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POSITIVE SOLUTIONS OF AN INTEGRODIFFERENTIAL MULTI–POINT INITIAL VALUE PROBLEM WITH FRACTIONAL ORDER

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Abstract. The aim of this paper is to obtain existence results for a nonlinear integrodifferential multi-point initial value problem of fractional order by using fractional calculus and fixed point theory. Then, we study the positivity of the obtained solution.

1. Introduction

Fractional differential equations play an important role in describing many phenomena and processes with memory and hereditary properties in science and engineering such as in viscoelasticity, electrochemistry, control. This subject is gaining much attention caused by development of the theory of fractional calculus itself and its different applications. For more details about this theory, see the books by A. Kilbas et *al.* [10] and by Podlubny [12].

Recently, many authors investigated the existence of solutions for fractional differential problems in Banach spaces involving Riemann-Liouville or Caputo derivatives; subject to initial, nonlocal or boundary conditions, using fixed point concept sometimes combined with other concepts such as cone theoretic techniques [9, 11], contraction mapping principle [1, 2, 14], technique of measures of noncompactness [4, 11]. For examples and details, see [1]–[14] and the references therein.

This paper is concerned with the following nonlinear integrodifferential equation of fractional order $1 < \alpha < 2$,

$$ {}^{c}D^{\alpha}_{0^+}x(t) = F\left(t, x(t), x'(t), \int_0^t h\left(t, s, x(s), x'(s)\right) ds\right), \quad t \in [0,T]; \tag{1}$$

subject to multi-point initial conditions:

$$x(0) + \lambda x(\eta) = 0; \quad x'(0) + \beta x'(\eta) = 0; \tag{2}$$

where ${}^{c}D^{\alpha}_{0^+}$ denotes the Caputo fractional derivative of order α, $t \in [0,T]$; η is fixed in $[0,T]$, λ and β are real constants, $F : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ and $h : \Delta \times \mathbb{R}^2 \to \mathbb{R}$ where $\Delta = \{(t,s) : 0 \leqslant s \leqslant t \leqslant T\}$.

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In [6], Byszewski initiated the work on nonlocal initial value problems. He proved the existence and uniqueness of mild, strong and classical solution of first order initial value problem with nonlocal condition of the following form

$$x(0)+g(t_1,...,t_p,x(.))=x_0. \tag{3}$$

The symbol $g(t_1,...,t_p,x(.))$ means that in the place of "." we can substitute only elements of the set $\{t_1,...,t_p\}\subset(0,T]$. The nonlocal condition can give better results than the usual initial condition given alone $x(0)=x_0$ because it has more information from the beginning of the desired solution. For its importance in different fields we refer to [6] and the references therein.

Multi-point boundary value problems, initiated by IL'in and Moiseev [8], have been considered for the fractionnal order in Banach spaces by many authors [1, 3, 13]. Including for example the work of B. Ahmad [1] where he obtained existence results for the multi-point boundary value problem of nonlinear fractional differential equation

$$\begin{cases} {}^cD_{0^+}^q x(t)=f(t,x(t)), \quad 0<t<1,\ 1<q\leqslant 2, \\ \alpha_1 x(0)-\beta_1 x'(0)=\gamma_1 x(\eta_1); \quad \alpha_2 x(1)+\beta_2 x'(1)=\gamma_2 x(\eta_2); \end{cases}$$

where $f:[0,1]\times X\to X$, X is a Banach space. While for the initial value problems, the researchers [2, 7, 14] used nonlocal condition under a more general form $x(0)+g(x)=x_0$ where g is a given function. For instance, J. Wu and Y. Liu established in [14] existence and uniqueness of solutions for fractional integrodifferential problem with nonlocal condition

$$\begin{cases} {}^cD_{0^+}^q x(t)=f\left(t,x(t),\int\limits_0^t k(t,s,x(s))\,ds\right), \quad t\in[0,1],\ 0<q\leqslant 1, \\ x(0)+g(x)=x_0; \end{cases}$$

where $f:[0,1]\times X\times X\to X$, $g:C(I,X)\to X$ and $k:\Delta_1\times X\to X$ such that $\Delta_1=\{(t,s):0\leqslant s\leqslant t\leqslant 1\}$, X is a Banach space.

In the present paper, we use a multi-point condition for an initial value problem of fractional integrodifferential equation which apparently has not yet been addressed by other authors. The multi-point conditions (2) is a special case of (3). Remark that if $\eta=T$ we are concerned by a boundary problem. If moreover, $\lambda=\beta=-1$, the problem is periodic.

This paper is organized as follows. In Section 2, we present necessary definitions and notations of fractional calculus with some basic properties. In Section 3, we establish two existence results for fractional integrodifferential equation (1) subject to two-point initial conditions (2), respectively based on Banach fixed point theorem and Krasnoselskii fixed point theorem. Then, we investigate the positivity of the obtained solutions. Finally, we give an example that illustrates the first result.

2. Preliminaries

First, we will give necessary definitions and properties from fractional calculus.

DEFINITION 1. [10] The Riemann-Liouville fractional integral operator of order $\alpha > 0$ for a given function f on $[0,T]$ is defined by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds; \text{ for } t > 0$$

$\Gamma(\alpha)$ is the classical Euler's gamma function.

DEFINITION 2. [10] The Caputo fractional derivative of order $\alpha > 0$ for a given function $f(t)$ on $[0,T]$ is defined by

$$^{c}D_{0^+}^{\alpha} f(t) = D_{0^+}^{\alpha} \left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k \right] \tag{4}$$

where $n = [\alpha] + 1$, $[\alpha]$ means the integer part of α and $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative operator of order $\alpha > 0$ defined by

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-1-\alpha} f(s)\, ds = D^n I_{0^+}^{n-\alpha} f(t), \text{ for } t > 0.$$

Note that fractional integrals and derivatives exist provided that the integral parts in each of their definitions are finite for "sufficiently good" functions. For example the fractional integral $I_{0^+}^{\alpha} f(t)$ exists and is bounded for $f(t) \in L_p(0,T)$, $1 \leqslant p \leqslant \infty$ or for $t^{\gamma} f(t) \in C[0,T]$ with $0 \leqslant \gamma < 1$. The Caputo fractional derivative $^{c}D_{0^+}^{\alpha} f(t)$ exists for $f(t)$ belonging to the space $C^n[0,T]$ and if $f(t)$ is in the space $AC^n[0,T]$ of functions which have continuous derivatives up to order $(n-1)$ on $[0,T]$ such that $f^{(n-1)} \in AC[0,T]$, the space of absolutely continuous functions, then $^{c}D_{0^+}^{\alpha} f(t)$ exists and can be defined by

$$^{c}D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-1-\alpha} f^{(n)}(s)\, ds = I_{0^+}^{n-\alpha} D^n f(t), \text{ for } t > 0.$$

More details on fractional calculus can be found in [10, 12].

We derive these useful lemmas directly from results in [10].

LEMMA 3. *For $f \in C^{n-1}[0,T]$ with a Caputo fractional derivative of order α that belongs to $C[0,T]$*

$$I_{0^+}^{\alpha c} D_{0^+}^{\alpha} f(t) = f(t) + c_0 + c_1 t + \dots + c_{n-1} t^{n-1}; \;\; c_i \in \mathbb{R}, \;\; i = 0, \dots n-1, \tag{5}$$

LEMMA 4. *Let* $f \in C^n[0,T]$ *then* ${}^cD^{\alpha}_{0+}I^{\alpha}_{0+}f(t) = f(t)$. *Also, let* $f \in C^n[0,T]$, *then the Caputo fractional derivatives* ${}^cD^{\alpha}_{0+}f(t)$ *is continuous on* $[0,T]$.

Moreover, $D^m I^{\alpha}_{0+}f(t) = I^{\alpha-m}_{0+}f(t)$ *with* $m \in \mathbb{N}$, ${}^cD^{0}_{0+}f(t) = f(t)$, $I^{0}_{0+}f(t) = f(t)$ *and* ${}^cD^{n}_{0+}f(t) = f^{(n)}(t)$.

Our results are based on the well known Banach contraction theorem and the following Krasnoselskii theorem.

THEOREM 5. (Krasnoselskii theorem) *Let* M *be a closed, bounded, convex and nonempty subset of a Banach space* E. *Let* A,B *be operators such that*

(i) $Ax + By \in M$ *whenever* $x;y \in M$; *(ii)* A *is compact and continuous; (iii)* B *is a contraction mapping.*

Then, there exists $z \in M$ *such that* $z = Az + Bz$.

$C^1([0,T],\mathbb{R})$ denotes the Banach space of all continuously differentiable functions from $[0,T]$ into $\mathbb{R}$, endowed with the norm $\|x\|_1 = \|x\| + \|x'\|$ where $\|x\| = \sup\limits_{t\in[0,T]} |x(t)|$.

Let us give the following definition.

DEFINITION 6. A function $x \in C^1([0,T],\mathbb{R})$ with its $\alpha-$fractional derivative existing on $[0,T]$ for $1 < \alpha < 2$, is said to be a solution of (1)–(2) if x satisfies the equation (1) for $t \in [0,T]$ and the multi-point conditions (2).

3. Main results

We set the following assumptions:

(A1) $F : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous for each $t \in [0,T]$, strongly measurable for all $x,y,z \in \mathbb{R}$.

(A2) There exist constants $M_1, M_2 > 0$, such that for each $t \in [0,T]$ and all $x_i, y_i, z_i \in \mathbb{R}$, $i = 1,2$ we have

$$|F(t,x_1,y_1,z_1) - F(t,x_2,y_2,z_2)| \leqslant M_1\left[|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|\right];$$

and $M_2 = \sup\limits_{t\in[0,T]} \left| F\left(t,0,0,\int\limits_0^t h(t,s,0,0)\,ds\right)\right|$.

(A3) $h : \Delta \times \mathbb{R}^2 \to \mathbb{R}$ is continuous for $(t,s) \in \Delta$ and there exists constant $L > 0$, such that for each $(t,s) \in \Delta$ and all $x_i, y_i \in \mathbb{R}$, $i = 1,2$ we have

$|h(t,s,x_1,y_1) - h(t,s,x_2,y_2|) \leqslant L[|x_1 - x_2| + |y_1 - y_2|]$.

Let us define for each $t \in [0,T]$, $x \in C^1([0,T],\mathbb{R})$ the operator $\overline{F}$ by

$$\overline{F}(t,x(t)) = F\left(t, x(t), x'(t), \int\limits_0^t h\left(t,s,x(s),x'(s)\right) ds\right)$$

and note that the fractional integrals $I^{\alpha}_{0+}\overline{F}(t,x(t))$ and $I^{\alpha-1}_{0+}\overline{F}(t,x(t))$ exist with $1 < \alpha < 2$.

LEMMA 7. *Let assumptions **(A1)–(A3)** be satisfied. Then for each $t \in [0,T]$ and all $x_1, x_2 \in C^1([0,T],\mathbb{R})$ there exists a positive constant Ω such that*

$$\left|\overline{F}(t,x_1(t)) - \overline{F}(t,x_2(t))\right| \leqslant \Omega \|x_1 - x_2\|_1 . \tag{6}$$

Proof. By using (**A3**) we get for each $t \in [0,T]$ and all $x_1, x_2 \in C^1([0,T],\mathbb{R})$

$$\begin{aligned}
&\left|\int_0^t \left[h\left(t,s,x_1(s),x_1'(s)\right) - h\left(t,s,x_2(s),x_2'(s)\right)\right] ds\right| \\
&\leqslant \int_0^t L[|x_1(s) - x_2(s)| + \left|x_1'(s) - x_2'(s)\right|] ds \\
&\leqslant L\left(\sup_{t\in[0,T]} |x_1(t) - x_2(t)| + \sup_{t\in[0,T]} \left|x_1'(t) - x_2'(t)\right|\right) t
\end{aligned}$$

to use in

$$\begin{aligned}
&\left|\overline{F}(t,x_1(t)) - \overline{F}(t,x_2(t))\right| \\
&\leqslant M_1\left(|x_1(t) - x_2(t)| + \left|x_1'(t) - x_2'(t)\right|\right) \\
&\quad + M_1 L\left(\sup_{t\in[0,T]} |x_1(t) - x_2(t)| + \sup_{t\in[0,T]} \left|x_1'(t) - x_2'(t)\right|\right) t \\
&\leqslant \Omega \|x_1 - x_2\|_1 .
\end{aligned}$$

where $\Omega := M_1[1 + LT]$. □

Now, we give the integral equation satisfied by the solution of the problem (1)–(2).

LEMMA 8. *Assume that **(A1)** is satisfied and $\lambda \neq -1$, $\beta \neq -1$, $0 < \eta < T$. Then, $x \in C^1([0,T],\mathbb{R})$ is a solution of the problem (1)–(2) if and only if it satisfies the following integral equation, for each $t \in [0,T]$*

$$x(t) = \frac{1}{\Gamma(\alpha)}\int_0^\eta \Lambda_{\lambda,\beta,\eta}(t,s)\,\overline{F}(s,x(s))\,ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,\overline{F}(s,x(s))\,ds, \tag{7}$$

where

$$\Lambda_{\lambda,\beta,\eta}(t,s) = \left[\frac{-\lambda}{(\lambda+1)}(\eta-s)^{\alpha-1} + \left(\frac{\lambda}{(\lambda+1)}\eta - t\right)\frac{\beta(\alpha-1)}{(\beta+1)}(\eta-s)^{\alpha-2}\right]. \tag{8}$$

Proof. First we prove the necessity. By applying $I_{0^+}^{\alpha}$ to (1) and from (5) we get for each $t \in [0,T]$

$$x(t) + c_0 + c_1 t = I_{0^+}^{\alpha}\overline{F}(t,x(t)) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,\overline{F}(s,x(s))\,ds. \tag{9}$$

Putting alternately $t=0$ and $t=\eta$ in (9), we obtain from condition (2)

$$c_0=\lambda x(\eta)=\lambda\frac{1}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}\overline{F}(s,x(s))\,ds-\lambda c_0-\lambda c_1\eta. \tag{10}$$

In addition, the differentiation of (9) gives

$$x'(t)+c_1=I_{0^+}^{\alpha-1}\overline{F}(t,x(t))=\frac{\alpha-1}{\Gamma(\alpha)}\int_0^{t}(t-s)^{\alpha-2}\overline{F}(s,x(s))\,ds, \tag{11}$$

likewise, from (2) we get $c_1=\beta x'(\eta)$. Also, by putting $t=\eta$ in (11) we have

$$x'(\eta)=\frac{\alpha-1}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-2}\overline{F}(s,x(s))\,ds-c_{1,} \tag{12}$$

it follows

$$c_1=\frac{\beta(\alpha-1)}{(\beta+1)\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-2}\overline{F}(s,x(s))\,ds. \tag{13}$$

Substituting (13) in (10) gives

$$c_0=\frac{\lambda}{(\lambda+1)\Gamma(\alpha)}\int_0^{\eta}\left[(\eta-s)^{\alpha-1}-\eta\frac{\beta(\alpha-1)}{(\beta+1)}(\eta-s)^{\alpha-2}\right]\overline{F}(s,x(s))\,ds. \tag{14}$$

Consequently, replacing the constant c_0 and c_1 respectively by (14) and (13) in the equation (9) we get

$$\begin{aligned}x(t)&=\frac{1}{\Gamma(\alpha)}\int_0^{\eta}\left[\frac{-\lambda}{\lambda+1}(\eta-s)^{\alpha-1}+\frac{\beta(\alpha-1)}{(\beta+1)}\left(\frac{\lambda}{\lambda+1}\eta-t\right)(\eta-s)^{\alpha-2}\right]\\&\quad\times\overline{F}(s,x(s))\,ds+\frac{1}{\Gamma(\alpha)}\int_0^{t}(t-s)^{\alpha-1}\overline{F}(s,x(s))\,ds.\end{aligned}$$

According to the notation (8), we remark that $x(t)$ satisfies (7).

For $t=0$ we have from (2)

$$\begin{aligned}x(0)&=-\lambda x(\eta)\\&=-\lambda\left[\frac{1}{\Gamma(\alpha)}\int_0^{\eta}\Lambda_{\lambda,\beta,\eta}(t,s)\overline{F}(s,x(s))\,ds+\frac{1}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}\overline{F}(s,x(s))\,ds\right]\end{aligned}$$

$$= -\lambda \frac{1}{\Gamma(\alpha)} \int_0^{\eta} \left[\frac{(\eta - s)^{\alpha-1}}{(\lambda+1)} - \eta \frac{\beta(\alpha-1)}{(\lambda+1)(\beta+1)} (\eta - s)^{\alpha-2} \right] \overline{F}(s, x(s))\, ds$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^{\eta} \left[\frac{-\lambda}{(\lambda+1)} (\eta - s)^{\alpha-1} + \frac{\lambda}{(\lambda+1)} \eta \frac{\beta(\alpha-1)}{(\beta+1)} (\eta - s)^{\alpha-2} \right] \overline{F}(s, x(s))\, ds$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^{\eta} \Lambda_{\lambda,\beta,\eta}(0,s) \overline{F}(s, x(s))\, ds;$$

which implies that for $t = 0$ the integral equation (7) is satisfied by solution of problem (1)–(2) .

On the other sense, remark that x which satisfies (7) belongs to $C^1([0,T], \mathbb{R})$ under (**A1**), in fact

$$x'(t) = \frac{-(\alpha-1)}{\Gamma(\alpha)} \int_0^{\eta} \frac{\beta}{(\beta+1)} (\eta - s)^{\alpha-2} \overline{F}(s, x(s))\, ds \qquad (15)$$
$$+ \frac{\alpha-1}{\Gamma(\alpha)} \int_0^{t} (t-s)^{\alpha-2} \overline{F}(s, x(s))\, ds.$$

Also, its α-fractional derivative exists on $(0, T]$ for $1 < \alpha < 2$. Indeed, by applying ${}^{c}D_{0^+}^{\alpha}$ to (7) we get

$$ {}^{c}D_{0^+}^{\alpha} x(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^{t} (t-s)^{1-\alpha} \frac{\partial^2}{\partial s^2} \left(\frac{1}{\Gamma(\alpha)} \int_0^{\eta} \Lambda_{\lambda,\beta,\eta}(s,r) \overline{F}(r, x(r))\, dr \right) ds$$
$$+ {}^{c}D_{0^+}^{\alpha} I_{0^+}^{\alpha} \overline{F}(t, x(t)) = \overline{F}(t, x(t))$$

since $\frac{\partial^2}{\partial s^2} \left(\frac{1}{\Gamma(\alpha)} \int_0^{\eta} \Lambda_{\lambda,\beta,\eta}(s,r) \overline{F}(r, x(r))\, dr \right) = 0$. To check the first initial condition, we set $t = 0$ in (7) and obtain

$$x(0) = \frac{1}{\Gamma(\alpha)} \int_0^{\eta} \Lambda_{\lambda,\beta,\eta}(0,s) \overline{F}(s, x(s))\, ds$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^{\eta} \left[\frac{-\lambda}{(\lambda+1)} (\eta - s)^{\alpha-1} + \left(\frac{\lambda}{(\lambda+1)} \eta \right) \frac{\beta(\alpha-1)}{(\beta+1)} (\eta - s)^{\alpha-2} \right] \overline{F}(s, x(s))\, ds.$$

Then we set $t=\eta$ in (7) and get

$$x(\eta)=\frac{1}{\Gamma(\alpha)}\int_0^{\eta}\left[\frac{-\lambda}{\lambda+1}(\eta-s)^{\alpha-1}+\frac{\beta(\alpha-1)}{(\beta+1)}\left(\frac{\lambda}{\lambda+1}\eta-\eta\right)(\eta-s)^{\alpha-2}\right]$$

$$\times\overline{F}(s,x(s))\,ds+\frac{1}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}\overline{F}(s,x(s))\,ds$$

$$=\frac{1}{\Gamma(\alpha)}\int_0^{\eta}\left[\frac{1}{\lambda+1}(\eta-s)^{\alpha-1}+\frac{\beta(\alpha-1)}{(\beta+1)}\frac{(-\eta)}{\lambda+1}(\eta-s)^{\alpha-2}\right]\overline{F}(s,x(s))\,ds.$$

It follows by simple calculus that $x(0)=-\lambda x(\eta)$. For the second initial condition, we put $t=0$ then $t=\eta$ in (15) and obtain respectively

$$x'(0)=\frac{1}{\Gamma(\alpha)}\int_0^{\eta}\left[-\frac{\beta(\alpha-1)}{(\beta+1)}(\eta-s)^{\alpha-2}\right]\overline{F}(s,x(s))\,ds,$$

$$x'(\eta)=\frac{1}{\Gamma(\alpha)}\int_0^{\eta}\left[-\frac{\beta(\alpha-1)}{(\beta+1)}(\eta-s)^{\alpha-2}\right]\overline{F}(s,x(s))\,ds$$

$$+\frac{\alpha-1}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-2}\overline{F}(s,x(s))\,ds$$

$$=\frac{1}{\Gamma(\alpha)}\int_0^{\eta}\frac{(\alpha-1)}{\beta+1}(\eta-s)^{\alpha-2}\overline{F}(s,x(s))\,ds.$$

This gives $x'(0)=-\beta x'(\eta)$ which leads to the second initial condition. This completes the proof. □

THEOREM 9. *The function $\Lambda_{\lambda,\beta,\eta}(t,s)$ defined by (8) satisfies the following properties: $\Lambda_{\lambda,\beta,\eta}(t,s)$ is continuous for each $(t,s)\in[0,T]\times[0,\eta)$ and there exist two positive constants Λ_1, Λ_2 such that*

$$\sup_{t\in(0,T)}\int_0^{\eta}\left|\Lambda_{\lambda,\beta,\eta}(t,s)\right|ds=\Lambda_1 \quad \text{and} \quad \sup_{t\in(0,T)}\int_0^{\eta}\left|\frac{\partial\Lambda_{\lambda,\beta,\eta}(t,s)}{\partial t}\right|ds=\Lambda_2.$$

Proof. It's clear that $\Lambda_{\lambda,\beta,\eta}(t,s)$ is a continuous function for each $(t,s)\in[0,T]\times[0,\eta)$ and we have

$$\int_0^{\eta}\left|\Lambda_{\lambda,\beta,\eta}(t,s)\right|ds\leqslant\left|\frac{-\lambda}{(\lambda+1)}\right|\int_0^{\eta}(\eta-s)^{\alpha-1}ds$$

$$+\left|\left(\frac{\lambda}{(\lambda+1)}\eta-t\right)\frac{\beta(\alpha-1)}{(\beta+1)}\right|\int_0^{\eta}(\eta-s)^{\alpha-2}ds$$

$$\leqslant \left|\frac{\lambda}{\lambda+1}\right| \frac{\eta^{\alpha}}{\alpha} + \left(\left|\frac{\lambda}{\lambda+1}\right| \eta + T\right) \left|\frac{\beta}{\beta+1}\right| \eta^{\alpha-1}.$$

Thus, $\Lambda_1 := \left|\frac{\lambda}{\lambda+1}\right| \frac{\eta^{\alpha}}{\alpha} + \left(\left|\frac{\lambda}{\lambda+1}\right| \eta + T\right) \left|\frac{\beta}{\beta+1}\right| \eta^{\alpha-1}$.
By differentiation of $\Lambda_\eta(t,s)$ we have

$$\frac{\partial \Lambda_{\lambda,\beta,\eta}(t,s)}{\partial t} = -\frac{\beta(\alpha-1)}{(\beta+1)}(\eta-s)^{\alpha-2},$$

and it's easy to get

$$\int_0^\eta \left|\frac{\partial \Lambda_{\lambda,\beta,\eta}(t,s)}{\partial t}\right| ds = \int_0^\eta \left|\frac{\beta(\alpha-1)}{\beta+1}\right| (\eta-s)^{\alpha-2} ds \leqslant \left|\frac{\beta}{\beta+1}\right| \eta^{\alpha-1}.$$

Then $\Lambda_2 := \left|\frac{\beta}{\beta+1}\right| \eta^{\alpha-1}$. This completes the proof. □

Now, give the existence and uniqueness result obtained via the Banach fixed point theorem.

THEOREM 10. *Assume that (**A1**)–(**A3**) are satisfied. If*

$$\frac{\Omega}{\Gamma(\alpha)}\left[\Lambda_1 + \Lambda_2 + \frac{T^{\alpha}}{\alpha} + T^{\alpha-1}\right] < 1; \tag{16}$$

then the initial value problem (1)–(2) has a unique solution in $C^1([0,T],\mathbb{R})$.

Proof. First, define the operator Φ by

$$\Phi x(t) = \frac{1}{\Gamma(\alpha)}\int_0^\eta \Lambda_{\lambda,\beta,\eta}(t,s)\overline{F}(s,x(s))\,ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\overline{F}(s,x(s))\,ds \tag{17}$$

where $\Lambda_\eta(t,s)$ is defined by (8). Under (**A1**) Φ maps clearly $C^1([0,T],\mathbb{R})$ into itself. We have to show that Φ is a contraction. Let $x,y \in C^1([0,T],\mathbb{R})$, then for each $t \in [0,T]$ we have by virtue of (6)

$$\begin{aligned}|\Phi x(t) - \Phi y(t)| \leqslant{}& \frac{1}{\Gamma(\alpha)}\int_0^\eta \left|\Lambda_{\lambda,\beta,\eta}(t,s)\right| \left|\overline{F}(s,x(s)) - \overline{F}(s,y(s))\right| ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|\overline{F}(s,x(s)) - \overline{F}(s,y(s))\right| ds\end{aligned}$$

$$\leqslant \frac{\Omega \|x-y\|_1}{\Gamma(\alpha)} \left[\int_0^\eta \left| \Lambda_{\lambda,\beta,\eta}(t,s) \right| ds + \int_0^t (t-s)^{\alpha-1} ds \right]$$
$$\leqslant \frac{\Omega}{\Gamma(\alpha)} \left[\Lambda_1 + \frac{T^\alpha}{\alpha} \right] \|x-y\|_1 .$$

After differentiation of $(\Phi x(t) - \Phi y(t))$ we get for each $t \in [0,T]$

$$\begin{aligned} |(\Phi x(t) - \Phi y(t))'| &\leqslant \frac{1}{\Gamma(\alpha)} \int_0^\eta \left| \frac{\partial \Lambda_\eta(t,s)}{\partial t} \right| \left| \overline{F}(s,x(s)) - \overline{F}(s,y(s)) \right| ds \\ &\quad + \frac{\alpha-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2} \left| \overline{F}(s,x(s)) - \overline{F}(s,y(s)) \right| ds \\ &\leqslant \frac{\Omega \|x-y\|_1}{\Gamma(\alpha)} \left[\int_0^\eta \left| \frac{\partial \Lambda_{\lambda,\beta,\eta}(t,s)}{\partial t} \right| ds + (\alpha-1) \int_0^t (t-s)^{\alpha-2} ds \right]. \end{aligned}$$

In view of theorem 9, we have

$$\left|(\Phi x(t) - \Phi y(t))'\right| \leqslant \frac{\Omega}{\Gamma(\alpha)} \left[\Lambda_2 + T^{\alpha-1}\right] \|x-y\|_1 .$$

Thus

$$\|\Phi x - \Phi y\|_1 \leqslant \frac{\Omega}{\Gamma(\alpha)} \left[\Lambda_1 + \Lambda_2 + \frac{T^\alpha}{\alpha} + T^{\alpha-1} \right] \|x-y\|_1 .$$

Consequently, Φ is a contraction together with the condition (16). We conclude that Φ has unique fixed point in $C^1([0,T],\mathbb{R})$ which is the unique solution of the problem (1)–(2). The proof is complete. □

In the second existence result based on Krasnoselskii theorem, we reduce the condition (16) but we lose the uniqueness of the solution.

THEOREM 11. *Assume that **(A1)**–**(A3)** are satisfied. If*

$$\frac{\Omega}{\Gamma(\alpha)} [\Lambda_1 + \Lambda_2] < 1, \tag{18}$$

then the problem (1)–(2) has at least one solution in $C^1([0,T],\mathbb{R})$.

Proof. Let us define the operators $\Phi_i : C^1([0,T],\mathbb{R}) \to C^1([0,T],\mathbb{R})$, $i = 1,2$ by

$$\Phi_1 x(t) = \frac{1}{\Gamma(\alpha)} \int_0^\eta \Lambda_{\lambda,\beta,\eta}(t,s) \overline{F}(s,x(s)) ds \tag{19}$$

and

$$\Phi_2 x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \overline{F}(s, x(s))\, ds. \tag{20}$$

Set $M_\theta = \left\{ x \in C^1([0,T], \mathbb{R}) : \|x\|_1 \leqslant \theta \right\}$, a bounded, closed and convex subset of $C^1([0,T], \mathbb{R})$, where θ is some positive constant such that

$$\frac{K_\theta}{\Gamma(\alpha)} \left[\Lambda_1 + \Lambda_2 + \frac{T^\alpha}{\alpha} + T^{\alpha-1} \right] \leqslant \theta,$$

with $K_\theta = \max_{\|x\|_1 \leqslant \theta} \sup_{t\in[0,T]} \left| \overline{F}(t, x(t)) \right| := \Omega\theta + M_2$, from the fact that for each $t \in [0,T]$ and all $x \in C^1([0,T], \mathbb{R})$

$$\begin{aligned} \left| \overline{F}(t,x(t)) \right| &\leqslant \left| \overline{F}(t,x(t)) - \overline{F}(t,0) \right| + \left| \overline{F}(t,0) \right| \\ &\leqslant \Omega \|x\|_1 + M_2. \end{aligned}$$

For all $x, y \in M_\theta$ and each $t \in [0,T]$, we get

$$\begin{aligned} &\left| \Phi_1 x(t) + \Phi_2 y(t) \right| \\ &\leqslant \frac{1}{\Gamma(\alpha)} \int_0^\eta \left| \Lambda_{\lambda,\beta,\eta}(t,s) \right| \left| \overline{F}(s,x(s)) \right| ds + \frac{1}{\Gamma(\alpha)} \int_0^t \left| (t-s)^{\alpha-1} \right| \left| \overline{F}(s,y(s)) \right| ds \\ &\leqslant \frac{K_\theta}{\Gamma(\alpha)} \left[\int_0^\eta \left| \Lambda_{\lambda,\beta,\eta}(t,s) \right| ds + \int_0^t \left| (t-s)^{\alpha-1} \right| ds \right] \\ &\leqslant \frac{K_\theta}{\Gamma(\alpha)} \left[\Lambda_1 + \frac{T^\alpha}{\alpha} \right]. \end{aligned}$$

By the same arguments we get for all $x, y \in M_\theta$ and each $t \in [0,T]$

$$\begin{aligned} &\left| (\Phi_1 x(t) + \Phi_2 y(t))' \right| \\ &\leqslant \frac{1}{\Gamma(\alpha)} \int_0^\eta \left| \frac{\partial \Lambda_{\lambda,\beta,\eta}(t,s)}{\partial t} \right| \left| \overline{F}(s,x(s)) \right| ds + \frac{\alpha-1}{\Gamma(\alpha)} \int_0^t \left| (t-s)^{\alpha-2} \right| \left| \overline{F}(s,y(s)) \right| ds \\ &\leqslant \frac{K_\theta}{\Gamma(\alpha)} \left[\int_0^\eta \left| \frac{\partial \Lambda_{\lambda,\beta,\eta}(t,s)}{\partial t} \right| ds + (\alpha-1) \int_0^t \left| (t-s)^{\alpha-2} \right| ds \right] \\ &\leqslant \frac{K_\theta}{\Gamma(\alpha)} \left[\Lambda_2 + T^{\alpha-1} \right]. \end{aligned}$$

Thus,

$$\|\Phi_1 x + \Phi_2 y\|_1 \leqslant \frac{K_\theta}{\Gamma(\alpha)} \left[\Lambda_1 + \Lambda_2 + \frac{T^\alpha}{\alpha} + T^{\alpha-1} \right] \leqslant \theta.$$

This shows that $\Phi_1 x + \Phi_2 y \in M_\theta$ for all $x, y \in M_\theta$. To prove that Φ_1 is a contraction, let $x, y \in M_\theta$, then we get for each $t \in [0,T]$

$$\begin{aligned}
&\left|\Phi_1 x(t) - \Phi_1 y(t)\right| \\
&\qquad \leqslant \frac{1}{\Gamma(\alpha)} \int_0^\eta \left|\Lambda_{\lambda,\beta,\eta}(t,s)\right| \left|\overline{F}(s,x(s)) - \overline{F}(s,y(s))\right| ds \\
&\qquad \leqslant \frac{\Omega}{\Gamma(\alpha)} \Lambda_1 \|x-y\|_1 .
\end{aligned}$$

After differentiation, we obtain for each $t \in [0,T]$

$$\begin{aligned}
&\left|(\Phi_1 x(t) - \Phi_1 y(t))'\right| \\
&\qquad \leqslant \frac{1}{\Gamma(\alpha)} \int_0^\eta \left|\frac{\partial \Lambda_{\lambda,\beta,\eta}(t,s)}{\partial t}\right| \left|\overline{F}(s,x(s)) - \overline{F}(s,y(s))\right| ds \\
&\qquad \leqslant \frac{\Omega}{\Gamma(\alpha)} \Lambda_2 \|x-y\|_1 .
\end{aligned}$$

From the previous we deduce

$$\|\Phi_1 x - \Phi_1 y\|_1 \leqslant \frac{\Omega}{\Gamma(\alpha)} [\Lambda_1 + \Lambda_2] \|x-y\|_1 .$$

In view of (18), we conclude that Φ_1 is a contraction.

Now, we will show that Φ_2 is compact and continuous. Let $\{x_n\}_{n \geqslant 1}$ be a sequence in M_θ such that $x_n(t) \to x(t)$, $x_n'(t) \to x'(t)$ in M_θ, for each $t \in [0,T]$. Then, we have $\|x_n - x\|_1 \to 0$ when $n \to \infty$. We infer that for each $t \in [0,T]$

$$\begin{aligned}
&\left|\Phi_2 x_n(t) - \Phi_2 x(t)\right| \\
&\qquad \leqslant \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|\overline{F}(s,x_n(s))) - \overline{F}(s,x(s))\right| ds \\
&\qquad \leqslant \frac{\Omega}{\Gamma(\alpha)} \|x_n - x\|_1 \int_0^t (t-s)^{\alpha-1} ds \\
&\qquad \leqslant \frac{\Omega T^\alpha}{\alpha \Gamma(\alpha)} \|x_n - x\|_1
\end{aligned}$$

and after differentiation of $(\Phi_2 x_n(t) - \Phi_2 x(t))$ we have for each $t \in [0,T]$

$$\begin{aligned}
&\left|(\Phi_2 x_n(t) - \Phi_2 x(t))'\right| \\
&\qquad \leqslant \frac{\alpha-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2} \left|\overline{F}(s,x_n(s)) - \overline{F}(s,x(s))\right| ds \\
&\qquad \leqslant \frac{\Omega}{\Gamma(\alpha)} \|x_n - x\|_1 (\alpha-1) \int_0^t (t-s)^{\alpha-2} ds \\
&\qquad \leqslant \frac{\Omega}{\Gamma(\alpha)} T^{\alpha-1} \|x_n - x\|_1 .
\end{aligned}$$

Consequently,

$$\|\Phi_2 x_n - \Phi_2 x\|_1 \leqslant \frac{\Omega}{\Gamma(\alpha)} \left[\frac{T^\alpha}{\alpha} + T^{\alpha-1}\right] \|x_n - x\|_1 .$$

The term in the right-hand side of the previous inequality tends clearly, to zero when $n \to \infty$, which means that Φ_2 is continuous. $\Phi_2 x(t)$ is uniformly bounded in M_θ from the fact that

$$|\Phi_2 x(t)| \leqslant \frac{K_\theta}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \leqslant \frac{K_\theta}{\Gamma(\alpha)} \frac{T^\alpha}{\alpha}$$

and

$$\left|(\Phi_2 x(t))'\right| \leqslant \frac{K_\theta (\alpha-1)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2} ds \leqslant \frac{K_\theta}{\Gamma(\alpha)} T^{\alpha-1}.$$

Hence,

$$\|\Phi_2 x\|_1 \leqslant \frac{K_\theta}{\Gamma(\alpha)} \left[\frac{T^\alpha}{\alpha} + T^{\alpha-1}\right] \leqslant \theta.$$

To show that $\Phi_2 x(t)$ is equicontinuous, let $0 < \tau_1 < \tau_2 < T$, then we have

$$\begin{aligned}
&|\Phi_2 x(\tau_2) - \Phi_2 x(\tau_1)| \\
&\quad \leqslant \frac{1}{\Gamma(\alpha)} \left| \int_0^{\tau_2} (\tau_2 - s)^{\alpha-1} \overline{F}(s, x(s))\, ds - \int_0^{\tau_1} (\tau_1 - s)^{\alpha-1} \overline{F}(s, x(s))\, ds \right| \\
&\quad \leqslant \frac{1}{\Gamma(\alpha)} \int_0^{\tau_1} \left[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}\right] \left|\overline{F}(s, x(s))\right| ds \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} \left|\overline{F}(s, x(s))\right| ds \\
&\quad \leqslant \frac{K_\theta}{\Gamma(\alpha)} \int_0^{\tau_1} \left[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}\right] ds + \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} ds \\
&\quad \leqslant \frac{K_\theta}{\Gamma(\alpha)} \left[\frac{-(\tau_2 - \tau_1)^\alpha}{\alpha} + \frac{\tau_2^\alpha}{\alpha} - \frac{\tau_1^\alpha}{\alpha} + \frac{(\tau_2 - \tau_1)^\alpha}{\alpha}\right] \\
&\quad \leqslant \frac{K_\theta}{\alpha \Gamma(\alpha)} (\tau_2^\alpha - \tau_1^\alpha).
\end{aligned} \tag{21}$$

Also, we get

$$\begin{aligned}
&|(\Phi_2 x(\tau_2))' - (\Phi_2 x(\tau_1))'| \\
&\qquad \leqslant \frac{\alpha-1}{\Gamma(\alpha)} \int_0^{\tau_1} \left[(\tau_2 - s)^{\alpha-2} - (\tau_1 - s)^{\alpha-2}\right] \left|\overline{F}(s, x(s))\right| ds \\
&\qquad\quad + \frac{\alpha-1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-2} \left|\overline{F}(s, x(s))\right| ds \\
&\qquad \leqslant \frac{K_\theta}{\Gamma(\alpha)} \left(\tau_2^{\alpha-1} - \tau_1^{\alpha-1}\right).
\end{aligned} \tag{22}$$

Thanks to (21)–(22) it follows

$$\|\Phi_2 x(\tau_2) - \Phi_2 x(\tau_1)\|_1 \leqslant \frac{K_\theta}{\Gamma(\alpha)} \left[\frac{1}{\alpha}\left(\tau_2^\alpha - \tau_1^\alpha\right) + \left(\tau_2^{\alpha-1} - \tau_1^{\alpha-1}\right)\right]$$

which tends to zero when $\tau_2 \to \tau_1$. So, $\Phi_2(M_\theta)$ is relatively compact and together with Arzelà-Ascoli theorem Φ_2 is compact.

Finally, we conclude by Krasnoselskii's theorem that $\Phi = \Phi_1 + \Phi_2$ has at least one fixed point in $M_\theta \subset C^1([0,T],\mathbb{R})$, which is a solution of (1)–(2). □

REMARK 12. The solution of multipoint-initial value problem (1)–(2) with integer order $\alpha = 2$, belongs to $C^2((0,T],\mathbb{R})$ and satisfies integral equation (7) for $\alpha = 2$.

Morover, the results obtained for non-integer order $1 < \alpha < 2$ stay true.

In the sequel, we discuss the positivity of the obtained solution in $C^1([0,T],\mathbb{R})$ by the Banach fixed point theorem. To this end, we add adequate assumptions and expound the following theorem.

Note that we mean by a positive solution of the problem (1)–(2) in $C^1([0,T],\mathbb{R})$ that $x(t) > 0$ and $x'(t) > 0$ for each $t \in [0,T]$.

THEOREM 13. *Assume that (**A1**)–(**A3**) are fulfilled in $\mathbb{R}_+$ such that $F:[0,T]\times {\mathbb{R}_+}^3 \to \mathbb{R}_+$, $h:\Delta\times{\mathbb{R}_+}^2 \to \mathbb{R}_+$ where $\Delta = \{(t,s): 0 \leqslant s \leqslant t \leqslant T\}$.*

If (16) is satisfied for $-1 < \lambda < 0$; $-1 < \beta < 0$, then the unique solution in $C^1([0,T],\mathbb{R})$ of the problem (1)–(2) is positive.

Proof. In view of theorem 10 and the fact that for $-1 < \lambda < 0$; $-1 < \beta < 0$, (16) is a particular case, then (1)–(2) admits a unique solution in $C^1([0,T],\mathbb{R})$.

Moreover, since $1 < \alpha < 2$, $0 < \eta < T$ we have for each $t \in [0,T], s \in [0,\eta)$

$$\frac{-\lambda}{(\lambda+1)}(\eta - s)^{\alpha-1} > 0; \quad \left(\frac{\lambda}{(\lambda+1)}\eta - t\right) < 0$$

and

$$\frac{\beta(\alpha-1)}{(\beta+1)}(\eta-s)^{\alpha-2} > 0.$$

Thus, for each $(t,s) \in [0,T]\times[0,\eta)$

$$\begin{aligned}&\Lambda_{\lambda,\beta,\eta}(t,s)\\ &= \left[\frac{-\lambda}{(\lambda+1)}(\eta-s)^{\alpha-1} + \left(\frac{\lambda}{(\lambda+1)}\eta - t\right)\frac{\beta(\alpha-1)}{(\beta+1)}(\eta-s)^{\alpha-2}\right] > 0\end{aligned}$$

and

$$\frac{\partial \Lambda_{\lambda,\beta,\eta}(t,s)}{\partial t} = -\frac{\beta(\alpha-1)}{(\beta+1)}(\eta-s)^{\alpha-2} > 0.$$

From integration properties it results that $x(t)$ the unique solution of (1)–(2) which satisfies (7) is positive for each $t \in [0,T]$ and the same for $x'(t)$ which satisfies (15). This completes the poof. □

A similar argument is used to provide the positivity of each solution obtained via Krasnoselskii fixed point theorem.

THEOREM 14. *Assume that (**A1**)–(**A3**) are fulfilled in* $\mathbb{R}_+$ *such that* $F : [0,T] \times {\mathbb{R}_+}^3 \to \mathbb{R}_+$, $h : \Delta \times {\mathbb{R}_+}^2 \to \mathbb{R}_+$ *where* $\Delta = \{(t,s) : 0 \leqslant s \leqslant t \leqslant T\}$.

If (18) *is satisfied for* $-1 < \lambda < 0$; $-1 < \beta < 0$ *then each solution in* $C^1([0,T],\mathbb{R})$ *of the problem* (1)–(2) *is positive.*

4. Example

Let us consider the following fractional initial value problem

$$
\begin{aligned}
{}^{c}D_{0^+}^{3/2} x(t) &= \frac{x(t)}{(t+10)^4 (1+|x(t)|)} + \frac{x'(t)}{(t+10)^4 (1+|x'(t)|)} \\
&\quad + \int_0^t \left[\frac{e^{-s}(x(s)+1)}{(t+10)^{10}(2+|x(s)|)} + \frac{e^{-s}(x'(s)+1)}{(t+10)^{10}(2+|x'(s)|)} \right] ds,
\end{aligned} \tag{23}
$$

subject to the multi-point initial conditions:

$$
x(0) + x\left(\frac{1}{2}\right) = 0; \quad x'(0) + x'\left(\frac{1}{2}\right) = 0. \tag{24}
$$

Here $\alpha = \frac{3}{2}$, $\lambda = \beta = 1$, $\eta = \frac{1}{2}$, $T = 1$

$$
\begin{aligned}
F(t,x,y,z) &= \frac{1}{(t+10)^4} \left[\frac{x}{1+|x|} + \frac{y}{1+|y|} + z \right], \\
h(t,s,x,y) &= \frac{e^{-s}}{(t+10)^6} \left[\frac{x+1}{2+|x|} + \frac{y+1}{2+|y|} \right].
\end{aligned}
$$

Observe that (A1)-(A3) are satisfied, for all $x_i, y_i, z_i \in \mathbb{R}$, $i = 1,2$ and each $t \in [0,1]$ we have

$$
|F(t,x_1,y_1,z_1) - F(t,x_2,y_2,z_2)| \leqslant \frac{1}{(t+10)^4} [|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|],
$$

thus, $M_1 = \dfrac{1}{10^4}$ and $M_2 = \sup\limits_{t \in [0,1]} \dfrac{1 - e^{-t}}{(t+10)^{10}} = \dfrac{1 - e^{-1}}{10^{10}}$.

For all $x_i, y_i \in \mathbb{R}, i = 1,2$ and each $(t,s) \in \Delta$ we have

$$
|h(t,s,x_1,y_1) - h(t,s,x_2,y_2)| \leqslant \frac{e^{-s}}{(t+10)^6} [|x_1 - x_2| + |y_1 - y_2|],
$$

thus, $L_1 = \dfrac{1}{10^6}$.

Also, condition (16) of theorem 10 is satisfied in accordance with following calculus

$$\frac{\frac{1}{10^4}\left[1+\frac{1}{10^6}\right]}{\Gamma(3/2)}\left[\left|\frac{1}{2}\right|\frac{(1/2)^{3/2}}{3/2}+\left(\left|\frac{1}{2}\right|\frac{1}{2}+1+1\right)\left|\frac{1}{2}\right|\left(\frac{1}{2}\right)^{1/2}+\frac{1}{3/2}+1\right]$$
$$=\frac{0.000258001}{0.886226925}<1.$$

Then, there exists a unique solution of (23)–(24) in $C^1([0,1],\mathbb{R})$.

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