Fractional
Differential
Calculus
Volume 4, Number 2 (2014), 125–136
doi:10.7153/fdc-04-07

EXISTENCE OF AN UNBOUNDED SOLUTION FOR MULTI–POINT BOUNDARY VALUE PROBLEMS OF FRACTIONAL DIFFERENTIAL EQUATIONS ON AN INFINITE DOMAIN

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Abstract. In this paper, considering the fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t)+a(t)f(t,u(t),u'(t))=0; \ t\in(0,\infty), \ \alpha\in(2,3), \\ u(0)=u'(0)=0, \quad \lim_{t\to\infty} D_{0+}^{\alpha-1}u(t)=\sum_{i=1}^{m}\beta_i D_{0+}^{\alpha-1}u(t)\Big|_{t=\xi_i}, \\ 0<\xi_1<\xi_2<\ldots<\xi_m<\infty, \ \beta_i\in\mathbb{R}, \end{cases}$$

where D_{0+}^{α} represents *Riemann-Liouville* fractional derivative of order α and using famous *Leray-Schauder Nonlinear Alternative* theorem, we will obtain an unbounded solution of above BVP. At the end some examples illustrate.

1. Introduction

Fractional differential equations is a full applicable theory in almost whole sciences such as basic sciences, engineering, social sciences, medicine, economics, dynamical processes and so on [see more details in monographes [1], [2], [3]]. Every interested researcher can find a large number of attractive investigations in various fields of fractional calculus and related applications such as solvability and existence of multiplicity of positive solutions for a given boundary value problems of fractional differential equations such as [4], [5], [6], [7], [8], [9], and references therein.

Kazem Ghanbari and *Yousef Gholami* in [6] used some standard fixed point theorems in order to represent the existence of triple positive solutions of the following fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t)+\lambda a(t)f(t,u(t))=0, \quad t\in(0,\infty), \ \alpha\in(2,3) \\ u(0)+u'(0)=0, \quad \lim_{t\to\infty} D_{0+}^{\alpha-1}u(t)=\sum_{i=1}^{m-2}\beta_i u'(\xi_i), \\ 0<\xi_1<\xi_2<\ldots<\xi_{m-2}<\infty, \ \beta_i\in\mathbb{R}^+\cup 0, \ i=1,2,\ldots,m-2 \end{cases}$$

where D_{0+}^{α} represent the fractional *Riemann-Liouville* derivative of order α.

Mathematics subject classification (2010): Primary: 34A08, 34B18; Secondary: 34B10, 34B40.
Keywords and phrases: Multi-point boundary value problem, Riemann-Liouville fractional derivative, unbounded solution, infinite interval, fixed point theorem.

Paper FDC-04-07

In this paper we consider the following BVP:

$$\begin{cases} D_{0^+}^{\alpha}u(t)+a(t)f(t,u(t),u'(t))=0; \ t\in(0,\infty), \ \ \alpha\in(2,3), \\ u(0)=u'(0)=0, \quad \lim\limits_{t\to\infty} D_{0^+}^{\alpha-1}u(t)=\sum\limits_{i=1}^{m}\beta_i D_{0^+}^{\alpha-1}u(t)\Big|_{t=\xi_i}, \\ 0<\xi_1<\xi_2<\ldots<\xi_m<\infty, \ \ \beta_i\in\mathbb{R}. \end{cases} \tag{1.1}$$

By means of *Leray-Schauder Nonlinear Alternative* theorem, we show that the boundary value problem (1.1) has an unbounded solution.

Firstly assume that the following conditions are satisfy:

(C_1) $\sum\limits_{i=1}^{m}\beta_i<1.$

(C_2) $a\in C([0,\infty),[0,\infty))$ and there exist $\rho\in(0,\infty)$ such that

$$0<\int_{\rho}^{\infty}a(s)ds<\infty.$$

(C_3) $f\in C\left([0,\infty)\times\mathbb{R}^2,\mathbb{R}\right)$ and $f(t,0,0)$ dos not vanish identically zero on $[0,\infty)$.

(C_4) $F(t,u,u')=f(t,\exp(t^{\alpha-1})u,\exp(t^{\alpha-2})u')$ and

$$\lim_{\substack{|u|\to\infty \\ |u'|\to\infty}}\frac{|F(t,u,u')|}{\psi_1(|u|)\psi_2(|u'|)}=\phi(t),$$

where $\phi\in L^1[0,\infty)$ and $\psi_1\in C([0,\infty),(0,\infty))$ is nondecreasing on $[0,\infty)$ also there exist $M\in\mathbb{R}^+$ such that $0<|\psi_2(|u'|)|\leqslant M$ on $[0,\infty)$.

2. Technical background

In this section we introduce some standard definitions and lemmas that will be needed to prove the main result in the next section.

DEFINITION 2.1. Assume that $u\in L^1(0,\infty)$. The fractional *Riemann-Liouville* integral of order α for u is defined by

$$I_{0^+}^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}u(s)ds, \quad \alpha>0.$$

DEFINITION 2.2. The fractional Riemann-Liouville derivative of order α for a given real valued function u on $(0,\infty)$ is defined by

$$D_{0^+}^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t(t-s)^{n-\alpha-1}u(s)ds, \quad \alpha>0, \ \ n=[\alpha]+1,$$

provided that the right hand side is point-wise defined on $(0,\infty)$.

LEMMA 2.3. *If* $u \in C(0,\infty)$, $D^{\alpha}_{0^+}u(t) \in L^1(0,\infty)$, $\alpha > 0$, *then*

$$I^{\alpha}_{0^+}D^{\alpha}_{0^+}u(t) = u(t) + \sum_{i=1}^{n} c_i t^i, \quad c_i \in \mathbb{R}, \ n = [\alpha]+1.$$

LEMMA 2.4. *Assume that* $y \in C(0,\infty)$, $0 < \int_0^\infty y(s)ds < \infty$. *Then the boundary value problem*

$$D^{\alpha}_{0^+}u(t) + y(t) = 0, \quad t \in (0,\infty), \ \alpha \in (2,3), \tag{2.1}$$

$$u(0) = u'(0) = 0, \quad \lim_{t\to\infty} D^{\alpha-1}_{0^+}u(t) = \sum_{i=1}^{m} \beta_i D^{\alpha-1}_{0^+}u(t)\Big|_{t=\xi_i}, \tag{2.2}$$

has the unique solution as

$$u(t) = \int_0^\infty G(t,s)y(s)ds, \tag{2.3}$$

such that

$$G(t,s) = G_1(t,s) + G_2(t,s), \tag{2.4}$$

with

$$G_1(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1}; & 0 \leqslant s \leqslant t < \infty \\ t^{\alpha-1}; & 0 \leqslant t \leqslant s < \infty \end{cases} \tag{2.5}$$

and

$$G_2(t,s) = \frac{\sum_{i=1}^{m} \beta_i t^{\alpha-1}}{\Gamma(\alpha)(1-\sum_{i=1}^{m}\beta_i)} \begin{cases} 0; & 0 \leqslant s \leqslant \xi_i < \infty \\ 1; & 0 \leqslant \xi_i \leqslant s < \infty \end{cases} \tag{2.6}$$

Proof. By means of Lemma 2.3 and considering (2.1), we have

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds.$$

Implementing boundary conditions $u(0) = 0$, $u'(0) = 0$ we conclude that $c_3 = 0$, $c_2 = 0$ respectively. Now applying third boundary condition

$$\lim_{t\to\infty} D^{\alpha-1}_{0^+}u(t) = \sum_{i=1}^{m} \beta_i D^{\alpha-1}_{0^+}u(t)\Big|_{t=\xi_i},$$

we deduce that

$$c_1 = \frac{\int_0^\infty y(s)ds - \sum_{i=1}^{m}\beta_i \int_0^{\xi_i} y(s)ds}{\Gamma(\alpha)(1-\sum_{i=1}^{m}\beta_i)}.$$

Thus

$$
\begin{aligned}
u(t) &= c_1 t^{\alpha-1} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds \\
&= t^{\alpha-1} \frac{\int_0^\infty y(s)ds - \sum_{i=1}^m \beta_i \int_0^{\xi_i} y(s)ds}{\Gamma(\alpha)(1-\sum_{i=1}^m \beta_i)} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds \\
&= \int_0^\infty G_1(t,s)y(s)ds - \int_0^\infty \frac{t^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + t^{\alpha-1} \frac{\int_0^\infty y(s)ds - \sum_{i=1}^m \beta_i \int_0^{\xi_i} y(s)ds}{\Gamma(\alpha)(1-\sum_{i=1}^m \beta_i)} \\
&= \int_0^\infty G_1(t,s)y(s)ds + \frac{\sum_{i=1}^m \beta_i t^{\alpha-1} \int_0^\infty y(s)ds}{\Gamma(\alpha)(1-\sum_{i=1}^m \beta_i)} - \frac{\sum_{i=1}^m \beta_i t^{\alpha-1} \int_0^{\xi_i} y(s)ds}{\Gamma(\alpha)(1-\sum_{i=1}^m \beta_i)} \\
&= \int_0^\infty G_1(t,s)y(s)ds + \int_0^\infty G_2(t,s)y(s)ds = \int_0^\infty G(t,s)y(s)ds,
\end{aligned}
$$

where $G(t,s)$ that is called *Green's function* corresponds to (2.1) and defined by (2.4)–(2.6). Uniqueness of coefficients c_1, c_2, c_3, shows that (2.3) is the unique solution of boundary value problem (2.1), (2.2). The proof is complete. □

REMARK 2.5. The *Green function* of (2.1), (2.2) has the following properties:

(H_1) $G(t,s) \geqslant 0$ for $t,s \in (0,\infty)$.

(H_2) $\exp(-t^{\alpha-1})G(t,s) \leqslant L_0, \quad L_0 = \frac{1}{\Gamma(\alpha)} \frac{1}{1-\sum_{i=1}^m \beta_i}$.

(H_3) $\exp(-t^{\alpha-2}) \frac{\partial G(t,s)}{\partial t} \leqslant L_1, \quad L_1 = \frac{1}{\Gamma(\alpha-1)} \frac{1}{1-\sum_{i=1}^m \beta_i}$.

REMARK 2.6. Considering the following space

$$
X = \left\{ u \in C([0,\infty),\mathbb{R}) \;\middle|\; \lim_{t\to\infty} \exp(-t^{\alpha-1})u(t) < \infty \right\},
$$

such that equipped with the norm

$$
||u||_X = ||u||_\infty = \sup_{t\in[0,\infty)} \exp(-t^{\alpha-1})|u(t)|,
$$

define the space

$$Y=\left\{u(t)\in X\;\middle|\;u'(t)\in C([0,\infty),\mathbb{R}),\lim_{t\to\infty}\exp(-t^{\alpha-2})u'(t)<\infty\right\},$$

that endowed with the norm

$$||u||_Y=||u||_\infty+||u'||_\infty=\sup_{t\in[0,\infty)}\exp(-t^{\alpha-1})|u(t)|+\sup_{t\in[0,\infty)}\exp(-t^{\alpha-2})|u'(t)|,$$

and applying some standard arguments about properties of a given Banach space, we can show that X,Y are Banach spaces.

Basically in this paper, we use the Banach space Y defined above.

Define the operator $T:Y\to Y$ as follows

$$(Tu)(t)=\int_0^\infty G(t,s)a(s)f(s,u(s),u'(s))ds,\quad u\in Y. \tag{2.7}$$

Obviously fractional boundary value problem (1.1) has a solution u if and only if u solve the operator equation $u=Tu$.

THEOREM 2.7. [5] *Let C be a convex subset of a Banach space, U be a open subset of C with $0\in U$. Then every completely continuous map $T:\overline{U}\to C$ has at least one of the two following properties:*

(E_1) *There exist an $u\in\overline{U}$ such that $Tu=u$.*

(E_2) *There exist an $v\in\partial U$ and $\lambda\in(0,1)$ such that $v=\lambda Tv$.*

As a result of noncompactness of half line $[0,\infty)$, the *Arzela-Ascoli* theorem fails to work in space Y. Thus in order to show the compactness of the operator T defined by (2.7), we need to represent the following modified compactness criterion.

LEMMA 2.8. [9] *Assume that Z is a bounded subset of Y. Then Z is relatively compact in Y, provided that the following conditions hold:*

(i) *For $u(t)\in Z$, $\exp(-t^{\alpha-1})u(t)$ and $\exp(-t^{\alpha-2})u'(t)$ are equicontinuous on any compact subinterval of $[0,\infty)$.*

(ii) *For given $\varepsilon>0$, there exist $\nu=\nu(\varepsilon)$ such that for every $t_1,t_2\geqslant\nu$*

$$\exp(-t^{\alpha-1})|[u(t_2)-u(t_1)]|<\varepsilon,\ \exp(-t^{\alpha-2})|[u'(t_2)-u'(t_1)]|<\varepsilon,\ u(t)\in Z.$$

(ii) is called Equiconvegence at infinity for Z.

LEMMA 2.9. *If conditions $(C_1)-(C_4)$ hold, then operator $T:Y\to Y$ is completely continuous.*

Proof. In order to represent the proof, we divide it into the three steps as follows:

(**i**) In this step show that integral operator $T: Y \to Y$ is continuous. Assume that $\{u_n\}$ be a sequence in Y such that $u_n \to u$ and $u_n' \to u'$ as $n \to \infty$. Hence there exist positive constant δ such that

$$\max\{||u||_Y, \sup_{n\in\mathbb{N}} ||u_n||_Y\},\ \max\{||u'||_Y, \sup_{n\in\mathbb{N}} ||u_n'||_Y\} < \delta.$$

Then using *Lebesgue* dominated convergence theorem, we conclude that

$$\int_0^\infty f(s,u_n(s),u_n'(s))ds \to \int_0^\infty f(s,u(s),u'(s))ds, \quad n\to\infty.$$

Therefore considering Remark 2.5 and Remark 2.6, we can get

$$\begin{aligned}||Tu_n - Tu||_Y &= ||Tu_n - Tu||_\infty + ||(Tu_n)' - (Tu)'||_\infty \\ &\leqslant L_0 \int_0^\infty a(s)|f(s,u_n(s),u_n'(s)) - f(s,u(s),u'(s))|ds \\ &\quad + L_1 \int_0^\infty a(s)|f(s,u_n(s),u_n'(s)) - f(s,u(s),u'(s))|ds \\ &\to 0, \quad n\to\infty.\end{aligned}$$

So T is continuous.

(**ii**) Now in order to prove the relatively compactness of operator $T: Y \to Y$, assume that Ω is a bounded subset of Y. Thus there exist $\rho > 0$ such that $||u||_Y \leqslant \rho$ for $u \in \Omega$. Then using conditions $(C_2),(C_4)$ and Remark 2.5, Remark 2.6 we have

$$\begin{aligned}||Tu||_\infty &= \sup_{t\in[0,\infty)} \left| \int_0^\infty G(t,s)a(s)f(s,u(s),u'(s))ds \right| \\ &\leqslant L_0 \int_0^\infty a(s) \left| f\left(s, \exp(s^{\alpha-1})\frac{u(s)}{\exp(s^{\alpha-1})}, \exp(s^{\alpha-2})\frac{u'(s)}{\exp(s^{\alpha-2})} \right)\right| ds \\ &= L_0 \int_0^\infty a(s) \left| F\left(s, \frac{u(s)}{\exp(s^{\alpha-1})}, \frac{u'(s)}{\exp(s^{\alpha-2})} \right)\right| ds \\ &\leqslant L_0 \int_0^\infty a(s)\phi(s)\psi_1\left(\frac{|u|}{\exp(s^{\alpha-1})}\right)\psi_2\left(\frac{|u'|}{\exp(s^{\alpha-2})}\right) ds \\ &\leqslant L_0\psi_1(\rho)M\int_0^\infty a(s)\phi(s)ds < \infty, \quad u\in\Omega.\end{aligned}$$

Similarly we can show that $||(Tu)'||_\infty < \infty$ for $u \in \Omega$. It show that $T\Omega$ is uniformly bounded.

In order to use Lemma 2.8, we should prove that $T\Omega$ is equicontinuous on any compact subinterval of $[0,\infty)$. For $J > 0$, $t_1,t_2 \in [0,J]$ and for $u \in \Omega$, without lose generality, let $t_1 < t_2$. Indeed:

$$
\begin{aligned}
&|\exp(-t_2^{\alpha-1})(Tu)(t_2) - \exp(-t_1^{\alpha-1})(Tu)(t_1)| \\
&\leqslant \int_0^\infty a(s)|[\exp(-t_2^{\alpha-1})G(t_2,s) - \exp(-t_1^{\alpha-1})G(t_1,s)]f(s,u(s),u'(s))|ds \\
&\leqslant \int_0^\infty a(s)|[\exp(-t_2^{\alpha-1})G_1(t_2,s) - \exp(-t_1^{\alpha-1})G_1(t_1,s)]f(s,u(s),u'(s))|ds \\
&\quad + \frac{\sum_{i=1}^m \beta_i}{\Gamma(\alpha)(1-\sum_{i=1}^m \beta_i)} |\exp(-t_2^{\alpha-1})t_2^{\alpha-1} - \exp(-t_1^{\alpha-1})t_1^{\alpha-1}| \\
&\quad \times \int_{\xi_i}^\infty a(s)|f(s,u(s),u'(s))|ds \\
&\leqslant L_0\psi_1(\rho)M \int_0^\infty |[\exp(-t_2^{\alpha-1})G_1(t_2,s) - \exp(-t_1^{\alpha-1})G_1(t_1,s)]a(s)\phi(s)ds \\
&\quad + \frac{L_0\psi_1(\rho)M\sum_{i=1}^m \beta_i}{\Gamma(\alpha)(1-\sum_{i=1}^m \beta_i)} |\exp(-t_2^{\alpha-1})t_2^{\alpha-1} - \exp(-t_1^{\alpha-1})t_1^{\alpha-1}| \int_{\xi_i}^\infty a(s)\phi(s)ds.
\end{aligned}
$$

So we conclude that

$$
\begin{aligned}
&|\exp(-t_2^{\alpha-1})(Tu)(t_2) - \exp(-t_1^{\alpha-1})(Tu)(t_1)| \\
&\leqslant L_0\psi_1(\rho)M \int_0^\infty \exp(-t^{\alpha-1})|G_1(t_2,s) - G_1(t_1,s)|a(s)\phi(s)ds \\
&\quad + \frac{L_0\psi_1(\rho)M\sum_{i=1}^m \beta_i}{\Gamma(\alpha)(1-\sum_{i=1}^m \beta_i)} \exp(-t_1^{\alpha-1})|t_2^{\alpha-1} - t_1^{\alpha-1}| \int_{\xi^i}^\infty a(s)\phi(s)ds \\
&\quad \longrightarrow 0, \quad \text{as uniformly } t_1 \to t_2 \text{ for } u \in \Omega.
\end{aligned}
$$

Similarly we can prove that

$$
|\exp(-t_2^{\alpha-2})(Tu)'(t_2) - \exp(-t_1^{\alpha-2})(Tu)'(t_1)| \to 0,
$$

when uniformly $t_1 \to t_2$. Therefore $T\Omega$ is locally equicontinuous on $[0,\infty)$.

(**iii**) At last we must prove that $T\Omega$ is equiconvergent at infinity.
For $u \in \Omega$,we know that

$$
\int_0^\infty a(s)|f(s,u(s),u'(s))|ds \leqslant \psi_1(\rho)M \int_0^\infty a(s)\phi(s)ds < \infty.
$$

on the other hand considering (2.5), obviously we have

$$
\lim_{t\to\infty} \int_0^\infty \frac{G_1(t,s)}{\exp(t^{\alpha-1})} a(s)f(s,u(s),u'(s))ds = 0.
$$

Hence

$$\begin{aligned}
&\lim_{t\to\infty}\exp(-t^{\alpha-1})|(Tu)(t)|\\
&\quad=\lim_{t\to\infty}\int_0^{\infty}\frac{G_1(t,s)}{\exp(t^{\alpha-1})}a(s)f(s,u(s),u'(s))ds\\
&\qquad+\lim_{t\to\infty}\exp(-t^{\alpha-1})\frac{\sum_{i=1}^{m}\beta_i t^{\alpha-1}}{\Gamma(\alpha)(1-\sum_{i=1}^{m}\beta_i)}\int_{\xi_i}^{\infty}a(s)f(s,u(s),u'(s))ds\\
&\quad\leqslant\frac{L_0\psi_1(\rho)M\sum_{i=1}^{m}\beta_i}{\Gamma(\alpha)(1-\sum_{i=1}^{m}\beta_i)}\lim_{t\to\infty}\frac{t^{\alpha-1}}{\exp(t_1^{\alpha-1})}\int_{\xi^i}^{\infty}a(s)\phi(s)ds<\infty.
\end{aligned}$$

Similarly we can obtain the following

$$\lim_{t\to\infty}\exp(-t^{\alpha-2})|(Tu)'(t)|<\infty.$$

Therefore $T\Omega$ is equiconvrgent at infinity. Finally by means of compactness criterion in Lemma 2.8, we deduce that integral operator $T:Y\to Y$ is completely continuous. □

3. Main result

THEOREM 3.1. *Let that conditions $(C_1)-(C_4)$ hold and the following condition is satisfied:*

there exist positive constant μ such that

$$\left(\frac{1}{L_0}+\frac{1}{L_1}\right)>\frac{2\psi_1(\mu)M}{\mu}\int_0^{\infty}a(s)\phi(s)ds. \tag{3.1}$$

Then the fractional boundary value problem (1.1) has an unbounded solution $u=u(t)$ such that

$$0\leqslant\exp(-t^{\alpha-1})u(t)+\exp(-t^{\alpha-2})u'(t)\leqslant\mu,\quad t\in[0,\infty).$$

Proof. Let us consider the following fractional boundary value problem

$$\begin{cases}D_{0^+}^{\alpha}u(t)+\lambda a(t)f(t,u(t),u'(t));\ \ t\in(0,\infty),\ \ \alpha\in(2,3),\lambda\in(0,1),\\ u(0)=u'(0)=0,\quad \lim_{t\to\infty}D_{0^+}^{\alpha-1}u(t)=\sum_{i=1}^{m}\beta_iD_{0^+}^{\alpha-1}u(t)\Big|_{t=\xi_i},\\ 0<\xi_1<\xi_2<\ldots<\xi_m<\infty,\ \ \beta_i\in\mathbb{R}.\end{cases} \tag{3.2}$$

According to end part of Remark 2.6, we know that solving (3.2) is equivalent to solving the fixed point problem $u=\lambda Tu$.

Assume that

$$K = \{u \in Y |\ ||u||_Y \leqslant \mu\}.$$

We claim that there is no $u \in \partial K$ such that $u = \lambda T u$ for $\lambda \in (0,1)$.

The proof is immediate, because if there exist $u \in \partial K$ with $u = \lambda T u$, then for $\lambda \in (0,1)$ we have

$$\begin{aligned}
||u||_\infty &= ||\lambda (Tu)(t)||_\infty = \sup_{t\in[0,\infty)} \lambda \exp(-t^{\alpha-1})|(Tu)(t)| \\
&\leqslant \sup_{t\in[0,\infty)} \int_0^\infty \frac{G(t,s)}{\exp(t^{\alpha-1})} a(s)|f(s,u(s),u'(s))|ds \\
&\leqslant L_0 \int_0^\infty a(s) \left| f\left(s, \exp(s^{\alpha-1}) \frac{u(s)}{\exp(s^{\alpha-1})}, \exp(s^{\alpha-2}) \frac{u'(s)}{\exp(s^{\alpha-2})} \right) \right| ds \\
&= L_0 \int_0^\infty a(s) \left| F\left(s, \frac{u(s)}{\exp(s^{\alpha-1})}, \frac{u'(s)}{\exp(s^{\alpha-2})} \right) \right| ds \\
&\leqslant L_0 \psi_1(\mu) M \int_0^\infty a(s)\phi(s)ds.
\end{aligned}$$

Thus

$$\mu \leqslant L_0 \psi_1(\mu) M \int_0^\infty a(s)\phi(s)ds.$$

Therefore it is clear that

$$\frac{1}{L_0} \leqslant \frac{\psi_1(\mu) M \int_0^\infty a(s)\phi(s)ds}{\mu}. \tag{3.3}$$

Similarly we can show that

$$\frac{1}{L_1} \leqslant \frac{\psi_1(\mu) M \int_0^\infty a(s)\phi(s)ds}{\mu}. \tag{3.4}$$

Gathering (3.3) and (3.4), we conclude that

$$\left(\frac{1}{L_0} + \frac{1}{L_1} \right) \leqslant \frac{2\psi_1(\mu) M \int_0^\infty a(s)\phi(s)ds}{\mu},$$

which is contradiction with (3.1). Then by means of Remark 2.5, Theorem 2.7, the fractional boundary value problem (1.1) has an unbounded solution $u = u(t)$ such that

$$0 \leqslant \exp(-t^{\alpha-1})u(t) + \exp(-t^{\alpha-2})u'(t) \leqslant \mu, \quad t \in [0,\infty).$$

This completes the proof. □

4. Applications

EXAMPLE 4.1. Considering the fractional boundary value problem

$$\begin{cases} D_{0^+}^{\frac{5}{2}}u(t)+a(t)f(t,u(t),u'(t))=0, \;\; t\in[0,\infty), \\ u(0)=u'(0)=0, \quad \lim\limits_{t\in[0,\infty)} D_{0^+}^{\frac{3}{2}}u(t)=\frac{1}{4}D_{0^+}^{\frac{3}{2}}u\left(\frac{1}{4}\right)+\frac{1}{2}D_{0^+}^{\frac{3}{2}}u\left(\frac{1}{2}\right), \end{cases} \tag{4.1}$$

where

$$a(t)=\exp\left(-\frac{t}{2}\right), \quad f(t,u,u')=\frac{1}{\exp(t^2)}\frac{\exp(t^{\frac{3}{2}})+\exp(-t^{\frac{3}{2}})u^2}{\exp(t^{\frac{1}{2}})+\exp(-t^{\frac{1}{2}})(u')^2},$$

observing conditions $(C_1)-(C_4)$, it is clear that the conditions $(C_1)-(C_3)$ hold. Also by means of condition (C_4) we find that

$$F(t,\exp(t^{\frac{3}{2}})u,\exp(t^{\frac{1}{2}})u')=\exp(-t)\frac{1+u^2}{1+(u')^2}.$$

In this case, we conclude that

$$\lim_{\substack{|u|\to\infty \\ |u'|\to\infty}} \frac{|F(t,u,u')|}{\psi_1(|u|)\psi_2(|u'|)}=\phi(t),$$

where $\phi(t)=\exp(-t)\in L^1[0,\infty)$ and $\psi_1(u)=1+u^2$ is nondecreasing and continuous on $[0,\infty)$, also $\psi_2(u')=\frac{1}{1+(u')^2}\in[0,1)$ on $[0,\infty)$. Hence condition (C_4) holds.

At last according to (3.1) and choosing $\mu>\frac{20}{3[\Gamma(\frac{3}{2})+\Gamma(\frac{5}{2})]}$, by means of Theorem 3.1 we conclude that the fractional boundary value problem (4.1) has at least one positive solution $u=u(t)$ such that

$$0\leqslant \exp(-t^{\frac{3}{2}})u(t)+\exp(-t^{\frac{1}{2}})u'(t)\leqslant\mu, \quad t\in[0,\infty).$$

EXAMPLE 4.2. Let us consider the following three-point fractional boundary value problem on positive half line

$$\begin{cases} D_{0^+}^{\frac{7}{3}}u(t)+a(t)f(t,u(t),u'(t))=0, \;\; t\in[0,\infty), \\ u(0)=u'(0)=0, \quad \lim\limits_{t\in[0,\infty)} D_{0^+}^{\frac{4}{3}}u(t)=\sum_{i=1}^{3}\beta_i D_{0^+}^{\frac{4}{3}}u(\xi_i), \end{cases} \tag{4.2}$$

where

$$a(t)=\exp(-t), \;\; \beta_1=\frac{1}{8}, \;\; \beta_2=\frac{1}{4}, \;\; \beta_3=\frac{1}{2}, \;\; \xi_1=\frac{1}{27}, \;\; \xi_2=\frac{1}{9}, \;\; \xi_3=\frac{1}{3}, \tag{4.3}$$

with

$$f(t,u,u') = \begin{cases} \exp\left(-t+\exp\left(\dfrac{u}{\exp(t^{\frac{4}{3}})}\right)\right) \cdot \dfrac{\mathrm{tg}^{-1}\left(\frac{u'}{\exp(t^{\frac{1}{3}})}\right)}{1000}, & 0 < u \leqslant 1 \\ \exp\left(-t+\exp\left(\dfrac{u}{\exp(t^{\frac{4}{3}})}\right)\right) \left[\dfrac{\mathrm{tg}^{-1}\left(\frac{u'}{\exp(t^{\frac{1}{3}})}\right)}{1000} + 10^3(u-1)\right], & 1 \leqslant u \leqslant 10^3 \\ \exp\left(-t+\exp\left(\dfrac{u}{\exp(t^{\frac{4}{3}})}\right)\right) \left[\dfrac{\mathrm{tg}^{-1}\left(\frac{u'}{\exp(t^{\frac{1}{3}})}\right)}{1000} + 9.99\times 10^5\right], & 10^3 \leqslant u. \end{cases}$$

(4.3) ensures that the conditions $(C_1), (C_2)$ hold. Also construction of function $f(t,u,u')$ shows that f is continuous on $[0,\infty)$. Thus condition (C_3) hold.

Now we have

$$F(t,u,u') = f\left(t, \exp(t^{\frac{4}{3}})u, \exp(t^{\frac{1}{3}})u'\right),$$
$$|F(t,u,u')| \leqslant \phi(t)\psi_1(|u|)\psi_2(|u'|),$$

where $\phi(t) = \exp(-t)$ and $\psi_1(t) = \exp(\exp(u(t)))$ and

$$\psi_2(u') = \begin{cases} \dfrac{\mathrm{tg}^{-1} \frac{u'}{\exp(t^{\frac{1}{3}})}}{1000}, & 0 < u \leqslant 1 \\ \dfrac{\mathrm{tg}^{-1} \frac{u'}{\exp(t^{\frac{1}{3}})}}{1000} + 10^3(u-1), & 1 \leqslant u \leqslant 10^3 \\ \dfrac{\mathrm{tg}^{-1} \frac{u'}{\exp(t^{\frac{1}{3}})}}{1000} + 9.99\times 10^5, & 10^3 \leqslant u. \end{cases}$$

Hence $\phi_2(u') \in (0, \frac{\pi}{2000} + 10^6]$. Therefore condition (C_4) holds.

Now using (3.1) and choosing $\mu > \frac{9(\frac{\pi}{2000}+10^6)}{2[\Gamma(\frac{4}{3})+\Gamma(\frac{7}{3})]}$ and applying Theorem 3.1, we deduce that the fractional boundary value problem (4.2) has an unbounded solution u such that

$$0 \leqslant \exp(-t^{\frac{7}{3}})u(t) + \exp(-t^{\frac{4}{3}})u'(t) \leqslant \mu, \quad t \in [0,\infty).$$

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(Received April 28, 2013)

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