Fractional
Differential
Calculus
Volume 4, Number 2 (2014), 89–124 doi:10.7153/fdc-04-06

# ON A LOCAL SOLVABILITY OF THE MULTIDIMENSIONAL MUSKAT PROBLEM WITH A FRACTIONAL DERIVATIVE IN TIME ON THE BOUNDARY CONDITION

NATALIYA VASYLYEVA

*Abstract.* In this paper, we analyze anomalous diffusion version of the multidimensional Muskat problem without surface tension on a free boundary. We prove the existence and uniqueness of the classical solution to this moving boundary problem locally in time.

## 1. Introduction

Let $\Omega$ be a double-connected bounded open domain in $R^n$, $n \geqslant 2$ with the boundary $\partial\Omega = \Gamma_1 \bigcup \Gamma_2$, $\Gamma_1 \bigcap \Gamma_2 = \emptyset$. Let $\Upsilon(t)$, for each $t \in [0,T]$, be a surface $\Upsilon(t) \subset \Omega$ that separates $\Omega$ into two subdomains $\Omega_1(t)$ and $\Omega_2(t)$ such that $\Omega = \Omega_1(t) \bigcup \Upsilon(t) \bigcup \Omega_2(t)$, and $\partial\Omega_i = \Gamma_i \bigcup \Upsilon(t)$, $i = 1,2$.

In this paper we study the two-phase free boundary problem in the case of anomalous diffusion. We look for the functions $p_i(y,t)$, $y \in \Omega_i(t)$, $t \in [0,T]$, $i = 1,2$, and a moving boundary $\Upsilon(t)$ by the following conditions:

$$-\Delta_y p_i = 0 \quad \text{in} \quad \Omega_i(t), \; i = 1,2, \; t \in [0,T], \tag{1.1}$$

$$p_1 - p_2 = 0 \quad \text{on} \quad \Upsilon(t), \tag{1.2}$$

$$V_{n_t}^{\nu} = -k_1 \frac{\partial p_1}{\partial n_t} = -k_2 \frac{\partial p_2}{\partial n_t} \quad \text{on} \quad \Upsilon(t), \; \nu \in (0,1); \tag{1.3}$$

$$p_i = \psi_i(y) \quad \text{on} \quad \Gamma_{iT} = \Gamma_i \times [0,T], \tag{1.4}$$

$$\Omega_i(t)|_{t=0} = \Omega_i, \quad \Upsilon(t)|_{t=0} = \Upsilon \quad \text{are given.} \tag{1.5}$$

Here $\Delta_y = \nabla_y^2$, $\nabla_y = (\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n})$; $k_i$, $i = 1,2$, are given positive constants, $\psi_i(y)$, $i = 1,2$, are given positive functions; $n_t$ is the unit normal to $\Upsilon(t)$ directed in $\Omega_1(t)$; $V_{n_t}^{\nu}$ is the fractional velocity of the boundary $\Upsilon(t)$ in the direction of the normal $n_t$ and is represented by (see, e.g., [35]):

$$V_{n_t}^{\nu} = \langle \mathbf{D}_t^{\nu} \Upsilon(t), n_t \rangle, \tag{1.6}$$

*Mathematics subject classification* (2010): Primary 35R35, 35C15; Secondary 35R11, 35B65.
*Keywords and phrases*: Free boundary problem, Muskat problem, anomalous diffusion, Caputo derivative.


Paper FDC-04-06

where $\langle \cdot,\cdot \rangle$ is the notation of the scalar product. $\mathbf{D}_t^{\nu}$ denotes the Caputo fractional derivative with respect to $t$ and is defined by (see (2.4.6) in [12])

$$\mathbf{D}_t^{\nu} w(\cdot,t) = \frac{1}{\Gamma(1-\nu)} \frac{\partial}{\partial t} \int_0^t \frac{w(\cdot,\tau)d\tau}{(t-\tau)^{\nu}} - \frac{w(\cdot,0)}{\Gamma(1-\nu)t^{\nu}}, \quad \nu \in (0,1), \tag{1.7}$$

where $\Gamma(\nu)$ is the Gamma function.

One can see that (1.1) and (1.2) together with the second equality in (1.3) define the transmission problem with the interface $\Gamma(t)$, and the first equality in (1.3) serves to find the unknown curve $\Gamma(t)$ that is called the free boundary. Moreover, conditions (1.3) mean that the motion in problem (1.1)–(1.5) is subjected by anomalous diffusion.

Note that the anomalous diffusion means that the diffusive motion can not be modelled as standard Brownian motion [6], [22], and the mean square displacement of the diffusing species $\langle(\Delta y)^2\rangle$ scales as a nonlinear power law in time, i.e. $\langle(\Delta y)^2\rangle \sim t^{\nu}$ for some real number $\nu$. In the case $\nu \in (0,1)$, this is referred as a subdiffusion.

If $\nu = 1$, problem (1.1)–(1.5) is called as the Muskat problem (the two-phase Hele-Shaw problem), which was proposed by Muskat in 1934 [23]. This problem describes the evolution of an interface between two immiscible incompressible fluids (for example, water and oil). The motion of fluids is governed by the Darcy law (see (1.3) with $\nu = 1$), stating that the velocities of fluids are proportional to the pressure gradients, and the conservation of mass law. The Muskat problem with a regular initial interface $\Gamma$ was studied by L. Jiang and Y. Chen [11], F. Yi [36], F. Otto [25], S. Howison [10], D. Ambrose [1], M. Siegel, R. Caflish and S. Howison [30], J. Escher and B. V. Matioc [8]. The solvability of the two-phase Hele-Shaw problem in the case of nonsmooth $\Gamma$ was studied in the weighted Hölder classes by B.V. Bazaliy and N. Vasylyeva [4, 5].

In this paper we consider the Muskat problem governed by "fractional" Darcy law which is formulated in [24], [35]. That means the presence of the fractional temporal derivative in condition (1.3).

The one-phase variant of problem (1.1)–(1.5) (the one-phase fractional Hele-Shaw problem) can be called as the fractional quasistationary Stefan problem which arises under consideration of the materials with memory [35], drug release control [18], [17]. Note that if equation (1.1) in the one-phase variant of problem (1.1)–(1.5) is changed by the subdiffusion equation we will get the fractional Stefan problem, which was formulated and studied by C. Atkinson [2] for the motion of planar, cylindrical and spherical domains.

The presence of the fractional derivative in time in (1.3) complicates essentially investigations of problem like (1.1)–(1.5) because this condition is nonlocal and some useful properties of integer order derivative (i.g., product rules, chain rules and so on) are not carried over to the fractional derivative operator. To the best of our knowledge, some exact solutions of the one-phase fractional Hele-Shaw problem and analogous one were constructed in [18], [33], [35], [19] if $\Omega(t) \subset R^1$, $\forall\, t \in [0,T]$. In the case of $\Omega(t) \subset R^n$, the classical solvability of the fractional Hele-Shaw problem for small period of time has been proved in [32]. As for investigations of two-phase fractional moving boundary problems, a class of fractional one-dimensional two-phase free boundary problems was researched in [34], where a novel numeric method was developed to handle the moving interface as well as the history kernel of the subdiffusion. However, solvability of the two-phase moving boundary problem with the fractional velocity of the free boundary has not yet been studied. Thus, the existence and uniqueness of a solution of problem (1.1)–(1.5) will be a subject of our investigations here. In this paper the one of the main results is the following.

THEOREM 1.1. *Let* $\alpha,\nu\in(0,1)$, *the surfaces* $\Gamma_i$, $i=1,2$, $\Upsilon\in C^{l+\alpha}$, $l\geqslant 3$, $0<k_2<k_1$ *and*

$$\min_{\Gamma_2}\psi_2(x)>\max_{\Gamma_1}\psi_1(x);\quad \psi_i(x)|_{\Gamma_i}>0\quad\text{and}\quad \psi_i(x)\in C^{3+\alpha}(\Gamma_i),\ \ i=1,2. \tag{1.8}$$

*Then there exists a unique solution of problem* (1.1)–(1.5) *for some small* $T>0$ *such that* $p_i\in C([0,T];C^{2+\alpha}(\overline{\Omega_i(t)\times\{t\}}))$, $i=1,2$, $\cup_{t\in[0,T]}\Upsilon(t)\in C^{2+\alpha}$, $\cup_{t\in[0,T]}\boldsymbol{D}_t^{\nu}\Upsilon(t)$ $\in C^{1+\alpha}$.

It is obviously that Theorem 1.1 is a generalization of the known result [36] in the case of the normal diffusion ( $\nu=1$ ) to the subdiffusion case ( $\nu\in(0,1)$ ).

Moreover, in this paper we also study the local existence of more smooth solutions of problem (1.1)–(1.5). In Theorem 5.3 of this paper we prove the one-valued solvability in the Hölder classes $C^{k+\alpha,\beta,\alpha}$ with $\beta:=\frac{\alpha\nu}{2}$. Hence, the results obtained in the nonlocal case ( $\nu\in(0,1)$ ) represent marked difference with the local case (see Section 4 [4]) where the exponent $\beta:=\frac{\alpha}{2}$ is greater.

To prove Theorems 1.1 and 5.3 we adapt the classical approach which is used for a free boundary problem in the case of normal diffusion (see, e.g. [3]) to the subdiffusion case. This technique consists in:

1. Reduction of a free boundary problem to a nonlinear problem defined in a fixed domain;
2. Linearization of this nonlinear problem on an initial data $(v_{01},v_{02})$ and on a some special function $s(\omega,t)$ connected with initial shape of the free boundary $\Upsilon(t)$;
3. The proof that the linear problem has a unique solution;
4. Proving that the corresponding nonlinear mapping is a contraction, so that it has a unique fixed point.

Under this consideration, we have to solve a lot of technically difficulties which deal with the nonlocal behavior of the free boundary velocity (i.e. with fractional derivatives). Especially, it becomes apparent on the (2)–(4) steps. Indeed, to linearize the nonlinear problem on the second step we have to construct the appropriate function $s(\omega,t)$ as $s=-k_1\frac{t^{\nu}}{\Gamma(\nu+1)}\frac{\partial v_{10}}{\partial\bar{n}(\omega)}$ and to describe the main properties of this function (see (3.22)–(3.26)). As for (3) and (4) steps of the method mentioned above, we adjust the classical Schauder technique and the contraction argument to the case of the fractional derivatives (see proofs of Theorems 5.1 and 5.2 and Lemma 5.1). To this end we prove some properties of Caputo and Riemann-Liouville derivatives in Propositions 2.1–2.3.

However, the main analytical difficulties deal with the research of a nonclassical boundary value problem with a fractional dynamic boundary condition:

$$\Delta_x u_\pm = 0 \text{ in } R^n_{\pm T};\ \wp(x',0) = 0 \text{ in } R^{n-1},\ u_\pm|_{t=0} = 0 \text{ in } R^n_\pm;$$

$$u_-(x,t) - u_+(x,t) = -a_0\wp(x',t) \quad \text{on} \quad \bar{R}^{n-1}_T;$$

$$\mathbf{D}^\nu_t \wp(x',t) - a_1\left[\frac{\partial u_-}{\partial n} - \frac{\partial u_+}{\partial n}\right] - \langle a_2,\ \nabla_{x'}(u_- - u_+)\rangle = f_1(x',t) \quad \text{on} \quad \bar{R}^{n-1}_T;$$

$$\frac{\partial u_-}{\partial n} - k\frac{\partial u_+}{\partial n} - k\langle a_3,\ \nabla_{x'}(u_- - u_+)\rangle = 0 \quad \text{on} \quad \bar{R}^{n-1}_T.$$

Note that this is the principal model problem such that the nonlinear problem (1.1)–(1.6) will inherit the main feature of this problem. We remark that, the model problem in the case of an integer order derivative in time $(\nu = 1)$ has been well studied with different methods. The one of them consists in the getting exact representation of the solution and the proving some coercive estimates. In this paper we try to follow this method in the case of the fractional derivative in time, $\nu \in (0,1)$. Using Fourier and Laplace transformations, we obtain the solution of this problem as the convolutions (see (4.28)–(4.30)): $u_\pm = G_\pm \star f_1$, $\wp = G \star f_1$. The kernels $G_\pm$ and $G$ can be represented only as integrals which contain the Wright functions. Note that the Wright functions (see, e.g. [21]) play fundamental roles in various applications of the fractional calculus. Thus nonlocal forms of the kernels are the distinguishing feature of the fractional case, $\nu \in (0,1)$. As usual in the potential theory, to estimate the functions $u_\pm(x,t)$ and $\wp(x',t)$, it is necessary to describe well the properties of the kernels $G_\pm$ and $G$. Unfortunately, in virtue of nonlocal representations for these kernels, it is impossible to get the good local estimates as in the case of the integer order derivative. We can get just the integral estimates which are described in Lemma 4.1. Note that to get Lemma 4.1 we essential use the main properties of the Wright functions: asymptotic representations, estimates, formula for fractional differentiation and integration. Moreover, using representation $\wp = G \star f_1$, it is necessary to find a convenient formula of $\mathbf{D}^\nu_t \wp$ for further investigations. As it turns out (see Proposition 4.1), the suitable form is

$$\mathbf{D}^\nu_t \wp = f_1(x',t) + \int_0^t d\tau \int_{R^{n-1}} [f_1(x'-y',t-\tau) - f_1(x',t-\tau)]\partial^\nu_\tau G(y',\tau)dy',$$

where $\partial^\nu_\tau$ denotes the Riemann-Liouville fractional derivative in time. Note that the analogous representation is obvious in the non-fractional case ($\nu = 1$), but it is not evident in the case of fractional derivative. Generally speaking, Lemma 4.1 and Propositions 4.1 and 4.2 play a significant role in the investigation of the model problem and their proofs contain the main difficulties caused the presence of the fractional derivative.

The paper is organized as follows. In Section 2 we describe some auxiliary properties related to the fractional derivative in time, Propositions 2.1–2.3. Note that they are very useful in the technical plan and apply throughout the paper. In Section 3, we reduce the problem with an unknown boundary to a problem in a fixed domain and reformulate the main result as Theorem 3.1. In Subsection 3.2, we represent our nonlinear problem in the form $\mathscr{A}\mathbf{z} = F(x,t) + \mathscr{F}^1(\mathbf{z})$, where $\mathbf{z} = (w_1, w_2, \sigma)$ and $\mathscr{A}$ is a linear operator, the vector $F(x,t)$ is constructed by using the initial data and $\mathscr{F}^1$ is a nonlinear operator. Section 4 is devoted to investigation of the model boundary-value problem with fractional derivative in time in the boundary condition. Then in Section 5, we proved the main result, Theorem 3.1. In Subsection 5.1, using method of a parameter extension together with results of Sections 2 and 4, we get the one-to-one solvability to the linear problem $\mathscr{A}\mathbf{z} = F(x,t)$, Theorem 5.1 (classical solvability) and Theorem 5.2 (solvability in Hölder spaces). In Subsection 5.2, based on the results of Theorem 5.1, Proposition 2.1 and the fixed point theorem, we prove Theorem 3.1. Moreover, in this subsection we get solvability of the problem $\mathscr{A}\mathbf{z} = F(x,t) + \mathscr{F}^1(\mathbf{z})$ in the Hölder classes, Theorem 5.3. The proof of some auxiliary assertions which are applied in Section 4 is given in Appendix 6.

## 2. Functional spaces and preliminaries

Before proving Theorem 1.1 we need in some auxiliary results and some definitions.

Let $D$ be a given domain in $R^n$, $D_T = D\times(0,T)$; $\bar{x},x$ be any points in $\bar{D}$, $x\neq\bar{x}$; $t,\tau\in[0,T]$, $t\neq\tau$; $\alpha,\beta\in(0,1)$, $l$ be a nonnegative integer number. Denote by

$$\langle u\rangle^{(\alpha)}_{x,D_T} = \sup_{(x,t),(\bar{x},t)\in D_T} \frac{|u(x,t)-u(\bar{x},t)|}{|x-\bar{x}|^{\alpha}};$$

$$\langle u\rangle^{(\beta)}_{t,D_T} = \sup_{(x,t),(x,\tau)\in D_T} \frac{|u(x,t)-u(x,\tau)|}{|t-\tau|^{\beta}};$$

$$[u]^{(\alpha,\beta)}_{D_T} = \sup_{x,\bar{x}\in D,t,\ \tau\in[0,T]} \frac{|u(x,t)-u(\bar{x},t)-u(x,\tau)+u(\bar{x},\tau)|}{|x-\bar{x}|^{\alpha}|t-\tau|^{\beta}};$$

In this paper we will use the two types of the functional spaces $C([0,T],\ C^{l+\alpha}(\bar{D}))$ and $C^{l+\alpha,\beta,\alpha}(\bar{D}_T)$. Recall that the spaces $C([0,T],C^{l+\alpha}(\bar{D}))$ used by many authors (see, e.g., [20] and references there).

We define the class $\underset{0}{C}([0,T],C^{l+\alpha}(\bar{D}))$ as the subspace of $C([0,T],\ C^{l+\alpha}(\bar{D}))$ such that $D^j_x u|_{t=0} = 0$, $|j| = \overline{0,l}$.

DEFINITION 2.1. We will say that the function $u(x,t)\in C^{l+\alpha,\beta,\alpha}(\bar{D}_T)$ iff the following norm is finite

$$\|u\|_{C^{l+\alpha,\beta,\alpha}(\bar{D}_T)} = \|u\|_{C([0,T],C^{l+\alpha}(\bar{D}))} + \sum_{|j|=0}^{l}\left\{\langle D^j_x u\rangle^{(\beta)}_{t,D_T} + [D^j_x u]^{(\alpha,\beta)}_{D_T}\right\}.$$

In a similar way we introduce the spaces $C^{l+\alpha,\beta,\alpha}(\partial D_T)$ where $\partial D_T = \partial D\times[0,T]$. Moreover, we will use the usual Hölder classes $C^{l+\alpha}(\overline{D})$ and $C^{l+\alpha}(\partial D)$, their definitions can be found, for instance, in [16].

The following results which is the well known in the case of an integer order derivative will be essentially applied to prove Lemma 4.3, Theorems 5.1, 5.2 and Lemma 5.1.

PROPOSITION 2.1. *Let* $\alpha,\nu\in(0,1)$, $\Omega\subset R^n$, $\Omega_T=\Omega\times(0,T)$, $\psi,\mathbf{D}_t^\nu\psi\in C([0,T],C^\alpha(\bar{\Omega}))$, *then*

$$|\psi(\cdot,t_1)-\psi(\cdot,t_2)|\leqslant C|t_1-t_2|^\nu\sup_{\bar{\Omega}_T}|\mathbf{D}_t^\nu\psi|; \tag{2.1}$$

$$\langle\psi(\cdot,t_1)-\psi(\cdot,t_2)\rangle_{x,\Omega_T}^{(\alpha)}\leqslant C\langle\mathbf{D}_t^\nu\psi\rangle_{x,\Omega_T}^{(\alpha)}|t_1-t_2|^\nu\quad\forall\, t_1,t_2\in[0,T]. \tag{2.2}$$

*Proof.* Following the arguments of Theorem 3.1 [29], one can easily obtain inequality (2.1). Then to get estimate (2.2) it is enough to apply inequality (2.1) to the function $\frac{|\psi(x,t)-\psi(\bar{x},t)|}{|x-\bar{x}|^\alpha}$. □

Next, we define the fractional Riemann-Liouville integral and derivative of a function $g(\cdot,t)$ with respect to $t$ as (see, e.g., (2.1.1) and (2.1.8) [12]):

$$I_t^\theta g(\cdot,t):=\frac{1}{\Gamma(\theta)}\int_0^t\frac{g(\cdot,\tau)d\tau}{(t-\tau)^{1-\theta}},\qquad\theta>0,\ \ t>0; \tag{2.3}$$

$$\partial_t^\theta g(\cdot,t):=\frac{1}{\Gamma(1-\theta)}\frac{\partial}{\partial t}\int_0^t\frac{g(\cdot,\tau)d\tau}{(t-\tau)^\theta},\qquad\theta\in(0,1). \tag{2.4}$$

Repeating the arguments from the proofs of Lemma 2.10 and formula (2.4.10) in [12], we can deduce the following.

PROPOSITION 2.2. *Let* $\alpha,\ \nu\in(0,1)$, $\Omega\subseteq R^n$, $\Omega_T=\Omega\times(0,T)$, $\varphi(x,t)$, $\varphi_1(x,t)\in C([0,T],L_\infty(\Omega))$, $\partial_t^\nu\varphi_1(x,t)\in L^1(\Omega_T)$. *Then*

**(i)** $\mathbf{D}_t^\nu\varphi(x,t)=\partial_t^\nu\varphi(x,t)$, *if* $\varphi(x,0)=0$, *where* $\partial_t^\nu\varphi(x,t)$ *is given by* (2.4);

**(ii)** $\partial_t^\nu\int\limits_0^t\varphi_1(x,t-\tau)\varphi(x,\tau)d\tau=\int\limits_0^t\varphi(x,t-\tau)\partial_\tau^\nu\varphi_1(x,\tau)d\tau+\varphi(x,t)\lim_{z\to0}I_z^{1-\nu}\varphi_1(x,z)$, $\forall t\in[0,T]$.

After that we represent some properties of a solution to the transmission problem which depends on time $t$ as a parameter:

$$\begin{gathered}\Delta_x W_i=g_{0i}(x,t)\ \text{ in }\ \Omega_{iT};\quad W_i|_{t=0}=0\ \text{ in }\ \Omega_i;\\ W_i=g_i(x,t)\ \text{ on }\ \Gamma_{iT},\ \ i=1,2;\qquad W_1-W_2=g_3(x,t)\ \text{ on }\ \Upsilon_T,\\ \frac{\partial W_1}{\partial\bar{n}(\omega)}-k\frac{\partial W_2}{\partial\bar{n}(\omega)}-\sum_{j=1}^{n-1}c_j(x)\frac{\partial}{\partial\omega_j}(W_1-W_2)=g_4(x,t)\ \text{ on }\ \Upsilon_T,\end{gathered} \tag{2.5}$$

where $\omega_1,\dots,\omega_{n-1}$ are some coordinates on $\Upsilon$.

PROPOSITION 2.3. *Let* $\alpha,\beta,\nu\in(0,1)$, $\Upsilon,\Gamma_i\in C^{2+\alpha}$, $i=1,2$, $k>0$ *and* $c_j(x)\in C^{2+\alpha}(\Upsilon)$, $j=\overline{1,n-1}$, *and* $g_{0i}(x,0)$, $g_l(x,0)=0$, $l=\overline{1,4}$.

**(i)** *If* $g_{0i} \in C([0,T], C^{\alpha}(\bar{\Omega}_i))$, $g_i \in C([0,T], C^{2+\alpha}(\Gamma_i))$, $i=1,2$, $g_3 \in C([0,T],\ C^{2+\alpha}(\Upsilon))$, $g_4 \in C([0,T], C^{1+\alpha}(\Upsilon))$, *then there is a unique solution* $(W_1(x,t),\ W_2(x,t))$ *of* (2.5) *and*

$$\sum_{i=1}^{2} \|W_i\|_{C([0,T],C^{2+\alpha}(\bar{\Omega}_i))} \leqslant C\Big(\sum_{i=1}^{2}[\|g_{0i}\|_{C([0,T],C^{\alpha}(\bar{\Omega}_i))} + \|g_i\|_{C([0,T],C^{2+\alpha}(\Gamma_i))}] + \|g_3\|_{C([0,T],C^{2+\alpha}(\Upsilon))} + \|g_4\|_{C([0,T],C^{1+\alpha}(\Upsilon))}\Big).$$

**(ii)** *If* $g_{0i} \in C^{\alpha,\beta,\alpha}(\bar{\Omega}_{iT})$, $g_i \in C^{2+\alpha,\beta,\alpha}(\Gamma_{iT})$, $i=1,2$, $g_3 \in C^{2+\alpha,\beta,\alpha}(\Upsilon_T)$, $g_4 \in C^{1+\alpha,\beta,\alpha}(\Upsilon_T)$, *then there is a unique solution* $(W_1(x,t),\ W_2(x,t))$ *of* (2.5) *and*

$$\sum_{i=1}^{2} \|W_i\|_{C^{2+\alpha,\beta,\alpha}(\bar{\Omega}_{iT})} \leqslant C\Big(\sum_{i=1}^{2}[\|g_{0i}\|_{C^{\alpha,\beta,\alpha}(\bar{\Omega}_{iT})} + \|g_i\|_{C^{2+\alpha,\beta,\alpha}(\Gamma_{iT})}] + \|g_3\|_{C^{2+\alpha,\beta,\alpha}(\Upsilon_T)} + \|g_4\|_{C^{1+\alpha,\beta,\alpha}(\Upsilon_T)}\Big).$$

**(iii)** *Let* $g_{0i}, g_i, g_4 \equiv 0$, $i=1,2$, *and* $g_3 \in C([0,T],\ C^{2+\alpha}(\Upsilon))$, $\mathbf{D}_t^{\nu} g_3 \in C([0,T],\ C^{1+\alpha}(\Upsilon))$. *Then the following estimates hold*

$$\sup_{\bar{\Omega}_{iT}}|W_i| \leqslant CT^{\nu}\sup_{\bar{\Omega}_{iT}}|\mathbf{D}_t^{\nu} W_i| \leqslant C_2 T^{\nu}\|\mathbf{D}_t^{\nu} g_3\|_{C([0,T],C^{1+\alpha}(\Upsilon))}. \tag{2.6}$$

*Proof.* Statements **(i)** and **(ii)** of this proposition follow from results [28]. Thus, to finish the proof of Proposition 2.3 we should get (2.6).

After differentiation (2.5) with respect to time, we get the new transmission problem for the functions $V_i := \mathbf{D}_t^{\nu} W_i$, $i=1,2:$

$$\Delta_x V_i = 0 \text{ in } \Omega_{iT}; \quad V_i = 0 \text{ on } \Gamma_{iT},\ i=1,2; \qquad V_1 - V_2 = \mathbf{D}_t^{\nu} g_3 \text{ on } \Upsilon_T,$$

$$\frac{\partial V_1}{\partial \bar{n}(\omega)} - k\frac{\partial V_2}{\partial \bar{n}(\omega)} - \sum_{j=1}^{n-1} c_j(x)\frac{\partial}{\partial \omega_j}(V_1 - V_2) = 0 \text{ on } \Upsilon_T. \tag{2.7}$$

Then we apply results from [28] to problem (2.7) and get

$$\|V_i(\cdot,t)\|_{W_p^2(\bar{\Omega}_i)} \leqslant C\|\mathbf{D}_t^{\nu}\sigma(\cdot,t)\|_{W_p^{2-\frac{1}{p}}(\Upsilon)}, \quad \forall t \in [0,T],\ \forall p > 1,\ i=1,2. \tag{2.8}$$

Based on the embedding theorem and properties of the function $\mathbf{D}_t^{\nu} g_3$, we can deduce from (2.8)

$$\sup_{\bar{\Omega}_{iT}}|V_i| \leqslant C\|\mathbf{D}_t^{\nu} g_3\|_{C([0,T],C^{1+\alpha}(\Upsilon))}. \tag{2.9}$$

Returning to the functions $W_i$, $i=1,2$, and using estimates (2.1) together with (2.9), we obtain (2.6). □

The following result is a simple consequence of Proposition 2.3.

REMARK 2.1. Proposition 2.3 is true in the case $\Omega_1 \cup \Omega_2$ is an unbounded domain.

## 3. The nonlinear functional equation, linearization

### 3.1. Reduction of problem (1.1)–(1.5) to a problem in the fixed domain

To prove the solvability of problem (1.1)–(1.5), it is convenient to reduce the one to a problem in a fixed domain. To this end, we use the Hanzawa method [9].

Let $\omega=(\omega_1,\dots,\omega_{n-1})$ be some coordinates on $\Upsilon$. We represent $\Upsilon$ in the form $y=\bar{m}(\omega)$ and denote by $\bar{n}(\omega)$ the normal to $\Upsilon$ directed into $\Omega_1$.

For sufficiently small $\gamma_0>0$, $\omega$-surfaces: $\bar{m}(\omega)+\eta\bar{n}(\omega)$, $|\eta|<2\gamma_0$, do not intersect each other and $\Gamma_1\bigcup\Gamma_2$. On the set

$$N=\{y\in R^n:\ dist(y,\Upsilon)<3\gamma_0/2\}$$

we introduce the local coordinates $(\omega,\eta)$ by

$$y=(y_1,\dots,y_n)=\bar{m}(\omega)+\eta\bar{n}(\omega),\ \ \bar{m}(\omega)\subset\Upsilon.$$

We assume that the free boundary in problem (1.1)–(1.5) is given as

$$\Upsilon(t)=\{(y,t):y(\omega,t)=\bar{m}(\omega)+\rho(\omega,t)\bar{n}(\omega),\ t\in[0,T]\},\tag{3.1}$$

where $\rho(\omega,t)$ is an unknown function, and

$$|\rho(\omega,t)|<\gamma_0c_0,\quad 0<c_0<1,\quad \rho(\omega,0)=0.\tag{3.2}$$

It means that in the local variables the surface $\Upsilon(t)$ is given by

$$\Phi_\rho(y,t)=\eta(y)-\rho(\omega(y),t)=0.\tag{3.3}$$

Using (1.6), (3.1) and (3.3), we can rewrite the boundary conditions in (1.3) as

$$\begin{aligned}\mathbf{D}_t^\nu\rho\frac{\langle\nabla_y\Phi_\rho(y,0),\nabla_y\Phi_\rho(y,t)\rangle}{|\nabla_y\Phi_\rho(y,0)|}&=-k_1\langle\nabla_yp_1,\nabla_y\Phi_\rho(y,t)\rangle\\&=-k_2\langle\nabla_yp_2,\nabla_y\Phi_\rho(y,t)\rangle.\end{aligned}\tag{3.4}$$

Let $\chi(\lambda)\in C_0^\infty(R^1)$, $\chi(\lambda)=1$ if $|\lambda|<\gamma_0/4$ and $\chi(\lambda)=0$ if $|\lambda|>3\gamma_0/4$, $|\chi^{(k)}|\leqslant c_1/\gamma_0^k$, $k=1,2,3$. We choose $c_0$ in (3.2) such that $c_0<1/2c_1$ then $1+\chi\prime(\lambda)\mu\geqslant1/2$ if $|\mu|\leqslant\gamma_0c_0$. We will use the coordinates $(\omega,\eta)$ to define the diffeomorphism

$$e_\rho:(x,t)\to(y,t)$$

from $X_T=R^n\times[0,T]$ onto $Y_T=R^n\times[0,T]$ by setting

$$\begin{cases}y=x,\ \text{ if }\ dist(x,\Upsilon)>3\gamma_0/4,\\ \omega(y)=\omega(x),\eta(y)=\lambda(x)+\chi(\lambda(x))\rho(\omega(x),t),\ \text{ otherwise,}\end{cases}\tag{3.5}$$

such that the transform $e_\rho^{-1}$ maps $\Omega_i(t)$ onto $\Omega_i\times(0,T)=\Omega_{iT}$, $i=1,2$, and $\Upsilon(t)$ onto $\Upsilon\times[0,T]=\Upsilon_T$; the free boundary is given by

$$e_\rho(\{\lambda(x)=0\}),\tag{3.6}$$

and $\omega(x)$, $\lambda(x)$ are the coordinates in $X_T$ similar to the coordinates $\omega(y)$, $\eta(y)$ in $Y_T$. After the change of variables (3.5), we have the new desired functions

$$v_i(x_1,\dots,x_n,t) = p_i(y_1,\dots,y_n,t)\circ e_\rho(x,t), \;\; i=1,2. \tag{3.7}$$

Denote by $\nabla_\rho = (E_\rho^*)^{-1}\nabla_x$, where $\nabla_x = (\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n})$ and $E_\rho$ is the Jacobi matrix of the mapping $y = e_\rho(x,t)$, so that

$$\nabla_y = \nabla_\rho \;\text{ and }\; \Delta_y = \nabla_\rho^2.$$

Taking into account that $y=x$ near $\Gamma_{iT}$, $i=1,2$, we can deduce from (1.1), (1.2) and (1.4) that the functions $v_i(x,t)$, $i=1,2$, satisfy the equations:

$$\nabla_\rho^2 v_i(x,t) = 0 \quad \text{in} \quad \Omega_{iT}, \;\; i=1,2; \tag{3.8}$$

$$v_1(x,t) - v_2(x,t) = 0 \quad \text{on} \quad \Upsilon_T, \tag{3.9}$$

$$v_i = \psi_i(x) \quad \text{on} \quad \Gamma_{iT}. \tag{3.10}$$

Using (3.4), we can rewrite boundary conditions (1.3) as

$$\begin{aligned}\mathbf{D}_t^\nu \rho &= -k_1\left[S(\omega,\rho,\nabla_\omega\rho)\frac{\partial v_1}{\partial\lambda} + \sum_{i=1}^{n-1} S_i(\omega,\rho,\nabla_\omega\rho)\frac{\partial v_1}{\partial\omega_i}\right] \\ &= -k_2\left[S(\omega,\rho,\nabla_\omega\rho)\frac{\partial v_2}{\partial\lambda} + \sum_{i=1}^{n-1} S_i(\omega,\rho,\nabla_\omega\rho)\frac{\partial v_2}{\partial\omega_i}\right] \quad \text{on } \Upsilon_T, \end{aligned}\tag{3.11}$$

where $\nabla_\omega\rho = (\rho_{\omega_1},\dots,\rho_{\omega_{n-1}})$; $S(\omega,\rho,\nabla_\omega\rho)$ and $S_i(\omega,\rho,\nabla_\omega\rho)$, $i=\overline{1,n-1}$, are some specific smooth functions [9]:

$$\begin{aligned} S(\omega,\rho,\nabla_\omega\rho) &= \sum_{m=1}^{n}\left[\frac{\partial\lambda}{\partial y_m}(y(\omega,\rho(\omega,t))) - \sum_{j=1}^{n-1}\rho_{\omega_j}(\omega,t)\frac{\partial\omega_j}{\partial y_m}(y(\omega,\rho(\omega,t)))\right]^2, \\ S_i(\omega,\rho,\nabla_\omega\rho) &= \sum_{m=1}^{n}\left[\frac{\partial\omega_m}{\partial y_m}(y(\omega,\rho(\omega,t)))\frac{\partial\lambda}{\partial y_m}(y(\omega,\rho(\omega,t)))\right. \\ &\quad \left. - \sum_{j=1}^{n-1}\rho_{\omega_j}(\omega,t)\frac{\partial\omega_j}{\partial y_m}(y(\omega,\rho(\omega,t)))\frac{\partial\omega_m}{\partial y_m}(y(\omega,\rho(\omega,t)))\right], \end{aligned}\tag{3.12}$$

such that

$$S(\omega,0,0) = 1, \quad \frac{\partial S}{\partial\rho_{\omega_i}}(\omega,0,0) = 0, \quad S_i(\omega,0,0) = 0, \;\; i=\overline{1,n-1}. \tag{3.13}$$

Moreover, one can easily check that

$$\nabla_\rho^2|_{t=0} = \nabla_x^2 = \Delta_x. \tag{3.14}$$

Thus, free boundary problem (1.1)–(1.5) is reduced to the nonlinear problem in the fixed domain for the functions $v_i(x,t),\ i=1,2$, and $\rho(\omega,t)$ that satisfy conditions (see equations (3.8)–(3.11)):

$$-\nabla^2_\rho v_i(x,t)=0 \ \text{ in } \ \Omega_{iT},\ \ i=1,2;\ \ v_i(x,t)=\psi_i(x) \ \text{ on } \ \Gamma_{iT}, \quad \rho(\omega,0)=0;$$

$$v_1(x,t)-v_2(x,t)=0 \ \text{ on } \ \Upsilon_T;$$

$$\begin{aligned} -\mathbf{D}^\nu_t\rho &= k_1\left[S(\omega,\rho,\nabla_\omega\rho)\frac{\partial v_1}{\partial\lambda}+\sum_{i=1}^{n-1}S_i(\omega,\rho,\nabla_\omega\rho)\frac{\partial v_1}{\partial\omega_i}\right] \\ &= k_2\left[S(\omega,\rho,\nabla_\omega\rho)\frac{\partial v_2}{\partial\lambda}+\sum_{i=1}^{n-1}S_i(\omega,\rho,\nabla_\omega\rho)\frac{\partial v_2}{\partial\omega_i}\right] \ \text{ on } \ \Upsilon_T. \end{aligned} \tag{3.15}$$

We define the function $v_{i0}(x)$ as a solution of the following transmission problem

$$\Delta_x v_{i0}=0 \ \text{ in } \ \Omega_i,\ \ i=1,2;\ \ v_{i0}|_{\Gamma_i}=\psi_i(x);$$

$$v_{10}-v_{20}=0 \ \text{ and } \ k_1\frac{\partial v_{10}}{\partial\bar{n}(\omega)}=k_2\frac{\partial v_{20}}{\partial\bar{n}(\omega)} \ \text{ on } \ \Gamma. \tag{3.16}$$

We assume that conditions (1.8) hold.

By the theory of transmission problems for elliptic equations [28], there exists a unique solution $(v_{10}(x),v_{20}(x))$ to problem (3.16) and

$$\sum_{i=1}^{2}\|v_{i0}\|_{C^{3+\alpha}(\bar\Omega_i)}\leqslant C\sum_{i=1}^{2}\|\psi_i\|_{C^{3+\alpha}(\Gamma_i)},\ \ \alpha\in(0,1); \tag{3.17}$$

$$\frac{\partial v_{i0}}{\partial\bar{n}(\omega)}|_\Upsilon<0, \tag{3.18}$$

where $C$ is a positive constant.

Henceforward the letter $C$ will be used to denote different constants encountered in our formulae.

Thus Theorem 1.1 from Section 1 can be reformulated as follows:

THEOREM 3.1. (Reformulated form) *Let conditions of Theorem 1.1 hold; $k=\frac{k_2}{k_1}$ and*

$$0<k<1. \tag{3.19}$$

*Then for some small $T$, there exists a unique solution $(v_1(x,t),v_2(x,t),\rho(\omega,t))$ of nonlinear problem (3.15) for $t\in[0,T]$, such that*

$$v_i(x,t)\in C([0,T],C^{2+\alpha}(\bar\Omega_i)), \quad \rho(\omega,t)\in C([0,T],C^{2+\alpha}(\Upsilon)),$$

$$\boldsymbol{D}^\nu_t\rho(\omega,t)\in C([0,T],C^{1+\alpha}(\Upsilon)), \tag{3.20}$$

$$v_i(x,0)=v_{i0}(x),\ \ i=1,2, \tag{3.21}$$

*where $v_{i0}$ is given with (3.16).*

Note that, equalities (3.21) follow immediately from (3.13) and (3.14).

### 3.2. A perturbation form of system (3.15)

In this subsection, we linearize system (3.15) on the initial data and rewrite it as a system $\mathscr{A}\mathbf{z} = \mathscr{F}\mathbf{z}$, where $\mathscr{A}$ is a linear operator and $\mathscr{F}$ is a nonlinear perturbation.

From (3.13) and (3.15), (3.21), for $t = 0$, we have

$$\mathbf{D}_t^\nu \rho(\omega,0) = -k_1 S(\omega,0,0)\frac{\partial v_{10}}{\partial \lambda}|_\Upsilon = -k_2 S(\omega,0,0)\frac{\partial v_{20}}{\partial \lambda}|_\Upsilon, \tag{3.22}$$

or due to (3.13)

$$\mathbf{D}_t^\nu \rho(\omega,0) = -k_1 \frac{\partial v_{10}}{\partial \lambda}|_\Upsilon = -k_2 \frac{\partial v_{20}}{\partial \lambda}|_\Upsilon. \tag{3.23}$$

Let a function $s(\omega,t)$ be such that

$$s(\omega,0) = 0, \qquad \mathbf{D}_t^\nu s(\omega,0) = \mathbf{D}_t^\nu \rho(\omega,0) \quad \text{on} \quad \Upsilon. \tag{3.24}$$

As an example of the function $s(\omega,t)$, we can take

$$s(\omega,t) = \frac{t^\nu}{\Gamma(\nu+1)}\mathbf{D}_t^\nu \rho(\omega,0)|_\Upsilon. \tag{3.25}$$

Due to

$$\mathbf{D}_t^\nu \frac{t^\nu}{\Gamma(\nu+1)} = 1,$$

we can deduce from (3.25), (3.22) and (3.17) the following result.

COROLLARY 3.1. *The function $s(\omega,t)$ given by (3.25) satisfies (3.24) and*

$$\|s\|_{C([0,T],C^{2+\alpha}(\Upsilon))} + \|\mathbf{D}_t^\nu s\|_{C([0,T],C^{2+\alpha}(\Upsilon))} \leqslant \|s\|_{C^{2+\alpha,\nu,\alpha}(\Upsilon_T)} + \|\mathbf{D}_t^\nu s\|_{C^{1+\alpha,\nu,\alpha}(\Upsilon_T)}$$
$$\leqslant C\sum_{i=1}^{2}\|\psi_i\|_{C^{3+\alpha}(\Gamma_i)}. \tag{3.26}$$

Next, using equation (3.9), we reduce boundary conditions (3.11) to the form:

$$\mathbf{D}_t^\nu \rho = \frac{k_2}{1-k} S(\omega,\rho,\nabla_\omega \rho)\left[\frac{\partial v_1}{\partial \bar{n}(\omega)} - \frac{\partial v_2}{\partial \bar{n}(\omega)}\right] \quad \text{on} \quad \Upsilon_T, \tag{3.27}$$

$$S(\omega,\rho,\nabla_\omega\rho)\left[k_1\frac{\partial v_1}{\partial \bar{n}(\omega)} - k_2\frac{\partial v_2}{\partial \bar{n}(\omega)}\right] + \sum_{i=1}^{n-1} S_i(\omega,\rho,\nabla_\omega\rho)\left[k_1\frac{\partial v_1}{\partial \omega_i} - k_2\frac{\partial v_2}{\partial \omega_i}\right] = 0 \text{ on } \Upsilon_T. \tag{3.28}$$

After that, we introduce the new unknown functions $w_i(x,t)$, $i = 1,2$, and $\sigma(\omega,t)$ as:

$$\sigma(\omega,t) = \rho(\omega,t) - s(\omega,t); \tag{3.29}$$

$$w_i(x,t) = v_i(x,t) - v_{i0}(x) - \langle \nabla_x v_{i0}, \bar{e}_\sigma \rangle, \quad i = 1,2, \tag{3.30}$$

where

$$\bar{e}_\sigma = \frac{\partial x}{\partial \lambda}\chi(\lambda)\sigma(\omega,t), \quad x = (x_1,\dots,x_n). \tag{3.31}$$

Denote by

$$L_0 = -\nabla_x^2, \qquad L_\rho = -\nabla_\rho^2,$$

such that $(L_0 u)\circ e_\rho = L_\rho(u\circ e_\rho)$. Now, taking into account (3.27), (3.28) and (3.16), we rewrite system (3.15) in the terms of the functions $w_i$, $i=1,2$, $\sigma$, and after some tedious calculations, get the next problem:

$$\Delta_x w_i = \mathscr{F}_{0i}(w_i,\sigma) \text{ in } \Omega_{iT}, \ \ i=1,2; \tag{3.32}$$

$$w_1 - w_2 = -\langle \nabla_x v_{10} - \nabla_x v_{20}, \bar{e}_\sigma\rangle \equiv -A(x)\sigma \text{ on } \Upsilon_T; \tag{3.33}$$

$$\mathbf{D}_t^\nu \sigma - \frac{k_2}{1-k}\left[\frac{\partial w_1}{\partial \bar{n}(\omega)} - \frac{\partial w_2}{\partial \bar{n}(\omega)}\right] = \mathscr{F}_1(w_1,w_2,\sigma) \text{ on } \Upsilon_T; \tag{3.34}$$

$$k_1\frac{\partial w_1}{\partial \bar{n}(\omega)} - k_2\frac{\partial w_2}{\partial \bar{n}(\omega)} - k\sum_{i=1}^{n-1} b_i(x)\frac{\partial}{\partial \omega_i}(w_1-w_2) = \mathscr{F}_2(w_1,w_2,\sigma) \text{ on } \Upsilon_T; \tag{3.35}$$

$$w_i = 0 \quad \Gamma_{iT}, \ \ i=1,2; \tag{3.36}$$

$$\sigma(\omega,0) = \mathbf{D}_t^\nu \sigma(\omega,0) = 0 \text{ on } \Upsilon; \tag{3.37}$$

where

$$A(x) = -\frac{1-k}{k}\frac{\partial v_{10}}{\partial \bar{n}(\omega)}, \quad b_i(x) = \frac{1}{(1-k)\frac{\partial v_{10}}{\partial \bar{n}(\omega)}}\sum_{j=1}^{n-1}\frac{\partial S_j(\omega,0,0)}{\partial \rho_{\omega_i}}\frac{\partial v_{10}}{\partial \omega_j}; \tag{3.38}$$

$$\begin{aligned}
&\mathscr{F}_{0i}(w_i,\sigma)\\
&= -[(L_0 v_{i0})\circ e_\rho - (L_0 v_{i0})\circ e_s] - L_s v_{i0} - (L_\rho - L_0)(v_{i0} - v_{i0}\circ e_\rho)\\
&\quad + (L_s - L_0)(v_{i0} - v_{i0}\circ e_s) - L_0[v_{i0} + \langle \nabla v_{i0}, \bar{e}_\rho\rangle - v_{i0}\circ e_\rho]\\
&\quad + L_0[v_{i0} + \langle \nabla v_{i0}, \bar{e}_s\rangle - v_{i0}\circ e_\rho] - (L_\rho - L_0)(w_i + \langle \nabla v_{i0}, \bar{e}_\sigma\rangle);
\end{aligned} \tag{3.39}$$

$$\begin{aligned}
&\mathscr{F}_1(w_1,w_2,\sigma)\\
&= \frac{k_2}{1-k}\left[S(\omega,\rho,\nabla_\omega\rho) - S(\omega,0,0) - \sum_{i=1}^{n-1}\frac{\partial S}{\partial \rho_{\omega_i}}(\omega,0,0)(\sigma_{\omega_i} + s_{\omega_i}) - \frac{\partial S}{\partial \rho}(\omega,0,0)(\sigma+s)\right]\\
&\quad \times\left[\frac{\partial(w_1+v_{10})}{\partial \bar{n}(\omega)} - \frac{\partial(w_2+v_{20})}{\partial \bar{n}(\omega)}\right] - \frac{k_2}{1-k}\left[\frac{\partial^2 v_{10}}{\partial \bar{n}(\omega)^2} - \frac{\partial^2 v_{20}}{\partial \bar{n}(\omega)^2}\right]S(\omega,\rho,\nabla_\omega\rho)\sigma\\
&\quad + \frac{k_2}{1-k}\frac{\partial S}{\partial \rho}(\omega,0,0)\left[\frac{\partial(w_1+v_{10})}{\partial \bar{n}(\omega)} - \frac{\partial(w_2+v_{20})}{\partial \bar{n}(\omega)}\right](\sigma+s);
\end{aligned} \tag{3.40}$$

$$\begin{aligned}
&\mathscr{F}_2(w_1,w_2,\sigma)\\
&= -\left[S(\omega,\rho,\nabla_\omega\rho) - S(\omega,0,0) - \frac{\partial S}{\partial \rho}(\omega,0,0)\rho - \sum_{i=1}^{n-1}\frac{\partial S}{\partial \rho_{\omega_i}}(\omega,0,0)\rho_{\omega_i}\right]\\
&\quad \times\left[k_1\frac{\partial w_1}{\partial \bar{n}(\omega)} - k_2\frac{\partial w_2}{\partial \bar{n}(\omega)}\right] - (s+\sigma)\frac{\partial S}{\partial \rho}(\omega,0,0)\left[k_1\frac{\partial w_1}{\partial \bar{n}(\omega)} - k_2\frac{\partial w_2}{\partial \bar{n}(\omega)}\right]\\
&\quad - \sigma S(\omega,\rho,\nabla_\omega\rho)\left[k_1\frac{\partial^2 v_{10}}{\partial \bar{n}(\omega)^2} - k_2\frac{\partial^2 v_{20}}{\partial \bar{n}(\omega)^2}\right]
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{j=1}^{n-1}\Big[S_j(\omega,\rho,\nabla_\omega\rho)-S_j(\omega,0,0)-\rho\frac{\partial S_j}{\partial\rho}(\omega,0,0)-\sum_{i=1}^{n-1}\frac{\partial S_j}{\partial\rho_{\omega_i}}(\omega,0,0)\rho_{\omega_i}\Big]\\
&\times\Big[\frac{\partial v_1}{\partial\omega_j}+\frac{\partial v_{10}}{\partial\omega_j}+\frac{\partial^2 v_{10}}{\partial\bar{n}(\omega)\partial\omega_j}\sigma+\frac{\partial v_{10}}{\partial\bar{n}(\omega)}\sigma_{\omega_j}\Big]\\
&-\sum_{ij=1}^{n-1}\Big\{\frac{\partial S_j}{\partial\rho_{\omega_i}}(\omega,0,0)\rho_{\omega_i}\Big[\frac{\partial v_1}{\partial\omega_j}+\frac{\partial^2 v_{10}}{\partial\omega_j\partial\bar{n}(\omega)}\sigma+\frac{\partial v_{10}}{\partial\bar{n}(\omega)}\sigma_{\omega_j}\Big]\\
&+\frac{k-1}{k\frac{\partial v_{10}}{\partial\bar{n}(\omega)}}\frac{\partial^2 v_{10}}{\partial\omega_j\partial\bar{n}(\omega)}\sigma\frac{\partial S_j}{\partial\rho_{\omega_i}}(\omega,0,0)\Big\}+\sum_{ij=1}^{n-1}s_{\omega_i}\frac{\partial v_{10}}{\partial\omega_j}\frac{\partial S_j}{\partial\rho_{\omega_i}}(\omega,0,0).
\end{aligned}\tag{3.41}
$$

Thus system (3.15) can be written briefly in the form

$$\mathscr{A}\mathbf{z}=\mathscr{F}\mathbf{z},\quad \text{where}\quad \mathbf{z}=(w_1,w_2,\sigma).\tag{3.42}$$

Based on representations (3.38)–(3.41); properties (3.13), (3.14) and (3.17), (3.19); and Corollary 3.1, we can conclude the following.

COROLLARY 3.2. *The functions* $\mathscr{F}_{0i}(w_i,\sigma)$, $i=0,1$, $\mathscr{F}_j(w_1,w_2,\sigma)$, $j=1,2$, *contain the higher derivatives of* $w_i(x,t)$ *and* $\sigma(\omega,t)$ *with the coefficients that tend to zero as* $t\to 0$, *the "quadratic" terms with respect to* $w_i(x,t)$ *and* $\sigma(\omega,t)$, *and their derivatives, and the terms of minor differential orders of unknown functions. Moreover,*

$$\mathscr{F}_{0i}(w_i,\sigma)|_{t=0}=0,\ \ \mathscr{F}_j(w_1,w_2,\sigma)|_{t=0}=0,\ \ i,j=1,2;\tag{3.43}$$

$$A(x)\in C^{2+\alpha}(\Upsilon),\ \ b_i(x)\in C^{1+\alpha}(\Upsilon),\ \ A(x)>0.\tag{3.44}$$

Note that conditions (3.43) together with (3.32)–(3.37) lead to

$$w_i(x,0)=0,\ \ x\in\bar{\Omega}_i,\ \ i=1,2.\tag{3.45}$$

The next step of our investigation is a proof of the boundedness of the linear operator $\mathscr{A}$ in the corresponding functional spaces. To this end, we freeze the functional arguments in the functions $\mathscr{F}_{0i}(w_i,\sigma)$ and $\mathscr{F}_j(w_1,w_2,\sigma)$. Then system (3.32)–(3.37) or (3.42) will be a linear system with variable coefficients, which will be studied in detail in Subsection 5.1. We remaind that investigation of this linear system is based on the research of the corresponding model problem with a fractional dynamic boundary condition.

## 4. The model problem with a fractional dynamic boundary condition

As is known, to construct a model problem near the boundary by using the Schauder method, it is necessary to fix the coefficients of the original problem at the boundary point and, if necessary, straighten the boundary in some vicinity of the fixed point. In this section we study the model problem of the more general view than it is demanded by operator $\mathscr{A}$ (see (3.42)).

Let $a_0$ and $a_1$ be some given positive constants, and $a_2 = \{a_2^1, \dots, a_2^{n-1}\}$, $a_3 = \{a_3^1, \dots, a_3^{n-1}\}$ be given vectors;

$$R_+^n = \{(x', x_n) : x' \in R^{n-1}, x_n > 0\}, \quad R_-^n = \{(x', x_n) : x' \in R^{n-1}, x_n < 0\},$$

$$x' = (x_1, \dots, x_{n-1}); \quad R_{\pm T}^n = R_\pm^n \times (0,T); \qquad R_T^{n-1} = R^{n-1} \times (0,T).$$

We look for a solution $(u_+(x,t), u_-(x,t), \wp(x',t))$ bounded at the infinity by the following conditions:

$$\Delta_x u_\pm = f_0^\pm(x,t) \quad \text{in } R_{\pm T}^n; \tag{4.1}$$

$$u_-(x,t) - u_+(x,t) = -a_0 \wp(x',t) \text{ on } \bar{R}_T^{n-1}; \tag{4.2}$$

$$\mathbf{D}_t^\nu \wp(x',t) - a_1\left[\frac{\partial u_-}{\partial n} - \frac{\partial u_+}{\partial n}\right] - \langle a_2, \nabla_{x'}(u_- - u_+)\rangle = f_1(x',t) \text{ on } \bar{R}_T^{n-1}; \tag{4.3}$$

$$\frac{\partial u_-}{\partial n} - k\frac{\partial u_+}{\partial n} - k\langle a_3, \nabla_{x'}(u_- - u_+)\rangle = f_2(x',t) \text{ on } \bar{R}_T^{n-1}; \tag{4.4}$$

$$\wp(x',0) = 0 \text{ in } R^{n-1}, \quad u_\pm|_{t=0} = 0 \text{ in } R_\pm^n, \tag{4.5}$$

where $n$ is the unit normal to $R^{n-1}$ directed in $R_-^n$; $f_0^\pm$, $f_1$ and $f_2$ are some given functions:

$$f_0^\pm, f_i \equiv 0 \quad \text{if either} \quad t = 0 \text{ or } |x| > R_0, \tag{4.6}$$

for some positive number $R_0$.

If $\nu = 1$ and $n = 2$, problem (4.1)–(4.5) was studied by F. Yi [36], B. V. Bazaliy and N. Vasylyeva [4]; and the one-valued solvability of this problem was proved in the classes $C([0,T], C^{2+\alpha}(R_\pm^2))$ and $E^{2+\alpha,\alpha,\alpha}(\bar{R}_{\pm T}^2)$, $\alpha \in (0,1)$.

THEOREM 4.1. *Let $\alpha, \nu \in (0,1)$, $0 < k < 1$, condition (4.6) hold.*

**(i)** *If*

$$f_0^\pm \in C([0,T], C^\alpha(\bar{R}_\pm^n)), \quad f_i \in C([0,T], C^{1+\alpha}(R^{n-1})), \ i = 1,2, \tag{4.7}$$

*then there exists a unique solution $(u_+(x,t), u_-(x,t), \wp(x',t))$ of (4.1)–(4.5):*

$$\begin{aligned} &\|u_+\|_{C([0,T],C^{2+\alpha}(\bar{R}_+^n))} + \|u_-\|_{C([0,T],C^{2+\alpha}(\bar{R}_-^n))} \\ &+ \|\mathbf{D}_t^\nu \wp\|_{C([0,T],C^{1+\alpha}(R^{n-1}))} + \|\wp\|_{C([0,T],C^{2+\alpha}(R^{n-1}))} \\ \leqslant\ & C_1[\|f_0^+\|_{C([0,T],C^\alpha(\bar{R}_+^n))} + \|f_0^-\|_{C([0,T],C^\alpha(\bar{R}_-^n))} \\ &+ \|f_1\|_{C([0,T],C^{1+\alpha}(R^{n-1}))} + \|f_2\|_{C([0,T],C^{1+\alpha}(R^{n-1}))}]. \end{aligned} \tag{4.8}$$

**(ii)** *If the right-hand sides in (4.1)–(4.5) meet the requirements:*

$$f_i \in C^{1+\alpha,\frac{\alpha}{2}\nu,\alpha}(\bar{R}_T^{n-1}), \ i = 1,2, \ f_0^\pm \in C^{\alpha,\frac{\alpha}{2}\nu,\alpha}(\bar{R}_{\pm T}^n), \tag{4.9}$$

*then there is a unique solution* $(u_+, u_-, \wp)$ *of* (4.1)–(4.5), $u_\pm \in C^{2+\alpha,\frac{\alpha}{2}\nu,\alpha}\,(\bar{R}^n_{\pm T})$, $\wp \in C^{2+\alpha,\frac{\alpha}{2}\nu,\alpha}(\bar{R}^{n-1}_T)$

$$\begin{aligned}
&\|u_+\|_{C^{2+\alpha,\frac{\alpha}{2}\nu,\alpha}(\bar{R}^n_{+T})} + \|u_-\|_{C^{2+\alpha,\frac{\alpha}{2}\nu,\alpha}(\bar{R}^n_{-T})} \\
&+\|\mathbf{D}^\nu_t \wp\|_{C^{1+\alpha,\frac{\alpha}{2}\nu,\alpha}(\bar{R}^{n-1}_T)} + \|\wp\|_{C^{2+\alpha,\frac{\alpha}{2}\nu,\alpha}(\bar{R}^{n-1}_T)} \\
\leqslant\; & C_2[\|f_1\|_{C^{1+\alpha,\frac{\alpha}{2}\nu,\alpha}(\bar{R}^{n-1}_T)} + \|f_2\|_{C^{1+\alpha,\frac{\alpha}{2}\nu,\alpha}(\bar{R}^{n-1}_T)} \\
&+\|f_0^+\|_{C^{\alpha,\frac{\alpha}{2}\nu,\alpha}(\bar{R}^n_{+T})} + \|f_0^-\|_{C^{\alpha,\frac{\alpha}{2}\nu,\alpha}(\bar{R}^n_{-T})}],
\end{aligned} \tag{4.10}$$

*where* $C_i,\ i = 1,2,$ *are positive constants independent of the right-hand sides of problem* (4.1)–(4.5).

As it follows from results of Proposition 2.3, it is enough to prove Theorem 4.1 in the following case

$$f_0^\pm,\ \ f_2 \equiv 0. \tag{4.11}$$

### 4.1. Integral representation of the solution to problem (4.1)–(4.5) in the case of (4.11)

First of all we will construct the integral representations for $\wp$ and $u_\pm$, and then we will obtain estimate (4.10).

Let $\xi = (\xi_1,\dots,\xi_{n-1})$ and $|\xi| = \left(\sum_{k=1}^{n-1}\xi_k^2\right)^{1/2}$. We denote by $\tilde{w}(\xi,x_n,t)$ the Fourier transform of $w(x',x_n,t)$, and by $\hat{w}(\cdot,p)$ the Laplace transform of $w(\cdot,t)$, and use the notation "$*$" instead of "$\widehat{\sim}$". By applying the Fourier and Laplace transformations to problem (4.1)–(4.5), we get

$$\frac{\partial^2 u^*_\pm}{\partial x_n^2}(\xi,x_n,p) - 4\pi^2|\xi|^2 u^*_\pm(\xi,x_n,p) = 0, \tag{4.12}$$

$$u^*_-(\xi,0,p) - u^*_+(\xi,0,p) = -a_0\wp^*(\xi,p); \tag{4.13}$$

$$p^\nu\wp^*(\xi,p) + a_1\frac{\partial}{\partial x_n}(u^*_- - u^*_+) - 2\pi i\langle a_2,\xi\rangle(u^*_- - u^*_+) = f_1^*(\xi,p)\ \text{ if }\ x_n = 0, \tag{4.14}$$

$$\frac{\partial}{\partial x_n}(u^*_- - ku^*_+) + 2k\pi i\langle a_3,\xi\rangle(u^*_- - u^*_+) = 0\ \text{ if }\ x_n = 0. \tag{4.15}$$

Note that to get (4.14) we used formula (2.2.38) in [12]:

$$\widehat{\mathbf{D}^\nu_t w}(\cdot,t) = p^\nu\hat{w}(\cdot,p) - p^{\nu-1}w(\cdot,0).$$

To satisfy equations in (4.12), we set

$$u^*_-(\xi,x_n,p) = M^*_-(\xi,p)e^{2\pi|\xi|x_n},\quad u^*_+(\xi,x_n,p) = M^*_+(\xi,p)e^{-2\pi|\xi|x_n}, \tag{4.16}$$

and then, we look for the function $\wp^*(\xi,p)$ from (4.13) as

$$\wp^*(\xi,p) = -\frac{1}{a_0}[M^*_-(\xi,p) - M^*_+(\xi,p)]. \tag{4.17}$$

To find the unknown functions $M_-^*(\xi,p)$ and $M_+^*(\xi,p)$, we have two transmission equations (4.14) and (4.15). It is easy to show that

$$M_+^*(\xi,p) = \frac{a_0}{k+1}[1 + ik\langle a_3,\xi\rangle|\xi|^{-1}]f_1^*(\xi,p)Q^*(\xi,p); \tag{4.18}$$

$$M_-^*(\xi,p) = -\frac{ka_0}{k+1}[1 - i\langle a_3,\xi\rangle|\xi|^{-1}]f_1^*(\xi,p)Q^*(\xi,p), \tag{4.19}$$

where

$$Q^*(\xi,p) = \frac{1}{p^\nu + 2\pi a_1 a_0\frac{1-k}{1+k}|\xi| + i\frac{2\pi a_0}{1+k}\langle 2ka_1a_3 + (k+1)a_2,\xi\rangle}.$$

Note that if condition (3.19) holds, then $2\pi a_1 a_0\frac{1-k}{1+k} > 0$. Thus, Re $\frac{1}{Q^*(\xi,p)} > 0$ if Re $p^\nu > 0$ and Im $\xi = 0$, i.e. the function $Q^*(\xi,p)$ does not have any singularities in this case.

Denote by

$$A_0(k) = 2\pi a_1 a_0\frac{1-k}{1+k}, \quad A_1(k) = \frac{2\pi a_0}{1+k}[2ka_1a_3 + (k+1)a_2],$$

$$K_+^*(\xi,x_n,\eta) = \left(1 + ik\frac{\langle a_3,\xi\rangle}{|\xi|}\right)\exp\left\{-A_0(k)|\xi|\left(\eta + \frac{2\pi x_n}{A_0(k)}\right) - i\eta\langle A_1(k),\xi\rangle\right\}, \tag{4.20}$$

$$K_-^*(\xi,x_n,\eta) = \left(1 - i\frac{\langle a_3,\xi\rangle}{|\xi|}\right)\exp\left\{-A_0(k)|\xi|\left(\eta - \frac{2\pi x_n}{A_0(k)}\right) - i\eta\langle A_1(k),\xi\rangle\right\}, \tag{4.21}$$

$$K^*(\xi,\eta) = \exp\{-A_0(k)|\xi|\eta - i\eta\langle A_1(k),\xi\rangle\}. \tag{4.22}$$

Then, using (4.16)–(4.19), one can easily check that

$$u_+^*(\xi,x_n,p) = \frac{a_0}{k+1}f_1^*(\xi,p)\int_0^{+\infty} e^{-\eta p^\nu}K_+^*(\xi,x_n,\eta)d\eta \equiv \frac{a_0}{k+1}f_1^*(\xi,p)G_+^*(\xi,x_n,p); \tag{4.23}$$

$$u_-^*(\xi,x_n,p) = -\frac{ka_0}{k+1}f_1^*(\xi,p)\int_0^{+\infty} e^{-\eta p^\nu}K_-^*(\xi,x_n,\eta)d\eta$$

$$\equiv -\frac{ka_0}{k+1}f_1^*(\xi,p)G_-^*(\xi,x_n,p); \tag{4.24}$$

$$\wp^*(\xi,p) = f_1^*(\xi,p)\int_0^{+\infty} e^{-\eta p^\nu}K^*(\xi,\eta)d\eta \equiv f_1^*(\xi,p)G^*(\xi,p). \tag{4.25}$$

To get representations for the functions $u_\pm(x,t)$ and $\wp(x',t)$, we need formula (3.2.7) from [26]:

$$\widehat{e^{-cp^\nu}} = t^{-1}W(-ct^{-\nu};-\nu,0), \quad c > 0. \tag{4.26}$$

Here $W(z;\beta,\gamma)$ is the Wright function which is defined for $z,\ \beta,\ \gamma\in\mathbb{C}$ as (see, e.g., (1.8.1 (27)) in [7])

$$W(z;\beta,\gamma)=\sum_{k=0}^{\infty}\frac{z^k}{k!\Gamma(\beta k+\gamma)}. \tag{4.27}$$

Note that the main properties of the Wright functions are described in Chapter 18.1 [7]; Chapter 1.11 [12]; Chapter 2 [26].

After that, applying the inverse Laplace and Fourier transformations to (4.23)–(4.25), we obtain

$$u_+(x',x_n,t)=\frac{a_0}{k+1}\int_0^t d\tau\int_{R^{n-1}} G_+(y',x_n,\tau)f_1(x'-y',t-\tau)dy', \tag{4.28}$$

$$y'=(y_1,\dots,y_{n-1}),$$

$$u_-(x',x_n,t)=-\frac{ka_0}{k+1}\int_0^t d\tau\int_{R^{n-1}} G_-(y',x_n,\tau)f_1(x'-y',t-\tau)dy', \tag{4.29}$$

$$\wp(x',t)=\int_0^t d\tau\int_{R^{n-1}} G(y',\tau)f_1(x'-y',t-\tau)dy', \tag{4.30}$$

where

$$\begin{aligned}G(y',\tau)&=\int_0^\infty d\eta\frac{W(-\eta\tau^{-\nu};-\nu,0)}{\tau}K(y',\eta),\\ K(y',\eta)&=\int_{R^{n-1}} K^*(\xi,\eta)e^{2\pi i\langle\xi,y'\rangle}d\xi,\end{aligned} \tag{4.31}$$

$$\begin{aligned}G_\pm(y',x_n,\tau)&=\int_0^\infty\frac{W(-\eta\tau^{-\nu};-\nu,0)K_\pm(y',x_n,\eta)}{\tau}d\eta,\\ K_\pm(y',x_n,\eta)&=\int_{R^{n-1}} e^{2\pi i\langle\xi,y'\rangle}K^*_\pm(\xi,x_n,\eta)d\xi.\end{aligned} \tag{4.32}$$

As usual in the potential theory, to estimate the functions $u_\pm(x,t)$ and $\sigma(x',t)$, it is necessary to describe well the properties of the kernels $G_\pm(y',x_n,\tau)$, $G(y',\tau)$ and $K_\pm(y',x_n,\tau)$, $K(y',\tau)$.

### 4.2. Estimates of the functions $\wp(x',t)$ and $u_\pm(x,t)$ constructed in (4.28)–(4.30)

The next lemma describes the main properties of the kernels $G(y',t)$, $K(y',z)$ which will be essential used to get estimates (4.8) and (4.10) for $\wp(x',t)$ and $\mathbf{D}_t^\nu\wp(x',t)$ if $f_0^\pm,\ f_2\equiv 0$.

LEMMA 4.1. *Let* $\alpha,\ \nu\in(0,1);\ 0<k<1,\ y'\in R^{n-1};\ y''\in R^{n-2},\ y''=(y_1,\dots,y_{l-1},y_{l+1},\dots,\ y_{n-1}),\ l=\overline{1,n-1},\ \eta\in(0,+\infty);\ \varepsilon$ *and* $A$ *be positive numbers. Then functions* $K(y',\eta)$ *and* $G(y',t)$ *which are given by* (4.31) *satisfy to the following inequalities:*

**(i)**

$$|D^m_{y'}K(y',\eta)|\leqslant C\frac{\exp\left(-A\sum_{j=1}^{n-1}|y_j|/\eta\right)}{\eta^{n-1+|m|}},\ \ |m|=\sum_{i=1}^{n-1}m_i,\ \ |m|=0,1,2,\dots; \tag{4.33}$$

**(ii)**

$$|\partial K(y',\eta)/\partial\eta|\leqslant C\frac{\exp\left(-A\sum_{j=1}^{n-1}|y_j|/\eta\right)}{\eta^n}; \tag{4.34}$$

**(iii)**

$$\int_{R^{n-1}}K(y',\eta)dy'=1; \tag{4.35}$$

$$\int_{-\infty}^{+\infty}\frac{\partial K}{\partial y_i}(y',\eta)dy_i=0,\ \ i=\overline{1,n-1}; \tag{4.36}$$

**(iv)**

$$K(y',0)=\Pi_{j=1}^{n-1}\delta(-y_j), \tag{4.37}$$

*where* $\delta(y)$ *is the Dirac delta function;*

**(v)**

$$\int_{R^{n-1}}G(y',t)dy'=\frac{t^{\nu-1}}{\Gamma(\nu)}; \tag{4.38}$$

$$\int_0^t d\tau\int_{R_+^{n-1}}|G(y',\tau)|dy'\leqslant Ct^{\nu}; \tag{4.39}$$

**(vi)**

$$\int_0^t d\tau\int\limits_{R_+^{n-1}}y_j^{\alpha}\left|\frac{\partial G(y',\tau)}{\partial y_i}\right|dy'\leqslant Ct^{\alpha\nu},\quad i,j=\overline{1,n-1}; \tag{4.40}$$

**(vii)**

$$\int_0^t d\tau\int_{R_+^{n-2}}dy''\int_0^{\varepsilon}y_j^{\alpha}\left|\frac{\partial G(y',\tau)}{\partial y_i}\right|dy_l\leqslant C\varepsilon^{\alpha},\ \ l,i,j=\overline{1,n-1}; \tag{4.41}$$

$$\int_0^t d\tau\int_{R_+^{n-2}}dy''\int_{\varepsilon}^{+\infty}\left|\frac{\partial G(y',\tau)}{\partial y_i}\right|dy_l\leqslant C,\ \ l\neq i; \tag{4.42}$$

$$\int_0^t d\tau\int_{R_+^{n-2}}dy''\left|\int_{|y_j|\geqslant\varepsilon}\frac{\partial G(y',\tau)}{\partial y_j}dy_j\right|=0,\ \ j=\overline{1,n-1}; \tag{4.43}$$

**(viii)**

$$\int_0^t d\tau \int_{R_+^{n-2}} dy'' \int_\varepsilon^{+\infty} y_j^\alpha \left| \frac{\partial^2 G(y',\tau)}{\partial y_i \partial y_l} \right| dy_l \leqslant C\varepsilon^{\alpha-1}, \tag{4.44}$$

**(ix)**

$$\int_0^t d\tau \int_{R_+^{n-2}} dy'' \int_0^\varepsilon y_j^\alpha |\partial_\tau^\nu G(y',\tau)| dy_l \leqslant C\varepsilon^\alpha, \ \ l,j=\overline{1,n-1}, \tag{4.45}$$

$$\int_0^t d\tau \int_{R_+^{n-2}} dy'' \int_\varepsilon^{+\infty} |\partial_\tau^\nu G(y',\tau)| dy_l \leqslant Ct, \tag{4.46}$$

$$\int_0^t d\tau \int_{R_+^{n-2}} dy'' \int_\varepsilon^{+\infty} y_j^\alpha \left| \partial_\tau^\nu \frac{\partial G(y',\tau)}{\partial y_l} \right| dy_l \leqslant C\varepsilon^{\alpha-1}. \tag{4.47}$$

*Proof.* Note that statements **(i)–(viii)** of this lemma have been proved in Lemma 3.1 [32] if $A_1(k)$ in the representation of the function $K(y',\eta)$ (see (4.22) and (4.31)) is a null vector. To prove (4.33)–(4.44) in the case of an arbitrary $A_1(k)$ (i.e. $A_1(k)\neq \{0,\dots,0\}$ ) it is enough to take into account that the factor $\exp\{-i\eta\langle A_1(k),\xi\rangle\}$ does not influence essentially to the main properties of the functions $K(y',\eta)$ and $G(y',t)$. Thus, repeating all the arguments from Lemma 3.1 [32] in our case, we get statements **(i)–(viii)** of this lemma.

To prove estimates (4.45)–(4.47), we will obtain the representation of the function $\partial_\tau^\nu \frac{\partial^m G}{\partial y_l^m}$, $m=0,1,\ l=\overline{1,n-1}$. Using the well known formula for the Riemann-Liouville derivative of the Wright function (see, e.g. (9) [27] or (11) [13]):

$$\partial_t^\gamma [t^{\delta-1} W(-ct^{-\beta};-\beta,\delta)] = t^{\delta-1-\gamma} W(-ct^{-\beta};-\beta,\delta-\gamma), \tag{4.48}$$

we deduce

$$\partial_\tau^\nu \frac{\partial^m G}{\partial y_l^m}(y',\eta) = \int_0^{+\infty} \tau^{-1-\nu} W(-\eta\tau^{-\nu};-\nu,-\nu) \frac{\partial^m K}{\partial y_l^m}(y',\eta)d\eta, \ \ m=0,1. \tag{4.49}$$

Then, applying estimate (4.33) to the right-hand side in (4.49), we infer

$$\left|\partial_\tau^\nu \frac{\partial^m G}{\partial y_l^m}\right| \leqslant C\int_0^{+\infty} \tau^{-1-\nu} |W(-\eta\tau^{-\nu};-\nu,-\nu)| \frac{\exp\left\{-A\sum\limits_{j=1}^{n-1}|y_j|\eta^{-1}\right\}}{\eta^{n-1+m}} d\eta, \ \ m=0,1. \tag{4.50}$$

Further, we will use inequality (4.50) to obtain estimates (4.45), (4.46). Let $m=0$, we will get (4.45) if $l\neq j$. Note that the case $l=j$ can be proved with the same way. Inequality (4.50) gives:

$$\mathscr{I}_1 := \int_0^t d\tau \int_{R_+^{n-2}} dy'' \int_0^\varepsilon y_j^\alpha |\partial_\tau^\nu G(y',\tau)| dy_l \leqslant C\int_0^{+\infty} d\tau \int_0^{+\infty} d\eta \int_{R_+^{n-2}} dy'' y_j^\alpha \int_0^\varepsilon \tau^{-1-\nu}$$

$$|W(-\eta\tau^{-\nu};-\nu,-\nu)|\frac{\exp\left\{-A\sum_{j=1}^{n-1}|y_j|\eta^{-1}\right\}}{\eta^{n-1}}dy_l. \tag{4.51}$$

After that, doing the consecutive change of variables:

$$y_i = x_i\eta,\ \ \overline{1,n-1},\ \ i\neq l; \tag{4.52}$$

$$\eta = \tau^\nu z, \tag{4.53}$$

in the right-hand side of (4.51), we obtain

$$\begin{aligned}\mathscr{I}_1 \leqslant C\int\limits_{R_+^{n-2}} dx'' x_j^\alpha \exp\left\{-A\sum_{j=1,j\neq l}^{n-1} x_j\right\}\int\limits_0^\varepsilon dy_l\int\limits_0^{+\infty} dz|W(-z;-\nu,-\nu)| \\ \times\int\limits_0^{+\infty}\tau^{\nu\alpha-\nu-1}z^{\alpha-1}\exp\left\{-A\frac{y_l}{z\tau^\nu}\right\}d\tau.\end{aligned} \tag{4.54}$$

Then, the change of variable

$$\tau = \left(\frac{y_l}{zr}\right)^{1/\nu} \tag{4.55}$$

in the last integral in (4.54) allows us to deduce that

$$\begin{aligned}\mathscr{I}_1 &\leqslant C\int\limits_0^\varepsilon dy_l y_l^{\alpha-1}\int\limits_0^{+\infty} dz|W(-z;-\nu,-\nu)|\int\limits_0^{+\infty} e^{-Ar}r^{-\alpha}dr \\ &\leqslant C\varepsilon^\alpha\int\limits_0^{+\infty}|W(-z;-\nu,-\nu)|dz.\end{aligned} \tag{4.56}$$

To show a boundedness of the last integral in (4.56), we will use the next estimate for the Wright function (see Lemma 3 in [27]):

$$|W(-z;-\beta,\gamma)| \leqslant \frac{\text{const.}}{1+|z|^{-\frac{\gamma-1}{\beta}}},\quad \text{if}\quad \gamma<1, \tag{4.57}$$

where we put $\beta := \nu$ and $\gamma := -\nu$. Thus, inequalities (4.56), (4.57) lead to (4.45).

As for a proof of inequality (4.46), we again use inequality (4.50) and change of variable (4.52). Thus, we have

$$\mathscr{I}_2 := \int\limits_0^t d\tau \int\limits_{R_+^{n-2}} dy'' \int\limits_{\varepsilon}^{+\infty} |\partial_\tau^\nu G(y',\tau)| dy_l$$

$$\leqslant \text{const.} \int\limits_{R_+^{n-2}} dx'' \exp\left\{-A \sum_{j=1, j\neq l}^{n-1} x_j\right\} \int\limits_0^t d\tau \int\limits_0^{+\infty} d\eta |W(-\eta\tau^{-\nu}; -\nu, -\nu)| \tau^{-1-\nu}$$

$$\times \int\limits_{\varepsilon}^{+\infty} e^{-Ay_l/\eta} \frac{dy_l}{\eta}.$$

Then the simple calculations lead to the following inequality

$$\mathscr{I}_2 \leqslant C \int\limits_0^{+\infty} d\eta e^{-A\varepsilon/\eta} \int\limits_0^t |W(-\eta\tau^{-\nu}; -\nu, -\nu)| \tau^{-1-\nu} d\tau. \tag{4.58}$$

Doing the change of variable

$$\tau = (\eta/\rho)^{1/\nu} \tag{4.59}$$

in the inner integral in (4.58) and using inequality (4.57), we conclude

$$\mathscr{I}_2 \leqslant C \int\limits_0^{+\infty} d\eta e^{-A\varepsilon/\eta} \eta^{-1} \int\limits_{\eta t^{-\nu}}^{+\infty} |W(-\rho; -\nu, -\nu)| d\rho \leqslant C \int\limits_0^{+\infty} d\eta \frac{e^{-A\varepsilon/\eta}}{\eta} \int\limits_{\eta t^{-\nu}}^{+\infty} \frac{d\rho}{\rho^{1+\frac{1}{\nu}}}. \tag{4.60}$$

After that, estimate (4.46) follows from (4.60).

At last, to complete the proof of Lemma 4.1, we have to obtain (4.47). Using (4.50) with $m = 1$ and doing the change of variables (4.52) and (4.59), we infer after some calculations:

$$\mathscr{I}_3 := \int\limits_0^t d\tau \int\limits_{R_+^{n-2}} dy'' \int\limits_{\varepsilon}^{+\infty} y_j^\alpha \left|\partial_\tau^\nu \frac{\partial G}{\partial y_l}\right| dy_l$$

$$\leqslant C \int\limits_0^t d\tau \int\limits_{R_+^{n-2}} dx'' x_j^\alpha \exp\left\{-A \sum_{j=1, j\neq l}^{n-1} x_j\right\} \int\limits_0^{+\infty} d\eta \frac{|W(-\eta\tau^{-\nu}; -\nu, -\nu)|}{\eta^{1-\alpha}\tau^{1+\nu}} \int\limits_{\varepsilon/\eta}^{+\infty} \exp\{-Ax_l\} dx_l$$

$$\leqslant C \int\limits_0^{+\infty} \eta^{\alpha-2} e^{-A\varepsilon/\eta} d\eta \int\limits_0^{+\infty} |W(-\rho; -\nu, -\nu)| d\rho, \quad l \neq j. \tag{4.61}$$

We apply estimate (4.57) in the inner integral in the right-hand side of (4.61) and perform the change of variable: $\varepsilon/\eta = \zeta$. Thus, we deduce the following inequality

$$\mathscr{I}_3 \leqslant C\varepsilon^{\alpha-1} \int_0^{+\infty} \zeta^{-\alpha} e^{-A\zeta} d\zeta \int_0^{+\infty} \frac{d\rho}{1+\rho^{1+\frac{1}{\nu}}}$$

which proves (4.47). In the similar way we can deduce (4.47) if $l = j$. □

Next we repeat the arguments from Section 7 [32] and use the results of Lemma 4.1 to get the following view for $\mathbf{D}_t^\nu \wp$.

PROPOSITION 4.1. *Let conditions of Lemma 4.1 hold then there is the following representation for the function* $\mathbf{D}_t^\nu \wp$:

$$\mathbf{D}_t^\nu \wp = f_1(x',t) + \int_0^t d\tau \int_{R^{n-1}} [f_1(x'-y',t-\tau) - f_1(x',t-\tau)] \partial_\tau^\nu G(y',\tau) dy'. \quad (4.62)$$

Then Lemma 4.1, Proposition 4.1 and results of Chapter 3 [16] allow us to deduce the following result.

LEMMA 4.2. *Let* $\alpha, \nu \in (0,1)$, *conditions* (4.6) *and* (4.11) *hold, and* $f_1 \in C([0,T], C^{1+\alpha}(R^{n-1}))$. *Then the function* $\wp$ *given with* (4.30) *satisfies the following estimate*

$$\|\wp\|_{C([0,T],C^{2+\alpha}(R^{n-1}))} + \|\mathbf{D}_t^\nu \wp\|_{C([0,T],C^{1+\alpha}(R^{n-1}))} \leqslant C\|f_1\|_{C([0,T],C^{1+\alpha}(R^{n-1}))}. \quad (4.63)$$

*If in addition* $f_1 \in C^{1+\alpha,\alpha\nu/2,\alpha}(\bar{R}_T^{n-1})$ *then*

$$\sum_{|k|=0}^{2} [D_x^k \wp]_{R_T^{n-1}}^{(\alpha,\nu\alpha/2)} + \sum_{|k|=0}^{1} [\mathbf{D}_t^\nu D_x^k \wp]_{R_T^{n-1}}^{(\alpha,\nu\alpha/2)} \leqslant C\|f_1\|_{C^{1+\alpha,\alpha\nu/2,\alpha}(\bar{R}_T^{n-1})}. \quad (4.64)$$

Next step of our investigation is a proof of the corresponding estimates to the functions $\wp(x',t)$ and $\mathbf{D}_t^\nu \wp(x',t)$ with respect to time. To this end, we need the following result.

PROPOSITION 4.2. *Let* $\alpha, \nu \in (0,1)$, $T_1$, $T_2 \in [0,T]$, $T_2 > T_1$. *Then the function* $G(y',\tau)$ *which is represented by* (4.31) *satisfies the following inequalities:*

**(i)**

$$\int_{T_1}^{T_2} d\tau \int_{R_+^{n-1}} y_j^\alpha |\partial_\tau^\nu G(y',\tau)| dy' \leqslant C(T_2-T_1)^{\alpha\nu}, \quad j = \overline{1,n-1}, \quad (4.65)$$

**(ii)**

$$\int_0^{3T_1-2T_2} d\tau \int_{T_1}^{T_2} dt \int_{R_+^{n-1}} y_j^\alpha \left| \frac{\partial}{\partial(t-\tau)} \partial_{t-\tau}^\nu G(y',t-\tau) \right| dy' \leqslant C(T_2-T_1)^{\alpha\nu},$$
$$\text{if } T_1 > 2(T_2-T_1), \quad j = \overline{1,n-1}. \quad (4.66)$$

The obtaining of these results is technically tedious so we give their proof in the Appendix 6.1

LEMMA 4.3. *Let* $\alpha,\nu\in(0,1)$, *conditions* (4.6), (4.11) *and* (4.9) *hold. Then there is the following estimate:*

$$\sum_{|m|=0}^{2}\langle D_x^m\wp\rangle_{t,R_T^{n-1}}^{(\alpha\nu/2)}+\sum_{|m|=0}^{1}\langle \mathbf{D}_t^\nu D_x^m\wp\rangle_{t,R_T^{n-1}}^{(\alpha\nu/2)}\leqslant C\|f_1\|_{C^{1+\alpha,\nu\alpha/2,\alpha}(\bar R_T^{n-1})}. \tag{4.67}$$

*Proof.* Note that estimate of the term $\sum_{|m|=0}^{2}\langle D_x^m\wp\rangle_{t,R_T^{n-1}}^{(\alpha\nu/2)}$ follows from Proposition 2.1, Lemma 4.2 and the next interpolation inequality (see Lemma 3.1 [31] and Corollary 1.2.18 [20]):

$$\|V\|_{C^{l_0}(Q)}\leqslant C\|V\|_{C^{l_2}(Q)}^{\varepsilon}\|V\|_{C^{l_1}(Q)}^{1-\varepsilon},$$

where $\varepsilon\in(0,1),\ 0\leqslant l_1<l_2,\ l_0=l_1+\varepsilon(l_2-l_1)$. Thus, we have

$$\sum_{|m|=0}^{2}\langle D_x^m\wp\rangle_{t,R_T^{n-1}}^{(\alpha\nu/2)}\leqslant C\|f_1\|_{C([0,T],C^{1+\alpha}(R^{n-1}))}. \tag{4.68}$$

To finish the proof of Lemma 4.3, we have to evaluate the terms $\langle \mathbf{D}_t^\nu\frac{\partial^m\wp}{\partial x_i^m}\rangle_{t,R_T^{n-1}}^{(\alpha\nu/2)}$, $m=0,1;\ i=\overline{1,n-1}$. Let $t_1,t_2\in[0,T]$, $t_2>t_1$ and denote by

$$\Delta t:=t_2-t_1,\quad \Delta_t\mathbf{D}_t^\nu\frac{\partial^m\wp}{\partial x_i^m}:=\mathbf{D}_t^\nu\frac{\partial^m\wp}{\partial x_i^m}(x',t_2)-\mathbf{D}_t^\nu\frac{\partial^m\wp}{\partial x_i^m}(x',t_1).$$

As follows from (4.62), we can conclude that

$$\mathbf{D}_t^\nu\frac{\partial^m\wp}{\partial x_i^m}=\frac{\partial^m f_1}{\partial x_i^m}+\int_0^t d\tau\int_{R^{n-1}}\left[\frac{\partial^m f_1}{\partial x_i^m}(x'-y',\tau)-\frac{\partial^m f_1}{\partial x_i^m}(x',\tau)\right]\partial_{t-\tau}^\nu G(y',t-\tau)dy', \tag{4.69}$$

where $m=0,1;\ i=\overline{1,n-1}$.

First, we analyze the case $t_1>2\Delta t$ and represent the difference $\Delta_t\mathbf{D}_t^\nu\frac{\partial^m\wp}{\partial x_i^m}$ as

$$\begin{aligned}\Delta_t\mathbf{D}_t^\nu\frac{\partial^m\wp}{\partial x_i^m}=&\int_{t_1-2\Delta t}^{t_2}d\tau\int_{R^{n-1}}\left[\frac{\partial^m f_1}{\partial x_i^m}(x'-y',\tau)-\frac{\partial^m f_1}{\partial x_i^m}(x',\tau)\right]\partial_{t_2-\tau}^\nu G(y',t_2-\tau)dy'\\ &-\int_{t_1-2\Delta t}^{t_1}d\tau\int_{R^{n-1}}\left[\frac{\partial^m f_1}{\partial x_i^m}(x'-y',\tau)-\frac{\partial^m f_1}{\partial x_i^m}(x',\tau)\right]\partial_{t_1-\tau}^\nu G(y',t_1-\tau)dy'\\ &+\int_0^{t_1-2\Delta t}d\tau\int_{R^{n-1}}\left[\frac{\partial^m f_1}{\partial x_i^m}(x'-y',\tau)-\frac{\partial^m f_1}{\partial x_i^m}(x',\tau)\right][\partial_{t_2-\tau}^\nu G(y',t_2-\tau)\end{aligned}$$

$$-\partial^{\nu}_{t_1-\tau}G(y',t_1-\tau)]dy' + \left[\frac{\partial^m f_1}{\partial x_i^m}(x',t_2) - \frac{\partial^m f_1}{\partial x_i^m}(x',t_1)\right]$$

$$\equiv \sum_{l=1}^{4} \mathfrak{J}_l. \tag{4.70}$$

To evaluate the term $\mathfrak{J}_1$, we do the change of a variable: $z = t_2 - \tau$, in the inner integral and then apply inequality (4.65) with $T_2 := 3\Delta t$ and $T_1 := 0$. Thus, we have

$$|\mathfrak{J}_1| \leqslant C(\Delta t)^{\alpha\nu} \left\langle \frac{\partial^m f_1}{\partial x_i^m} \right\rangle^{(\alpha)}_{x,R_T^{n-1}}. \tag{4.71}$$

The estimate of $\mathfrak{J}_2$ is got with the same way. As for $\mathfrak{J}_3$, the mean-value theorem together with estimate (4.66), where $T_1 := t_1$ and $T_2 := t_2$,

$$|\mathfrak{J}_3| \leqslant C(\Delta t)^{\alpha\nu} \left\langle \frac{\partial^m f_1}{\partial x_i^m} \right\rangle^{(\alpha)}_{x,R_T^{n-1}}. \tag{4.72}$$

At last, the estimate of $\mathfrak{J}_4$ follows immediately from the properties of the function $f_1$. Thus, as it follows from (4.70)–(4.72), the following inequality is fulfilled:

$$\sum_{i=1}^{n-1} \left\langle \mathbf{D}_t^{\nu} \frac{\partial^m \wp}{\partial x_i^m} \right\rangle^{(\alpha\nu/2)}_{t,R_T^{n-1}} \leqslant C\|f_1\|_{C^{1+\alpha,\alpha\nu/2}(\bar{R}_T^{n-1})}, \quad m = 0,1, \quad \text{if } t_1 > 2\Delta t. \tag{4.73}$$

Let us prove estimate (4.73) if $t_1 < 2\Delta t$. To this end, we use another form to difference $\Delta_t \mathbf{D}_t^{\nu} \frac{\partial^m \wp}{\partial x_i^m}$:

$$\begin{aligned}
\Delta_t \mathbf{D}_t^{\nu} \frac{\partial^m \wp}{\partial x_i^m} &= \int_0^{t_1} d\tau \int_{R^{n-1}} \left[\frac{\partial^m f_1}{\partial x_i^m}(x'-y',t_2-\tau) - \frac{\partial^m f_1}{\partial x_i^m}(x',t_2-\tau)\right] \partial_\tau^{\nu} G(y',\tau)dy' \\
&\quad - \int_0^{t_1} d\tau \int_{R^{n-1}} \left[\frac{\partial^m f_1}{\partial x_i^m}(x'-y',t_1-\tau) - \frac{\partial^m f_1}{\partial x_i^m}(x',t_1-\tau)\right] \partial_\tau^{\nu} G(y',\tau)dy' \\
&\quad + \int_{t_1}^{t_2} d\tau \int_{R^{n-1}} \left[\frac{\partial^m f_1}{\partial x_i^m}(x'-y',t_2-\tau) - \frac{\partial^m f_1}{\partial x_i^m}(x',t_2-\tau)\right] \partial_\tau^{\nu} G(y',\tau)dy' \\
&\quad + \left[\frac{\partial^m f_1}{\partial x_i^m}(x',t_2) - \frac{\partial^m f_1}{\partial x_i^m}(x',t_1)\right] \\
&\equiv \sum_{j=1}^{4} \mathscr{B}_j, \quad i = \overline{1,n-1}, \quad m = 0,1.
\end{aligned} \tag{4.74}$$

Note that the estimate of $\mathscr{B}_2$ follows from inequality (4.65) where $T_1 := 0$ and $T_2 := t_1$:

$$|\mathscr{B}_2| \leqslant C\|f_1\|_{C([0,T],C^{1+\alpha}(R^{n-1}))} t_1^{\alpha\nu}.$$

Since we consider the case of $t_1 < 2\Delta t$, the last inequality is rewritten as

$$|\mathscr{B}_2| \leqslant C\|f_1\|_{C([0,T],C^{1+\alpha}(R^{n-1}))}(\Delta t)^{\alpha\nu}. \tag{4.75}$$

The estimates of $\mathscr{B}_1$ and $\mathscr{B}_3$ are obtained with the same arguments. The term $\mathscr{B}_4$ is evaluated like $\mathfrak{J}_4$. Thus, based on (4.74) and (4.75), we get estimate (4.73) in the case of $t_1 < 2\Delta t$.

Then, we collect these results with inequalities (4.68) and (4.73) to complete the proof of Lemma 4.3. □

The following statement follows immediately from the results of Lemmas 4.2 and 4.3.

LEMMA 4.4. *Let* $\alpha,\nu \in (0,1)$, *conditions* (4.6), (4.11) *and* (4.9) *hold, then next estimate is fulfilled:*

$$\|\wp\|_{C^{2+\alpha,\nu\alpha/2,\alpha}(\bar{R}_T^{n-1})} + \|\mathbf{D}_t^{\nu}\wp\|_{C^{1+\alpha,\nu\alpha/2,\alpha}(\bar{R}_T^{n-1})} \leqslant C\|f_1\|_{C^{1+\alpha,\nu\alpha/2,\alpha}(\bar{R}_T^{n-1})}. \tag{4.76}$$

Now we will prove the results similarly to Lemmas 4.2 and 4.4 for the functions $u_{\pm}(x,t)$. To this end, we need in the following properties of the kernels $G_{\pm}(x',0,\tau)$ and $K_{\pm}(x',0,\eta)$ represented with (4.32).

LEMMA 4.5. *Let the conditions of Lemma 4.1 and Proposition 4.2 hold. Then:*

- *The functions $K_{\pm}(x',0,\eta)$ satisfy inequalities* (4.33), (4.34) *and* (4.36).
- *Inequalities* (4.39)–(4.47) *hold for the functions $G_{\pm}(x',0,\tau)$.*
- $$\int_0^{3T_1-2T_2} d\tau \int_{T_1}^{T_2} dt \int_{R_+^{n-1}} y_j^{\alpha} \left| \frac{\partial^2 G_{\pm}(x',0,t-\tau)}{\partial(t-\tau)\partial y_i} \right| dy' \leqslant C(T_2-T_1)^{\alpha\nu}, \tag{4.77}$$

  *if* $T_1 > 2(T_2-T_1),\ j,i = \overline{1,n-1}$.

The proof of Lemma 4.5 is given in Appendix 6.2. After that, using the results of Lemma 4.5 and repeating arguments from the proofs of Lemmas 4.2 and 4.4, we get the following.

LEMMA 4.6. *Let the conditions of Lemma 4.2 hold, then the functions* $u_{\pm}(x',0,t) \in C([0,T],C^{2+\alpha}(R^{n-1}))$ *and*

$$\|u_{\pm}\|_{C([0,T],C^{2+\alpha}(R^{n-1}))} + \|\mathbf{D}_t^{\nu}u_{\pm}\|_{C([0,T],C^{1+\alpha}(R^{n-1}))} \leqslant C\|f_1\|_{C([0,T],C^{1+\alpha}(R^{n-1}))}. \tag{4.78}$$

*If the conditions of Lemma 4.4 hold, then the functions* $u_{\pm}(x',0,t) \in C^{2+\alpha,\nu\alpha/2,\alpha}(\bar{R}_T^{n-1})$

$$\|u_{\pm}\|_{C^{2+\alpha,\nu\alpha/2,\alpha}(\bar{R}_T^{n-1})} + \|\mathbf{D}_t^{\nu}u_{\pm}\|_{C^{1+\alpha,\nu\alpha/2,\alpha}(\bar{R}_T^{n-1})} \leqslant C\|f_1\|_{C^{1+\alpha,\nu\alpha/2,\alpha}(\bar{R}_T^{n-1})}. \tag{4.79}$$

### 4.3. Proof of Theorem 4.1

First of all we prove estimates (4.8) and (4.10) if either (4.7) or (4.9) holds. To this end, as it follows from results of Lemmas 4.2, 4.4 and 4.6, it is enough to extend estimates (4.78) and (4.79) into the functions $u_{\pm}(x',x_n,t)$, $(x,t)\in R^n_{\pm T}$. We represent $u_{\pm}(x',x_n,t)$ as $u_{\pm}(x',x_n,t)=\mathscr{K}\star u_{\pm}(x',0,t)$, where $\mathscr{K}$ is the kernel of the Dirichlet problem and $u_{\pm}(x',0,t)\in C^{2+\alpha}(R^{n-1})$ for all $t\in[0,T]$. Then, applying arguments from Chapter 3 [16] allows us to show that functions $u_{\pm}(x',x_n,t)$ satisfy estimate either (4.8) or (4.10). Moreover, the functions $u_{\pm}(x',x_n,t)$ represented by (4.28) and (4.29) satisfy equations (4.1). The direct calculations together with applications of Lemmas 4.1 and 4.5 assure that $\wp(x',t)$ and $u_{\pm}(x',x_n,t)$ given by (4.28)–(4.30) meet requirements (4.2)–(4.5). Note that the uniqueness of the constructed solution in the corresponding classes follows from coercive estimates (4.8), (4.10). All the written above proves Theorem 4.1 in the case of (4.9).

To get Theorem 4.1 in the case of (4.7) we have to show continuous of the functions $\wp(x',t)$ and $u_{\pm}(x',x_n,t)$ together with their derivatives with respect to time. To this end, we repeat the arguments from Section 4 [14] adapting them to our case. That completes the proof of Theorem 4.1.

If constant $k$ in condition (4.4) is changed by $k\delta$, $\delta\in[0,1]$, then we can repeat all the arguments from this section and obtain the results of Theorem 4.1, where the constants in estimates (4.8) and (4.10) are independent of $\delta$. The last statement follows from uniformly boundedness of $A_0(k\delta)$, $A_1(k\delta)$, $a_3k\delta$, $\frac{a_0}{k\delta+1}$, $\frac{k\delta a_0}{k\delta+1}$ (see representations of the solution (4.20)–(4.22), (4.28)–(4.32)). Moreover, it is easy to see that

$$0<A_0(k)<A_0(k\delta)<2\pi a_1a_0;\ \ 0<\frac{a_0}{k+1}<\frac{a_0}{k\delta+1}<a_0;\ \ 0\leqslant\frac{k\delta a_0}{k\delta+1}<ka_0.$$

Thus, we can conclude the following:

REMARK 4.1. Let boundary condition (4.4) be changed by

$$\frac{\partial u_-}{\partial n}-k\delta\frac{\partial u_+}{\partial n}-k\delta\langle a_3,\nabla_{x'}(u_--u_+)\rangle=f_2(x',t)\ on\quad R^{n-1}_T, \tag{4.80}$$

where $\delta\in[0,1]$. We assume that conditions of Theorem 4.1 hold, then problem (4.1)–(4.3), (4.80), (4.5), (4.6) has a unique solution $(u_-,u_+,\wp)$ which satisfies inequalities (4.8) and (4.10) with the constants independent of $\delta$.

## 5. Proof of the main results

Our argument splits into two steps. The first is related to the proof of the boundedness of the linear operator $\mathscr{A}$ (see (3.42)). The second step is connected with the proof of the nonlinear operator $\mathscr{A}^{-1}\mathscr{F}(z)$ is a contraction one.

### 5.1. Linear problem corresponding to (3.42)

Here we analyze the following linear problem

$$\Delta_x w_i=F_{0i}(x,t)\ \text{ in }\ \Omega_{iT},\ \ 1,2, \tag{5.1}$$

$$w_1 - w_2 = -A(x)\sigma \quad \Upsilon_T; \tag{5.2}$$

$$\mathbf{D}_t^\nu \sigma - \frac{k_2}{1-k}\left[\frac{\partial w_1}{\partial \bar{n}(\omega)} - \frac{\partial w_2}{\partial \bar{n}(\omega)}\right] - \sum_{j=1}^{n-1} a_j(x)\frac{\partial}{\partial \omega_j}(w_1 - w_2) = F_1(x,t) \text{ on } \Upsilon_T; \tag{5.3}$$

$$\frac{\partial w_1}{\partial \bar{n}(\omega)} - k\frac{\partial w_2}{\partial \bar{n}(\omega)} - \frac{k}{k_1}\sum_{j=1}^{n-1} b_j(x)\frac{\partial}{\partial \omega_j}(w_1 - w_2) = F_2(x,t) \text{ on } \Upsilon_T; \tag{5.4}$$

$$w_i(x,t) = 0 \text{ on } \Gamma_{iT}, \ 1,2; \tag{5.5}$$

$$\sigma(\omega,0) = 0 \text{ on } \Upsilon; \quad w_i(x,0) = 0, \ x \in \bar{\Omega}_i. \tag{5.6}$$

Here $F_{0i}, \ F_i, \ i = 1,2, \ A(x), \ a_j(x), \ b_j(x), \ j = \overline{1,n-1}$, are some given functions and

$$F_{0i}(x,0) = 0, \ x \in \Omega_i, \ 1,2; \quad F_j(x,0) = 0, \ x \in \Upsilon, \ j = 1,2;$$

$$F_{0i}(x,t) \in C([0,T],C^\alpha(\bar{\Omega}_i)), \quad F_j(x,t) \in C([0,T],C^{1+\alpha}(\Upsilon)); \tag{5.7}$$

$A(x), \ b_j(x)$ satisfy condition (3.44) and $a_j(x) \in C^{1+\alpha}(\Upsilon)$,

$$k = \frac{k_2}{k_1}, \quad 0 < k_2 < k_1. \tag{5.8}$$

THEOREM 5.1. *Let $\alpha, \nu \in (0,1)$; conditions (5.7), (5.8) and (3.44) hold; $\Gamma_i, \ \Upsilon \in C^{2+\alpha}$. Then, for a sufficiently small $T$, there exists a solution $(w_1,w_2,\sigma)$ of problem (5.1)–(5.6) and*

$$\sum_{i=1}^{2}\|w_i\|_{C([0,T],C^{2+\alpha}(\bar{\Omega}_i))} + \|\sigma\|_{C([0,T],C^{2+\alpha}(\Upsilon))} + \|\mathbf{D}_t^\nu\sigma\|_{C([0,T],C^{1+\alpha}(\Upsilon))}$$

$$\leqslant C[\sum_{i=1}^{2}\|F_{0i}\|_{C([0,T],C^\alpha(\bar{\Omega}_i))} + \sum_{j=1}^{2}\|F_j\|_{C([0,T],C^{1+\alpha}(\Upsilon))}]. \tag{5.9}$$

*Proof.* First of all we analyze problem (5.1)–(5.6) under conditions

$$F_{0i}, \ F_2 \equiv 0, \ 1,2. \tag{5.10}$$

We will use the method of parameter extension to solve problem (5.1)–(5.6), that is replacing condition (5.4) by

$$\frac{\partial w_1}{\partial \bar{n}(\omega)} - k\delta\frac{\partial w_2}{\partial \bar{n}(\omega)} - \frac{k\delta}{k_1}\sum_{j=1}^{n-1} b_j(x)\frac{\partial}{\partial \omega_j}(w_1 - w_2) = 0 \text{ on } \Upsilon_T, \ \delta \in [0,1]. \tag{5.11}$$

Let us consider problem (5.1)–(5.3), (5.11), (5.5) and (5.6). If $\delta = 1$, this problem is just problem (5.1)–(5.6). When $\delta = 0$, this problem splits into two problems:

$$\Delta_x w_1 = 0 \text{ in } \Omega_{1T}, \ \frac{\partial w_1}{\partial \bar{n}(\omega)}|_{\Gamma_T} = 0, \ w_1|_{\Gamma_{1T}} = 0, \ w_1(x,0) = 0, \ x \in \Omega_1, \tag{5.12}$$

so $w_1 \equiv 0$ in $\bar{\Omega}_{1T}$; and

$$\Delta_x w_2 = 0 \text{ in } \Omega_{2T}; \ \ w_2|_{\Gamma_{2T}} = 0; \ \ w_2(x,0) = 0, \ x \in \Omega_2; \ \ \sigma(\omega,0) = 0, \ \ \omega \in \Upsilon;$$

$$w_2 = A(x)\sigma \text{ and } \mathbf{D}_t^{\nu}\sigma + \frac{k_2}{1-k}\frac{\partial w_2}{\partial \bar{n}(\omega)} + \sum_{j=1}^{n-1} a_j(x)\frac{\partial w_2}{\partial \omega_j} = F_1 \text{ on } \Upsilon_T. \tag{5.13}$$

Problem (5.13) with $a_j(x) \equiv 0, \ j = \overline{1, n-1}$, has been studied in Section 4 [32]. Thus, it is not hard to prove, using arguments and results from Section 4 [32], one-to-one solvability of (5.13): $w_2 \in C([0,T], C^{2+\alpha}(\bar{\Omega}_2)), \ \sigma \in C([0,T], C^{2+\alpha}(\Upsilon))$ :

$$\begin{aligned} &\|w_2\|_{C([0,T],C^{2+\alpha}(\bar{\Omega}_2))} + \|\mathbf{D}_t^{\nu} w_2\|_{C([0,T],C^{1+\alpha}(\bar{\Omega}_2))} + \|\sigma\|_{C([0,T],C^{2+\alpha}(\Upsilon))} \\ &\qquad + \|\mathbf{D}_t^{\nu}\sigma\|_{C([0,T],C^{1+\alpha}(\Upsilon))} \leqslant C\|F_1\|_{C([0,T],C^{1+\alpha}(\Upsilon))}, \end{aligned} \tag{5.14}$$

where the constant $C$ depends only on $k_1, \ k_2, \ \|A(x)\|_{C^{2+\alpha}(\Upsilon)}, \ \|a_j\|_{C^{1+\alpha}(\Upsilon)}$, and measure of $\Upsilon, \ \Gamma_2, \ \Omega_2$.

In order to get the well-posedness of problem (5.1)–(5.6), we have to obtain a uniform a priori estimate with respect to $\delta$ of the solution $w_i \in C([0,T], C^{2+\alpha}(\bar{\Omega}_i))$, $i = 1,2, \ \sigma \in C([0,T], C^{2+\alpha}(\Upsilon))$ of problem (5.1)–(5.3), (5.5), (5.6) and (5.11).

Adapting the standard Schauder technique to the case of a fractional derivative (to this end we essential use Proposition 2.1) and applying the results of Theorem 4.1 and Remark 4.1, we deduce:

$$\begin{aligned} &\sum_{i=1}^{2} \|w_i\|_{C([0,T],C^{2+\alpha}(\bar{\Omega}_i))} + \|\sigma\|_{C([0,T],C^{2+\alpha}(\Upsilon))} + \|\mathbf{D}_t^{\nu}\sigma\|_{C([0,T],C^{1+\alpha}(\Upsilon))} \\ &\qquad \leqslant C_1[\|F_1\|_{C([0,T],C^{1+\alpha}(\Upsilon))} + \sup_{\bar{\Omega}_{1T}}|w_1| + \sup_{\bar{\Omega}_{2T}}|w_2|], \end{aligned} \tag{5.15}$$

where $C_1$ is independent of $\delta$.

As for estimates of $\sup\limits_{\bar{\Omega}_{iT}}|w_i|, \ i = 1,2$, we apply statement (iii) from Proposition 2.3 where we put $g_3 := -A(x)\sigma$, and $W_i := w_i$. Thus we have

$$\sup_{\bar{\Omega}_{iT}}|w_i| \leqslant CT^{\nu}\sup_{\bar{\Omega}_{iT}}|\mathbf{D}_t^{\nu} w_i| \leqslant C_2 T^{\nu}\|\mathbf{D}_t^{\nu}\sigma\|_{C([0,T],C^{1+\alpha}(\Upsilon))}. \tag{5.16}$$

Then we collect (5.15) and (5.16) and, choosing $T$ such that

$$C_1 C_2 T^{\nu} < 1/4, \tag{5.17}$$

we get the uniform estimate

$$\begin{aligned} &\sum_{i=1}^{2}[\|w_i\|_{C([0,T],C^{2+\alpha}(\bar{\Omega}_i))} + \|\mathbf{D}_t^{\nu} w_i\|_{C([0,T],C^{\alpha}(\bar{\Omega}_i))}] + \|\sigma\|_{C([0,T],C^{2+\alpha}(\Upsilon))} \\ &\qquad + \|\mathbf{D}_t^{\nu}\sigma\|_{C([0,T],C^{1+\alpha}(\Upsilon))} \leqslant C\|F_1\|_{C([0,T],C^{1+\alpha}(\Upsilon))}, \end{aligned} \tag{5.18}$$

where constant $C$ is independent of $\delta$.

In this way, we have proved the well-posedness of problem (5.1)–(5.6) in the case of (5.10). At last results of statement **(i)** from Proposition 2.3 allow us to remove restriction (5.10). $\square$

Next, using results of Theorem 5.1 and Proposition 2.1 together with inequality (4.9), we can get an existence of the more smooth solution of problem (5.1)–(5.6).

THEOREM 5.2. *Let conditions of Theorem 5.1 hold and* $F_{0i}(x,t) \in C^{\alpha,\frac{\alpha\nu}{2},\alpha}(\bar{\Omega}_{iT})$, $F_j(x,t) \in C^{1+\alpha,\frac{\alpha\nu}{2},\alpha}(\Upsilon_T)$. *Then, for a sufficiently small* $T$, *there exists a solution* $(w_1,w_2,\sigma)$ *of problem* (5.1)–(5.6) *and*

$$\sum_{i=1}^{2}\|w_i\|_{C^{2+\alpha,\frac{\alpha\nu}{2},\alpha}(\bar{\Omega}_{iT})}+\|\sigma\|_{C^{2+\alpha,\frac{\alpha\nu}{2},\alpha}(\Upsilon_T)}+\|\mathbf{D}_t^{\nu}\sigma\|_{C^{1+\alpha,\frac{\alpha\nu}{2},\alpha}(\Upsilon_T)}$$

$$\leqslant C[\sum_{i=1}^{2}\|F_{0i}\|_{C^{\alpha,\frac{\alpha\nu}{2},\alpha}(\bar{\Omega}_{iT})}+\sum_{j=1}^{2}\|F_j\|_{C^{1+\alpha,\frac{\alpha\nu}{2},\alpha}(\Upsilon_T)}]. \tag{5.19}$$

Note that Theorem 5.1 gives the local classical solvability of linear problem (5.1)–(5.6) with fractional derivative. The same result takes place in the case $\nu=1$ (see Section 4 from [36]). As for results of Theorem 5.2, they mean the local one-valued solvability in Hölder spaces $C^{k+\alpha,\beta,\alpha}$, $\beta := \alpha\nu/2$. These results represent a marked difference with the case $\nu=1$ where exponent $\beta$ is greater, $\beta := \alpha$, $\alpha \in (0,1/2)$ (see Section 4 [4]).

### 5.2. Solvability of nonlinear problem (3.32)–(3.37)

We introduce the functional spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, such that $\mathbf{z}\in\mathscr{H}_1$ and $\mathscr{F}\mathbf{z}\in\mathscr{H}_2$,

$$\begin{aligned}\mathscr{H}_1 = {}& \underset{0}{C}([0,T],C^{2+\alpha}(\bar{\Omega}_1))\times\underset{0}{C}([0,T],C^{2+\alpha}(\bar{\Omega}_2))\times\underset{0}{C}([0,T],C^{2+\alpha}(\Upsilon))\\ &\times\underset{0}{C}([0,T],C^{1+\alpha}(\Upsilon));\\ \mathscr{H}_2 = {}& \underset{0}{C}([0,T],C^{\alpha}(\bar{\Omega}_1))\times\underset{0}{C}([0,T],C^{\alpha}(\bar{\Omega}_2))\times\underset{0}{C}([0,T],C^{2+\alpha}(\Upsilon))\times\underset{0}{C}([0,T],C^{1+\alpha}(\Upsilon))\\ &\times\underset{0}{C}([0,T],C^{1+\alpha}(\Upsilon))\times\underset{0}{C}([0,T],C^{2+\alpha}(\Gamma_1))\times\underset{0}{C}([0,T],C^{2+\alpha}(\Gamma_2));\end{aligned}$$

and

$$\begin{aligned}\|\mathbf{z}\|_{\mathscr{H}_1} &= \|(w_1,w_2,\sigma)\|_{\mathscr{H}_1}\\ &= \sum_{i=1}^{2}\|w_i\|_{C([0,T],C^{2+\alpha}(\bar{\Omega}_i))}+\|\sigma\|_{C([0,T],C^{2+\alpha}(\Upsilon))}+\|\mathbf{D}_t^{\nu}\sigma\|_{C([0,T],C^{1+\alpha}(\Upsilon))};\\ \|\mathscr{F}\mathbf{z}\|_{\mathscr{H}_2} &= \|(\mathscr{F}_{01}(\mathbf{z}),\mathscr{F}_{02}(\mathbf{z}),0,\mathscr{F}_1(\mathbf{z}),\mathscr{F}_2(\mathbf{z}),0,0)\|_{\mathscr{H}_2}\\ &= \sum_{i=1}^{2}\|\mathscr{F}_{0i}(\mathbf{z})\|_{C([0,T],C^{\alpha}(\bar{\Omega}_i))}+\sum_{j=1}^{2}\|\mathscr{F}_j(\mathbf{z})\|_{C([0,T],C^{1+\alpha}(\Upsilon))}.\end{aligned}$$

Based on (3.42) and representations (3.39)–(3.41), we can rewrite problem (3.32)–(3.37) in the form

$$\mathscr{A}\mathbf{z} = \mathscr{F}\mathbf{z} = F(x,t) + \mathscr{F}^1(\mathbf{z}), \tag{5.20}$$

where $\mathbf{z} = (w_1, w_2, \sigma)$, $\mathscr{A}$ is the linear operator which has been studied in Subsection 5.1, $\mathscr{A} : \mathscr{H}_1 \to \mathscr{H}_2$; the vector $F(x,t)$ is constructed by initial data; $\mathscr{F}^1(\mathbf{z})$ contains the elements described in Corollary 3.2.

Since the operator $\mathscr{A}$ satisfies the conditions of Theorem 5.1, nonlinear problem (5.20) can be represented as

$$\mathbf{z} = \mathscr{A}^{-1}F(x,t) + \mathscr{A}^{-1}\mathscr{F}^1(\mathbf{z}) \equiv \mathscr{P}(\mathbf{z}).$$

LEMMA 5.1. *Let $B_d$, $B_d \subset \mathscr{H}_1$, be a ball with the center located in the origin and the radius of $d$. For $\mathbf{z} \in B_d$ the following estimates hold*

$$\|\mathscr{F}(0)\|_{\mathscr{H}_2} \leqslant C_1(T); \tag{5.21}$$

$$\|\mathscr{F}(\mathbf{z}_1) - \mathscr{F}(\mathbf{z}_2)\|_{\mathscr{H}_2} \leqslant C_2(T,d)\|\mathbf{z}_1 - \mathbf{z}_2\|_{\mathscr{H}_1}, \tag{5.22}$$

*where constants $C_1(T)$ and $C_2(T,d)$ vanish if $T,d \to 0$.*

*Proof.* To prove Lemma 5.1 we adapt the arguments from Section 5 [4] to the case of the fractional derivative. On this route we use essentially representations (3.25), (3.39)–(3.41) and results of Theorem 5.1 and Corollaries 3.1, 3.2, Proposition 2.1.

First we get inequality (5.21) for $\mathscr{F}_1(\mathbf{z})$. As for estimates of $\mathscr{F}_{0i}(\mathbf{z})$ and $\mathscr{F}_2(\mathbf{z})$ which are contained in $\mathscr{F}(\mathbf{z})$, they are evaluated with the same way.

From (3.40), one can see that the “worst” term under evaluating $\mathscr{F}_1(0)$ is $s_{\omega_i}$. So that we get, using (3.17), (3.25) and (3.26),

$$\|s_{\omega_i}\|_{C([0,T],C^{1+\alpha}(\Upsilon))} \leqslant \text{const.}\, T^{\nu} \sum_{i=1}^{2} \|\psi_i\|_{C^{3+\alpha}(\Gamma_i)}. \tag{5.23}$$

In virtue of the appropriate choose the function $s$, inequality (5.23) will be hold in the case $\nu = 1$ ( i.e. in the case of integer order derivative).

Thereby we conclude from Corollary 3.2 and (5.23) that

$$\|\mathscr{F}_1(0)\|_{C([0,T],C^{1+\alpha}(\Upsilon))} \leqslant CT^{\nu} \sum_{i=1}^{2} \|\psi_i\|_{C^{3+\alpha}(\Gamma_i)}.$$

Then we show that $\mathscr{F}_2(\mathbf{z})$ satisfies inequality (5.22). As it follows from Corollary 3.2, the main difficulties deal with the linear terms in the difference $[\mathscr{F}_2(\mathbf{z}_1) - \mathscr{F}_2(\mathbf{z}_2)]$. The “worst” term comes from $\sum_{i,j=1}^{n-1} \frac{\partial S_j}{\partial \rho_{\omega_i}}(\omega,0,0) s_{\omega_i} \frac{\partial v_{10}}{\partial \bar{n}(\omega)} \sigma_{\omega_j}$, and one is

$$\left.\frac{\partial S_j}{\partial \rho_{\omega_i}}\right|_{t=0} s_{\omega_i} \frac{\partial v_{10}}{\partial \bar{n}(\omega)} \sigma_{\omega_j} = \Phi(x) t^{\nu} \sigma_{\omega_j}, \quad \Phi(x) \in C^{2+\alpha}(\Upsilon). \tag{5.24}$$

Note that to deduce (5.24) we apply representation (3.25) for the function $s$ and estimates (3.17) together with (3.26).

Thus, we can conclude from (5.24) and Theorem 5.1

$$\left\| \sum_{i,j=1}^{n-1} \frac{\partial S_j}{\partial \rho_{\omega_i}}(\omega,0,0) s_{\omega_i} \frac{\partial v_{10}}{\partial \bar{n}(\omega)} \left[ \frac{\partial \sigma_1}{\partial \omega_j} - \frac{\partial \sigma_2}{\partial \omega_j} \right] \right\|_{C([0,T],C^{1+\alpha}(\Upsilon))}$$

$$\leqslant CT^{\nu} \|\sigma_1 - \sigma_2\|_{C([0,T],C^{2+\alpha}(\Upsilon))}. \tag{5.25}$$

The other terms in the difference $[\mathscr{F}_2(\mathbf{z}_1) - \mathscr{F}_2(\mathbf{z}_2)]$ can be estimated more easily, using again Proposition 2.1, Corollaries 3.1 and 3.2, Theorem 5.1, so that we get inequality (5.22) for $\mathscr{F}_2(\mathbf{z})$. In the same way one can obtain estimate (5.22) for $\mathscr{F}_1(\mathbf{z})$ and $\mathscr{F}_{0i}(\mathbf{z})$. That completes the proof of Lemma 5.1. □

Inequalities (5.22) mean that for sufficiently small $T$ and $d$ the nonlinear operator $\mathscr{P}(\mathbf{z})$ satisfies the conditions of the fixed point theorem for a contraction operator. Hence, the fixed point of the operator is the solution of problem (3.15), and Theorem 3.1 has been proved.

To get the local solvability of nonlinear problem (3.15) in the more smooth classes, we repeat the arguments above and apply the results of Theorem 5.2 together with the second inequality in (3.29). Thus, we get the following result.

THEOREM 5.3. *Let conditions of Theorem 3.1 hold. Then for some small $T$, there is a unique solution $(v_1(x,t), v_2(x,t), \rho(\omega,t))$ of nonlinear problem (3.15) for $t \in [0,T]$, such that*

$$v_i(x,t) \in C^{2+\alpha,\frac{\alpha\nu}{2},\alpha}(\bar{\Omega}_{iT}), \quad \rho(\omega,t) \in C^{2+\alpha,\frac{\alpha\nu}{2},\alpha}(\Upsilon_T), \ \ \mathbf{D}_t^{\nu}\rho(\omega,t) \in C^{1+\alpha,\frac{\alpha\nu}{2},\alpha}(\Upsilon_T),$$

*and equality (3.21) takes place.*

## 6. Appendix

### 6.1. The proof of Proposition 4.2

First, we get estimate (4.65). Using inequalities (4.50) with $m = 0$, we deduce

$$\mathscr{L}_1 := \int_{T_1}^{T_2} d\tau \int_{R_+^{n-1}} y_j^{\alpha} |\partial_\tau^{\nu} G(y',\tau)| dy'$$

$$\leqslant C \int_{R_+^{n-1}} y_j^{\alpha} dy' \int_{T_1}^{T_2} d\tau \int_0^{+\infty} \tau^{-1-\nu} |W(-\eta\tau^{-\nu}; -\nu, -\nu)| \exp\left\{ -A \sum_{i=1}^{n-1} \frac{y_i}{\eta} \right\} \eta^{1-n} d\eta.$$

Then the change of variables (4.52) and (4.53) leads to

$$\mathscr{L}_1 \leqslant C \int\limits_{R_+^{n-1}} x_j^\alpha \exp\left\{-A\sum_{i=1}^{n-1} x_i\right\} dx' \int\limits_{T_1}^{T_2} d\tau \int\limits_0^{+\infty} \tau^{\nu\alpha-1} z^\alpha |W(-z;-\nu,-\nu)| dz$$
$$\leqslant C(T_2-T_1)^{\nu\alpha}. \tag{6.1}$$

To obtain the last inequality in (6.1), we applied estimate (4.57) with $\beta := \nu,\ \gamma := -\nu$ to the function $|W(-z;-\nu,-\nu)|$.

Next, we prove (4.66). Using the following formula from[26]

$$\frac{d^n}{dt^n}(t^{\gamma-1}W(ct^{-\beta};-\beta,\gamma)) = t^{\gamma-1-n}W(ct^{-\beta};-\beta,\gamma-n), \tag{6.2}$$

we represent $\frac{\partial}{\partial t}\partial_t^\nu G(y',t)$ as

$$\frac{\partial}{\partial t}\partial_t^\nu G(y',t) = \int\limits_0^{+\infty} t^{-2-\nu} W(-\eta t^{-\nu};-\nu,-\nu-1) K(y',\eta) d\eta. \tag{6.3}$$

After that we apply inequality (4.33) with $m=0$ to the right-hand side in (6.3) and infer

$$\left|\frac{\partial}{\partial t}\partial_t^\nu G(y',t)\right| \leqslant C \int\limits_0^{+\infty} t^{-2-\nu} |W(-\eta t^{-\nu};-\nu,-\nu-1)| \frac{\exp\left\{-A\sum\limits_{i=1}^{n-1}|y_i|\eta^{-1}\right\}}{\eta^{n-1}} d\eta. \tag{6.4}$$

Based on estimate (6.4), one can easily deduce that

$$\mathscr{L}_2 := \int\limits_0^{3T_1-2T_2} d\tau \int\limits_{T_1}^{T_2} dt \int\limits_{R_+^{n-1}} y_j^\alpha \left|\frac{\partial}{\partial(t-\tau)}\partial_{t-\tau}^\nu G(y',t-\tau)\right| dy'$$
$$\leqslant C \int\limits_0^{3T_1-2T_2} d\tau \int\limits_{T_1}^{T_2} dt \int\limits_0^{+\infty} (t-\tau)^{-2-\nu} |W(-\eta(t-\tau)^{-\nu};-\nu,-1-\nu)|$$
$$\times \int\limits_{R_+^{n-1}} y_j^\alpha \exp\left\{-A\sum_{i=1}^{n-1}\frac{y_i}{\eta}\right\} \eta^{1-n} dy', \tag{6.5}$$

or, after the change of variables (4.52), (4.53),

$$\mathscr{L}_2 \leqslant C \int\limits_{R_+^{n-1}} x_j^\alpha \exp\left\{-A\sum_{i=1}^{n-1} x_i\right\} dy' \int\limits_0^{3T_1-2T_2} d\tau \int\limits_{T_1}^{T_2} dt \int\limits_0^{+\infty} (t-\tau)^{-2+\alpha\nu} z^\alpha |W(-z;-\nu,-1-\nu)|$$
$$\leqslant C \int\limits_0^{3T_1-2T_2} d\tau \int\limits_{T_1}^{T_2} dt (t-\tau)^{-2+\alpha\nu}. \tag{6.6}$$

Here we applied again estimate (4.57) with $\beta := \nu$, $\gamma := -1-\nu$. After some simple calculations, we get from (6.6) that

$$\mathscr{L}_2 \leqslant C[T_2^{\alpha\nu} - T_1^{\alpha\nu} + (1-2^{\alpha\nu})(T_2-T_1)^{\alpha\nu}]. \tag{6.7}$$

Since we consider the case $\frac{T_2-T_1}{T_1} \leqslant \frac{1}{2}$, then

$$T_2^{\alpha\nu} - T_1^{\alpha\nu} = \left(\left[\frac{T_2-T_1}{T_1}+1\right]^{\alpha\nu} - 1\right) T_1^{\alpha\nu} \leqslant C(T_2-T_1)^{\alpha\nu}. \tag{6.8}$$

Hence, inequality (6.7) together with (6.8) lead to estimate (4.66).

### 6.2. Proof of Lemma 4.5

As follows from representation (4.20)–(4.22).

$$\begin{aligned} K_+^*(\xi,0,\eta) &= \left(1+ik\frac{\langle a_3,\xi\rangle}{|\xi|}\right)K^*(\xi,\eta),\\ K_-^*(\xi,0,\eta) &= \left(1-i\frac{\langle a_3,\xi\rangle}{|\xi|}\right)K^*(\xi,\eta), \end{aligned} \tag{6.9}$$

where the function $\frac{\langle a_3,\xi\rangle}{|\xi|}$ does not change essentially properties of the function $K^*(\xi,\eta)$.

Moreover, representations (4.31) and (4.32) together with (6.9) lead to

$$G_+(y',0,\tau) = G(y',\tau) + \int_0^{+\infty} d\eta\,\tau^{-1}W(-\eta\tau^{-\nu};-\nu,0)\int_{R^{n-1}}\frac{ik\langle a_3,\xi\rangle}{|\xi|}K^*(\xi,\eta)e^{2i\pi\langle\xi,y'\rangle}d\xi;$$

$$G_-(y',0,\tau) = G(y',\tau) - \int_0^{+\infty} d\eta\,\tau^{-1}W(-\eta\tau^{-\nu};-\nu,0)\int_{R^{n-1}}K^*(\xi,\eta)\frac{i\langle a_3,\xi\rangle}{|\xi|}e^{2i\pi\langle\xi,y'\rangle}d\xi.$$

Thus, to get statements **(i)** and **(ii)** of Lemma 4.5, it is enough to repeat the corresponding arguments from Lemma 4.1. As for inequality (4.77), we can represent $\frac{\partial^2 G_\pm}{\partial\tau\partial y_i}$ as

$$\begin{aligned} \frac{\partial^2 G_\pm}{\partial\tau\partial y_i}(y',0,\tau) &= \int_0^{+\infty}\frac{\partial}{\partial\tau}(\tau^{-1}W(-\eta\tau^{-\nu};-\nu,0))\frac{\partial K_\pm}{\partial y_i}(y',0,\eta)d\eta\\ &= \int_0^{+\infty}\tau^{-2}W(-\eta\tau^{-\nu};-\nu,-1)\frac{\partial K_\pm}{\partial y_i}(y',0,\eta)d\eta. \end{aligned} \tag{6.10}$$

Here we used again formula (6.2). Based on statement **(i)** of Lemma 4.5 and representation (6.10), one can easily infer that

$$\left|\frac{\partial^2 G_\pm}{\partial\tau\partial y_i}(y',0,\tau)\right| \leqslant C\int\limits_0^{+\infty}\tau^{-2}|W(-\eta\tau^{-\nu};-\nu,-1)|\exp\left\{-A\sum_{i=1}^{n-1}\frac{|y_i|}{\eta}\right\}\eta^{-n}d\eta. \quad (6.11)$$

Then, we repeat the proof of estimate (4.66) and, using (6.11) together with change of variables (4.52) and (4.53), have

$$\begin{aligned}\mathscr{L}_3 &:= \int\limits_0^{3T_1-2T_2}d\tau\int\limits_{T_1}^{T_2}dt\int\limits_{R_+^{n-1}}y_j^\alpha\left|\frac{\partial^2 G_\pm}{\partial(t-\tau)\partial y_i}(y',0,t-\tau)\right|dy' \\ &\leqslant C\int\limits_{R_+^{n-1}}x_j^\alpha\exp\left\{-A\sum_{i=1}^{n-1}x_i\right\}dx'\int\limits_0^{3T_1-2T_2}d\tau\int\limits_{T_1}^{T_2}dt(t-\tau)^{\nu\alpha-2} \\ &\times\int\limits_0^{+\infty}z^{\alpha-1}|W(-z;-\nu,-1)|dz. \end{aligned} \quad (6.12)$$

Applying inequality (4.57) with $\beta:=\nu,\ \gamma:=-1$ to the last integral in (6.12), we deduce after some simple calculations:

$$\mathscr{L}_3 \leqslant C\int\limits_0^{3T_1-2T_2}d\tau\int\limits_{T_1}^{T_2}dt(t-\tau)^{\nu\alpha-2}. \quad (6.13)$$

Note that estimate (6.13) is the same as (6.6). Hence, repeating the end of arguments from the proof of Proposition 4.2, we can get estimate (4.77).

*Acknowledgements.* This work is partially supported by Grant FP7-People-2011-IRSES project number 295164 (EUMLS: EU-Ukrainian Mathematicians for Life Sciences).

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(Received November 14, 2014)

*Nataliya Vasylyeva*
*Institute of Applied Mathematics and Mechanics*
*of NAS of Ukraine*
*R. Luxemburg, str. 74*
*83114 Donetsk, Ukraine*
*e-mail:* vasylyeva@iamm.ac.donetsk.ua
nataliy_v@yahoo.com