}.$$

Let

$$r = \frac{\|u_0\| + \frac{\beta}{\sqrt{3\pi}}}{1 - \sum_{i=1}^{n} |c_i|}.$$

Then from Corollary 3.1, it follows that Eq. (4.1) with nonlocal initial condition (1.3) has at least one mild solution in Y_r, and from Corollary 3.3, it follows that Eq. (4.1) with nonlocal initial condition (1.3) has at least one mild solution in $C([0,1],X)$.

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