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EXISTENCE AND UNIQUENESS OF MILD SOLUTION FOR NONLOCAL IMPULSIVE INTEGRO–DIFFERENTIAL EQUATION WITH STATE DEPENDENT DELAY

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Abstract. In this article, we first establish the mild solution for an impulsive fractional integro-differential equation with state dependent delay. Then we prove the existence and uniqueness of solution by applying well known classical fixed point theorems. The obtained results are illustrated with an example.

1. Introduction

We consider the following semi-linear nonlocal impulsive fractional integro-differential equation with state dependent delay

$$^{C}D_t^{\alpha}x(t) = Ax(t) + f(t, x_{\rho(t,x_t)}) + \int_0^t q(t-s)g(s,x_s)ds, \ \ t \in J, \tag{1.1}$$

$$\Delta x(t_k) = I_k(x(t_k^-)), \ \ t \neq t_k, \quad k = 1,2,\dots,m, \tag{1.2}$$

$$x(t) + (h(x_{t_1},\cdots,x_{t_p}))(t) = \phi(t), \ \ t \in (-\infty, 0], \tag{1.3}$$

where $J = (0,T]$, $^{C}D_t^{\alpha}$, $\alpha \in (0,1)$ is the Caputo's fractional derivative, $0 < t_1 < \cdots < t_p \leqslant T$, $A : D(A) \subset X \to X$ is the sectorial operator defined on a complex Banach space X. The functions $f, g : J \times \mathbf{B}_h \to X$, $h : \mathbf{B}_h{}^p \to X$, $q : J \to X$, $\rho : J \times \mathbf{B}_h \to (-\infty, T]$ and $\phi \in \mathbf{B}_h$ are given and satisfies some assumptions, where $\mathbf{B}_h$ is a phase space introduced in section 2. The history function $x_t : (-\infty, 0] \to X$ is defined by $x_t(\theta) = x(t+\theta)$, $\theta \in (-\infty, 0]$ belongs to abstract phase space $\mathbf{B}_h$. Here $0 \leqslant t_0 < t_1 < \ldots < t_m < t_{m+1} \leqslant T$, the functions $I_k \in C(X,X)$, $k = 1,2,\dots,m$, are bounded and the notation $\triangle x(t_k) = x(t_k^+) - x(t_k^-)$ where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left- hand limits of $x(t)$ at $t = t_k$ respectively, also we take $x(t_i^-) = x(t_i)$.

Fractional differentiation and integration are the generalization of the ordinary differentiation and integration to an arbitrary non-integral order. The study of fractional order differential equations have become a very important field of research at present due to its numerous applications in engineering, physics and economics. Further, fractional differential equations with impulsive effect come up due to the sudden, discontinues

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phenomena such equations are take place in biology, physics, electrical engineering, etc. For more details, we refer the reader to [10, 11, 12, 15, 21, 22, 25, 26].

Functional differential equations with state dependent delay often seen in most important areas such as classical electrodynamics, population dynamics, modeling of price fluctuations and blood cells production, etc. A significant focus study of such type models one can see [1, 2, 6, 8, 9, 13, 14, 23, 19, 24].

Many of the physical systems can better be described by using the nonlocal conditions to describe, for instance, the diffusion phenomenon of a small amount of gas in a transparent tube can give better results than using the usual initial condition $x(0) = x_0$, for details, we refer [4, 27].

Benchohra et al. [8] discussed the existence and uniqueness of solution to the following impulsive Caputo's fractional differential equation

$$\begin{aligned} {}^C D_t^{\alpha} y(t) &= f(t, y_{\rho(t,y_t)}), \ \ t \in [0,b], \ \ t \neq t_k, \ \ k = 1,2,\ldots,m, \\ \Delta y(t_k) &= I_k(y(t_k^-)), \quad k = 1,2,\ldots,m, \\ y(t) &= \phi(t), \ \ t \in (-\infty, 0], \end{aligned}$$

by using classical fixed point theorems author's established their results.

Chauhan et al. [12] established the existence uniqueness and continuous dependence of mild solution to a class of impulsive fractional functional differential equations with infinite delay

$$\begin{aligned} {}^C D_t^{\alpha}[x(t) + g(t,x_t)] &= A[x(t) + g(t,x_t)] + J_t^{1-\alpha} f(t, x_t, Bx(t)), \ \ t \in [0,T], \\ \Delta x(t_k) &= I_k(x(t_k^-)), \ \ t \neq t_k, \ \ k = 1,2,\ldots,m, \\ x(t) &= \phi(t), \ \ t \in [-\infty, 0], \end{aligned}$$

where ${}^C D_t^{\alpha}$, $\alpha \in (0,1)$, denotes Caputo's fractional derivative and operator $A : D(A) \subset X \to X$ is the infinitesimal generator of solution operator $S_\alpha(t)_{t \geqslant 0}$ on a complex Banach space X. The author's obtained the results by using the fixed point technique. In [10], author's established the existence and uniqueness of solution for fractional differential equation with an integral boundary and impulsive conditions.

Our present work is motivated by the results in [8, 10, 12]. We adopt the idea which is used in [12] to solve the system (1.1)–(1.3) and the concept of mild solution is modified from the papers [3, 11, 21, 22, 25]. For detail to the concept of modification one can see the cited papers [12, 15, 18].

This paper is divided into four sections. In Section 2. we include the setting of function spaces, some basic definitions and preliminaries results. Third section includes the state and proof of our main theorems. In the last section, an example is given to verify our results.

2. Preliminaries

Let $(X,\|.\|_X)$ be a complex Banach space and $L(X)$ represents the Banach space of all bounded linear operators from X into X and the corresponding norm is denoted by $\|.\|_{L(X)}$. The notation $C(J,X)$ stand for Banach space of all continuous functions from J into X with the norm

$$\|x\|_{C(J,X)} = \sup_{t\in J}\{\|x(t)\|_X : x \in C(J,X)\}.$$

The abstract phase space $\mathbf{B}_h$ (see for instance in [17]) defined as follows: Assume that $h:(-\infty,0]\to(0,\infty)$ is a continuous function with $l=\int_{-\infty}^{0}h(t)dt<\infty$. For any $a>0$, we define $\mathbf{B}=\{\psi:[-a,0]\to X$ such that $\psi(t)$ is bounded and measurable$\}$, equipped with the norm

$$\|\psi\|_{[-a,0]} = \sup_{s\in[-a,0]}|\psi(s)|,\ \forall\ \psi\in\mathbf{B}.$$

Let us define $\mathbf{B}_h=\{\psi:(-\infty,0]\to X$, such that for any $a\geqslant c>0$, $\psi\mid_{[-c,0]}\in\mathbf{B}$ with $\psi(0)=0$ and $\int_{-\infty}^{0}h(s)\|\psi\|_{[s,0]}ds<\infty\}$. If $\mathbf{B}_h$ is endowed with the norm

$$\|\psi\|_{\mathbf{B}_h} = \int_{-\infty}^{0}h(s)\|\psi\|_{[s,0]}ds,\ \forall\ \psi\in\mathbf{B}_h,$$

then it is clear that $(\mathbf{B}_h,\|.\|_{\mathbf{B}_h})$ is a Banach space. Now we consider the space

$$\mathbf{B}_h' = \{x:(-\infty,T]\to X \text{ such that } x\mid_{J_k}\in C(J_k,X) \text{ and there exist } x(t_k^+) \text{ and } x(t_k^-) \text{ with } x(t_k)=x(t_k^-),\ x_0=\phi\in\mathbf{B}_h,\ k=1,2,\dots,m\},$$

where $x\mid_{J_k}$ is the restriction of x to $J_k=(t_k,t_{k+1}]$, $k=1,2,\dots,m$. Set $\|.\|_{\mathbf{B}_h'}$ to be a semi norm in $\mathbf{B}_h'$ defined by

$$\|x\|_{\mathbf{B}_h'} = \sup\{\|x(s)\|_X : s\in[0,T]\}+\|\phi\|_{\mathbf{B}_h},\ x\in\mathbf{B}_h'.$$

If $x:(-\infty,T]\to X$ such that $x\in\mathbf{B}_h'$, then for all $t\in J$, the following conditions hold:

(1) x_t is in $\mathbf{B}_h$.

(2) $\|x(t)\|_X\leqslant H\|x_t\|_{\mathbf{B}_h}$.

(3) $\|x_t\|_{\mathbf{B}_h}\leqslant K(t)\sup\{\|x(s)\|:0\leqslant s\leqslant t\}+M(t)\|\phi\|_{\mathbf{B}_h}$ where $H>0$ is constant; K, $M:[0,\infty)\to[0,\infty)$, $K(.)$ is continuous, $M(.)$ is locally bounded and H, K, M are independent of $x(t)$.

(H_ϕ) (From [9]) Let the function $t\to\phi_t$ is well defined and continuous from the set

$$\Re(\rho^-)=\{\rho(s,\psi):(s,\psi)\in[0,T]\times\mathbf{B}_h\}.$$

into $\mathbf{B}_h$ and there exists a continuous and bounded function $J^\phi:\Re(\rho^-)\to(0,\infty)$ such that $\|\phi_t\|_{\mathbf{B}_h}\leqslant J^\phi(t)\|\phi\|_{\mathbf{B}_h}$ for every $t\in\Re(\rho^-)$.

LEMMA 2.1. (Lemma 3.1 [9]) *Let $x:(-\infty,T]\to X$ such that $x\in \mathbf{B}_h'$, be piece-continuous on J and $x_0=\phi$, if (H_ϕ) hold, then*

$$\|x_s\|_{\mathbf{B}_h}\leqslant (M_b+J^\phi)\|\phi\|_{\mathbf{B}_h}+K_b\sup\{\|x(\theta)\|_X;\ \ \theta\in[0,\max\{0,s\}]\},\ \ s\in\Re(\rho^-)\cup J,$$

where $J^\phi=\sup_{t\in\Re(\rho^-)}J^\phi(t)$, $M_b=\sup_{s\in[0,T]}M(s)$ and $K_b=\sup_{s\in J}K(s)$.

DEFINITION 2.2. The Riemann-Liouville fractional integral operator of order $\alpha>0$, for a function $f\in L^1_{loc}(\mathbb{R}^+,X)$ is defined by

$$J_t^0 f(t)=f(t),\quad J_t^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds,\ \ \alpha>0,\ \ t>0, \tag{2.1}$$

where $\Gamma(\cdot)$ is the Euler gamma function.

DEFINITION 2.3. Caputo's derivative of order $\alpha>0$ for a function $f:[0,\infty)\to\mathbb{R}$ such that $f\in C^n(\mathbb{R}^+,X)$ is defined as

$$D_t^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s)ds=J^{n-\alpha}f^{(n)}(t), \tag{2.2}$$

for $n-1<\alpha<n$, $n\in N$. If $0<\alpha<1$, then

$$D_t^\alpha f(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}f^{(1)}(s)ds. \tag{2.3}$$

Obviously, Caputo's derivative of a constant is equal to zero.

DEFINITION 2.4. A two parameter function of the Mittag-Lefller type is defined by the series expansion

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}=\frac{1}{2\pi\imath}\int_c\frac{\mu^{\alpha-\beta}e^\mu}{\mu^\alpha-z}d\mu,\ \ \alpha,\ \beta>0,\ \ z\in\mathbb{C},$$

where c is a contour which starts and ends at ∞ and encircles the disc $|\mu|\leqslant|z|^{\frac{1}{\alpha}}$ counter clockwise. The most interesting properties of the Mittag-Lefller functions are associated with their Laplace integral

$$\int_0^\infty e^{-\lambda t}t^{\beta-1}E_{\alpha,\beta}(\omega t^\alpha)dt=\frac{\lambda^{\alpha-\beta}}{\lambda^\alpha-\omega},\ \ \mathrm{Re}\,\lambda>\omega^{\frac{1}{\alpha}},\ \ \omega>0.$$

For more details one can see the monographs of I. Podlubny [20].

DEFINITION 2.5. A closed and linear operator A is said to be sectorial if there are constants $\omega\in\mathbb{R}$, $\theta\in[\frac{\pi}{2},\pi]$, $M>0$, such that the following two conditions are satisfied:

(1) $$\Sigma_{(\theta,\omega)}=\{\lambda\in C:\lambda\neq\omega,\ \ |arg(\lambda-\omega)|<\theta\}\subset\rho(A),$$

$$(2) \qquad \|R(\lambda,A)\|_{L(X)} \leqslant \frac{M}{|\lambda-\omega|}, \quad \lambda \in \Sigma_{(\theta,\omega)},$$

where X is the complex Banach space with norm $\|.\|_X$. For details see [16].

DEFINITION 2.6. Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X. Let $\rho(A)$ be the resolvent set of A, we call A is the generator of an α-resolvent family if there exists $\omega \geqslant 0$ and a strongly continuous function $T_\alpha : \mathbb{R}^+ \to L(X)$ such that $\{\lambda^\alpha : \mathrm{Re}\,\lambda > \omega\} \subset \rho(A)$ and

$$(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{-\lambda t} T_\alpha(t)x dt, \ \ \mathrm{Re}\,\lambda > \omega, \ \ x \in X.$$

In this case, $T_\alpha(t)$ is called α-resolvent family generated by A. For details see [5].

DEFINITION 2.7. (See definition 2.1 in [2]). Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X and $\alpha > 0$. We say that A is the generator of a solution operator if there exists $\omega \geqslant 0$ and a strongly continuous function $S_\alpha : \mathbb{R}^+ \to L(X)$ such that $\{\lambda^\alpha : \mathrm{Re}\,\lambda > \omega\} \subset \rho(A)$ and

$$\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x dt, \ \ \mathrm{Re}\,\lambda > \omega, \ \ x \in X.$$

In this case, $S_\alpha(t)$ is called the solution operator generated by A.

THEOREM 2.8. [10] (Krasnoselkii's fixed point theorem) *Let B be a closed convex and nonempty subset of a Banach space X. Let P and Q be two operators such that (i) $Px + Qy \in B$, whenever $x, y \in B$. (ii) P is compact and continuous. (iii) Q is a contraction mapping. Then there exists $z \in B$ such that $z = Pz + Qz$.*

3. The main results

DEFINITION 3.1. A piecewise continuous function $x(t) \in \mathbf{B}'_h$ is called the mild solution of the system (1.1)–(1.3) if it satisfy the following integral equation

$$x(t) = \begin{cases} \phi(t) - (h(x_{t_1}, \cdots, x_{t_p}))(t), & t \in (-\infty, 0] \\ S_\alpha(t)(\phi(0) - (h(x_{t_1}, \cdots, x_{t_p}))(0)) & \\ + \sum_{i=1}^m S_\alpha(t - t_i)\chi_i(t) I_i(x(t_i^-)) & \\ + \int_0^t T_\alpha(t-s) f(s, x_{\rho(s,x_s)}) ds & \\ + \int_0^t T_\alpha(t-s) \left[\int_0^s q(s-\xi) g(\xi, x_\xi) d\xi\right] ds, & t \in J. \end{cases} \tag{3.1}$$

Where $E_{\alpha,1}(At^\alpha) = S_\alpha(t)$ and $t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha) = T_\alpha(t)$. For more detail of the stated formula one can see the paper [12].

To prove our results, we shall assume that $\rho : J \times \mathbf{B}_h \to (-\infty, T]$, $q : J \to X$ are continuous and the following axioms hold:

(H1) The function $f : J \times \mathbf{B}_h \to X$ is continuous and there exists a constant L_f such that $\|f(t,\psi) - f(t,\chi)\|_X \leqslant L_f \|\psi - \chi\|_{\mathbf{B}_h}$, $\forall\, \psi, \chi \in \mathbf{B}_h$.

(H2) The function $g : J \times \mathbf{B}_h \to X$ is continuous and there exists a constant L_g such that $\|g(t,x) - g(t,y)\|_X \leqslant L_g \|x - y\|_X$, $\forall\, x, y \in X$.

(H3) The function $I_k : X \to X$ is continuous and there exist a constant L_k such that $\|I_k(x) - I_k(y)\|_X \leqslant L_I \|x - y\|_X$, $\forall\, x, y \in X$, and $k = 1, 2, \ldots m$.

(H4) The function $h : \mathbf{B}_h{}^p \to X$ is continuous and there exists a constant L_h such that $\|h(u_{t_1}, \cdots, u_{t_p}) - h(v_{t_1}, \cdots, v_{t_p})\| \leqslant L_h \|u - v\|_{\mathbf{B}_h}$, for $u, v \in \mathbf{B}_h$.

(H5) The functions $T_\alpha(t), S_\alpha(t)$ are strongly continuous and there exist constants $\tilde{M}_T$, $\tilde{M}_S$ such that $\|T_\alpha(t)\|_{L(X)} \leqslant t^{\alpha-1} \tilde{M}_T$, $\|S_\alpha(t)\|_{L(X)} \leqslant \tilde{M}_S$.

(H6) The function $g : J \times \mathbf{B}_h \to X$ is continuous and there exists a continuous function $l_g : J \to \mathbb{R}^+$ such that $\|g(t,\psi)\|_X \leqslant l_g(t) \|\psi\|_{\mathbf{B}_h}$.

(H7) The function $f : J \times \mathbf{B}_h \to X$ is continuous and there exists a continuous function $l_f : J \to \mathbb{R}^+$ such that $\|f(t,\psi)\|_X \leqslant l_f(t) \|\psi\|_{\mathbf{B}_h}$.

(H8) The functions $I_k : X \to X$, are continuous and there exists a constant $\Omega > 0$, such that $\|I_k(x)\|_X \leqslant \Omega \|x\|_X$ for all $x \in X$ and $k = 1, 2 \ldots, m$.

Our first result is based on Banach contraction theorem.

THEOREM 3.2. Let the assumptions (H1)–(H5) hold and

$$\triangle = \left[\tilde{M}_S L_h + \tilde{M}_S m L_I + \tilde{M}_T (L_f K_b + L_g q^* K_b) \frac{T^\alpha}{\alpha} \right] < 1,$$

where $q* = \sup_{0 \leqslant t \leqslant T} \int_0^t \|q(t-s)\|_X ds$. Then there exists a unique mild solution of the system (1.1)–(1.3) on J.

Proof. We define an operator $N : \mathbf{B}_h^{'} \to \mathbf{B}_h^{'}$ by

$$Nx(t) = \begin{cases} \phi(t) - (h(x_{t_1}, \cdots, x_{t_p}))(t), & t \in (-\infty, 0], \\ S_\alpha(t)(\phi(0) - (h(x_{t_1}, \cdots, x_{t_p}))(0)) & \\ + \sum_{i=1}^m S_\alpha(t - t_i) \chi_i(t) I_i(x(t_i^-)) & \\ + \int_0^t T_\alpha(t-s) \left[f(s, x_{\rho(s,x_s)}) + \int_0^s q(s-\xi) g(\xi, x_\xi) d\xi \right] ds, & t \in J. \end{cases}$$

For $\phi \in \mathbf{B}_h$, let $y(.) : (-\infty, T]$ be the function defined by

$$y(t) = \begin{cases} \phi(t) - (h(x_{t_1}, \cdots, x_{t_p}))(t), & t \in (-\infty, 0], \\ 0, & t \in J. \end{cases}$$

For each $z:[0,T]\to X$ with $z|_{J_k}\in C(J_k,X)$, $k=1,2,\dots,m$ and $z(0)=0$, we denote by $\bar{z}$ as $\bar{z}(t)=\begin{cases}0, & t\in(-\infty,0],\\ z(t), & t\in J.\end{cases}$

If $x(.)$ satisfies the solution (3.1) then we can decompose $x(.)$ as $x(t)=y(t)+\bar{z}(t)$ which implies $x_t=y_t+\bar{z}_t$ for $t\in J$, and $z(.)$ satisfied

$$\begin{aligned}z(t) &= S_\alpha(t)(\phi(0)-(h(x_{t_1},\cdots,x_{t_p}))(0))+\sum_{0<t_i<t}S_\alpha(t-t_i)I_i(z(t_i^-))\\ &\quad+\int_0^t T_\alpha(t-s)\big[f(s,y_{\rho(s,y_s+\bar{z}_s)}+\bar{z}_{\rho(s,y_s+\bar{z}_s)})\\ &\quad+\int_0^s q(s-\xi)g(\xi,y_\xi+\bar{z}_\xi)d\xi\big]ds,\ \text{ for } t\in(t_k,t_{k+1}],\ k=1,2,\dots,m.\end{aligned}$$

Now, we define a set $\mathbf{B}_h^{''}=\{z\in\mathbf{B}_h^{'}$ such that $z_0=0\}$ and let $\|\cdot\|_{\mathbf{B}_h^{''}}$ be the semi-norm in $\mathbf{B}_h^{''}$ defined as

$$\|z\|_{\mathbf{B}_h^{''}}=\sup_{t\in J}\|z(t)\|_X+\|z_0\|_{\mathbf{B}_h}=\sup_{t\in J}\|z(t)\|_X,\qquad z\in\mathbf{B}_h^{''}.$$

Thus, the space $(\mathbf{B}_h^{''},\|\cdot\|_{\mathbf{B}_h^{''}})$ is Banach space. Now we define the operator $P:\mathbf{B}_h^{''}\to\mathbf{B}_h^{''}$ by

$$\begin{aligned}P(z)(t) &= S_\alpha(t)(\phi(0)-(h(x_{t_1},\cdots,x_{t_p}))(0))+\sum_{0<t_i<t}S_\alpha(t-t_i)I_i(z(t_i^-))\\ &\quad+\int_0^t T_\alpha(t-s)\left[f(s,y_{\rho(s,y_s+\bar{z}_s)}+\bar{z}_{\rho(s,y_s+\bar{z}_s)})+\int_0^s q(s-\xi)g(\xi,y_\xi+\bar{z}_\xi)d\xi\right]ds,\end{aligned}$$

for $t\in(t_k,t_{k+1}]$, $k=1,2,\dots,m$. It is clear that the operator N has a unique fixed point if and only if P has a unique fixed point. To prove that, let us assume $z,z^*\in\mathbf{B}_h^{''}$ and $t\in(t_k,t_{k+1}]$, $k=1,2,\dots,m$, we have the following estimate

$$\begin{aligned}&\|P(z)(t)-P(z^*)(t)\|_X\\ &\leqslant \|S_\alpha(t)\|_{L(X)}\|(h(z_{t_1},\cdots,z_{t_p}))(0))-(h(z^*_{t_1},\cdots,z^*_{t_p}))(0))\|_{\mathbf{B}_h}\\ &\quad+\sum_{0<t_i<t}\|S_\alpha(t-t_i)\|_{L(X)}\|I_i(z^*(t_i^-))-I_i(z(t_i^-))\|_X\\ &\quad+\int_0^t\|T_\alpha(t-s)\|_{L(X)}\left\|f(s,y_{\rho(s,y_s+\bar{z}^*_s)}+\bar{z}^*_{\rho(s,y_s+\bar{z}^*_s)})-f(s,y_{\rho(s,y_s+\bar{z}_s)}+\bar{z}_{\rho(s,y_s+\bar{z}_s)})\right\|_X\\ &\quad+\int_0^t\|T_\alpha(t-s)\|_{L(X)}\left[\int_0^s\|q(s-\xi)\|_X\|g(\xi,y_\xi+\bar{z}^*_\xi)-g(\xi,y_\xi+\bar{z}_\xi)\|_X d\xi\right]ds\\ &\leqslant \tilde{M}_S L_h\|z^*-z\|_{\mathbf{B_h}}+\tilde{M}_S m L_I\|z^*-z\|_{\mathbf{B}_h^{''}}\\ &\quad+\tilde{M}_T L_f\int_0^t(t-s)^{\alpha-1}\|\bar{z}^*_{\rho(s,y_s+\bar{z}^*_s)}-\bar{z}_{\rho(s,y_s+\bar{z}_s)}\|_{\mathbf{B}_h}\end{aligned}$$

$$+\tilde{M}_T L_g \int_0^t (t-s)^{\alpha-1} \int_0^s \|q(s-\xi)\|_X \| \bar{z}^*_\xi - \bar{z}_\xi \|_{\mathbf{B}_h} d\xi ds$$
$$\leqslant \left[\tilde{M}_S L_h + \tilde{M}_S m L_I + \tilde{M}_T (L_f K_b + L_g q^* K_b) \frac{T^\alpha}{\alpha}\right] \|z^* - z\|_{\mathbf{B}''_h}.$$

Since $\Delta < 1$, which implies that P is a contraction map and has a unique fixed point $z \in \mathbf{B}''_h$. Its means that the system (1.1)–(1.3) has a unique fixed point. This complete the proof of the theorem. □

Our second result based on the Krasnoselkii's fixed point theorem.

THEOREM 3.3. *Assume that the assumptions (H1), (H5)–(H8) and* (H_ϕ) *hold with*

$$\left[\tilde{M}_T (L_f K_b + L_g q^* K_b) \frac{T^\alpha}{\alpha}\right] < 1.$$

Then the system (1.1)–(1.3) has at least one solution on J.

Proof. Let us choose a positive number

$$r \geqslant \Big[\tilde{M}_S(\|\phi(0)\|_{\mathbf{B_h}} + L_h r + \|(h(x_{t_1}, \cdots, x_{t_p}))\|_{\mathbf{B_h}}$$
$$+\tilde{M}_S m \Omega r + \tilde{M}_T \|l_f\|_\infty [(M_b + J^\phi)\|\phi\|_{\mathbf{B}_h} + K_b r] \frac{T^\alpha}{\alpha}$$
$$+\tilde{M}_T \|l_g\|_\infty q^* (K_b r + M_b \|\phi\|_{\mathbf{B}_h}) \frac{T^\alpha}{\alpha}\Big],$$

and consider $B_r = \{z \in \mathbf{B}''_h : \|z\|_{\mathbf{B}''_h} \leqslant r\}$, then B_r is a bounded, closed convex subset in $\mathbf{B}''_h$. Let $P_1 : B_r \to B_r$ and $P_2 : B_r \to B_r$ be defined as:

$$(P_1 z)(t) = S_\alpha(t)(\phi(0) - (h(x_{t_1}, \cdots, x_{t_p}))(0)) + \sum_{0<t_i<t} S_\alpha(t-t_i) I_i(z(t_i^-)),$$

$$(P_2)z(t) = \int_0^t T_\alpha(t-s) \left[f(s, y_{\rho(s,y_s+\bar{z}_s)} + \bar{z}_{\rho(s,y_s+\bar{z}_s)}) + \int_0^s q(s-\xi) g(\xi, y_\xi + \bar{z}_\xi) d\xi\right] ds,$$

for $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$. We complete the proof in the following steps

Step 1. Let $z, z^* \in B_r$, and $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$,

$$\|(P_1 z)(t) + (P_2 z^*)(t)\|_X$$
$$\leqslant \|S_\alpha(t)\|_{L(X)} (\|\phi(0)\|_{\mathbf{B_h}} + \|(h(x_{t_1}, \cdots, x_{t_p}))(0)\|_{\mathbf{B_h}}$$
$$+ \sum_{0<t_i<t} \|S_\alpha(t-t_i)\|_{L(X)} \|I_i(z(t_i^-))\|_X + \int_0^t \|T_\alpha(t-s)\|_{L(X)}$$
$$\times \left[\|f(s, y_{\rho(s,y_s+\bar{z}^*_s)} + \bar{z}^*_{\rho(s,y_s+\bar{z}^*_s)})\|_X + \int_0^s \|q(s-\xi)\|_X \|g(\xi, y_\xi + \bar{z}^*_\xi)\|_X d\xi\right] ds$$
$$\leqslant \tilde{M}_S(\|\phi(0)\|_{\mathbf{B_h}} + L_h r + \|(h(x_{t_1}, \cdots, x_{t_p}))\|_{\mathbf{B_h}})$$

$$+\tilde{M}_S m\Omega r+\tilde{M}_T\|l_f\|_\infty[(M_b+J^\phi)\|\phi\|_{\mathbf{B}_h}+K_b r]\frac{T^\alpha}{\alpha}$$
$$+\tilde{M}_T\|l_g\|_\infty q^*(K_b r+M_b\|\phi\|_{\mathbf{B}_h})\frac{T^\alpha}{\alpha}\leqslant r,$$

which implies that $\|P_1z+P_2z\|_{\mathfrak{B}_h''}\leqslant r.$ i.e. $P_1z+P_2z^*\in B_r$, $\forall\, z,z^*\in B_r$.

Step 2. Here we show that the mapping $(P_1z)(t)$ is continuous on B_r. For this purpose, let $\{z^n\}_{n=1}^\infty$ be a sequence in B_r with $\lim z^n\to z$ in B_r. Then for $t\in(t_k,t_{k+1}]$, $k=1,2,\ldots,m,$ we have

$$\begin{aligned}&\|(P_1z^n)(t)-(P_1z)(t)\|_X\\ &\leqslant\|S_\alpha(t)\|_{L(X)}(\|(h(x^n_{t_1},\cdots,x^n_{t_p}))-(h(x_{t_1},\cdots,x_{t_p}))\|_{\mathbf{B_h}})\\ &\leqslant\sum_{0<t_i<t}\|S_\alpha(t-t_i)\|_{L(X)}\|I_i(z^n(t_i^-))-I_i(z(t_i^-))\|_X.\end{aligned}$$

Since the functions h, I_i, $i=1,2,\ldots,m,$ are continuous, hence $\lim_{n\to\infty}P_1z^n=P_1z$ in B_r. Which implies that the mapping P_1 is continuous on B_r.

Step 3. The family $\mathscr{W}(t)=\{(P_1z):z\in B_r\}$ is uniformly bounded follows by the following inequality. For $t\in(t_k,t_{k+1}]$, $k=1,2,\ldots,m,$ we get

$$\begin{aligned}&\|(P_1z)(t)\|_X\\ &\leqslant\|S_\alpha(t)\|_{L(X)}\|(\phi(0)-(h(x_{t_1},\cdots,x_{t_p}))(0))\|_X+\sum_{0<t_i<t}\|S_\alpha(t-t_i)\|_{L(X)}\|I_i(z(t_i^-))\|_X\\ &\leqslant\tilde{M}_S(\|\phi(0)\|_{\mathbf{B_h}}+L_hr+\|(h(x_{t_1},\cdots,x_{t_p}))\|_{\mathbf{B_h}})+\tilde{M}_S\|\phi(0)\|_X+\tilde{M}_Tm\Omega r.\end{aligned}$$

Step 4. To show that $\mathscr{W}(t)$ is equi-continuous. Let $u,v\in(t_k,t_{k+1}]$, $t_k\leqslant u<v\leqslant t_{k+1}$, $k=1,2,\ldots,m$, $z\in B_r$, we obtain

$$\begin{aligned}&\|(P_1z)(v)-(P_1z)(u)\|_X\\ &\leqslant\|S_\alpha(v)-S_\alpha(u)\|_{L(X)}\|(\phi(0)-(h(x_{t_1},\cdots,x_{t_p})(0))\|_X\\ &\quad+\sum_{0<t_i<t}\|S_\alpha(v-t_i)-S_\alpha(u-t_i)\|_{L(X)}\|I_i(z(t_i^-))\|_X.\end{aligned}$$

Since $S_\alpha(t)$ is strongly continuous and the continuity of the function $t\mapsto\|S(t)\|_{L(X)}$ allows us to conclude that $\lim_{u\to v}\|S_\alpha(v-t_i)-S_\alpha(u-t_i)\|_{L(X)}=0,$ which implies that $\mathscr{W}(t)$ is equi-continuous. Finally, combining step 2 to step 4 together with the Arzela Ascoli's theorem, $\mathscr{W}(t)$ is relatively compact therefore we conclude that the operator P_1 is Compact.

Step 5. To show that P_2 is a contraction mapping. Let $z,z^*\in B_r$ and $t\in(t_k,t_{k+1}]$, $k=1,2,\ldots,m,$ we have the following estimate

$$\begin{aligned}&\|(P_2z)(t)-(P_2z^*)(t)\|_X\\ &\leqslant\int_0^t\|T_\alpha(t-s)\|_{L(X)}\|f(s,y_{\rho(s,y_s+\bar{z}_s^*)}+\bar{z}^*_{\rho(s,y_s+\bar{z}_s^*)})-f(s,y_{\rho(s,y_s+\bar{z}_s)}+\bar{z}_{\rho(s,y_s+\bar{z}_s)})\|_Xds\end{aligned}$$

$$+\int_0^t \|T_\alpha(t-s)\|_{L(X)}\left[\int_0^s \|q(s-\xi)\|_X\|g(\xi,y_\xi+\bar{z}^*_\xi)-g(\xi,y_\xi+\bar{z}_\xi)\|_X d\xi\right]ds$$
$$\leqslant \left[\tilde{M}_T(L_fK_b+L_gq^*K_b)\frac{T^\alpha}{\alpha}\right]\|z^*-z\|_{\mathbf{B}''_h}.$$

Since $\left[\tilde{M}_T(L_fK_b+L_gq^*K_b)\frac{T^\alpha}{\alpha}\right]<1$, which implies that P_2 is a contraction mapping. Hence, by the Krasnoselkii's fixed point theorem, we can conclude that the system (1.1)–(1.3) has at least one solution on J. This completes the proof of the theorem. □

In the next theorem, we will show the solution of the considered problem continuously dependent on the initial data.

THEOREM 3.4. *Suppose that assumption (H1)–(H5) are satisfied and following inequality hold:*

$$L_h+m\tilde{M}_SL_I+\tilde{M}_T\frac{T^\alpha}{\alpha}[L_fK_b+q*LgK_b]<1.$$

Then for each $\phi,\phi^*\in\mathbf{B}_h$ and $x(t)=y(t)+\overline{z(t)},x^*(t)=y^*(t)+\overline{z^*}$ be the corresponding mild solution of system (1.1)–(1.3) on J, then following inequality hold

$$\|z(t)-z^*(t)\|_X\leqslant\frac{\tilde{M}_S}{1-(L_h+m\tilde{M}_SL_I+\tilde{M}_T\frac{T^\alpha}{\alpha}[L_fK_b+q*LgK_b])}\|\phi-\phi^*\|_{\mathbf{B}_h}$$

for $t\in J$.

Proof. For $t\in(t_k,t_{k+1}]$, we have

$$\begin{aligned}
&\|z(t)-z^*(t)\|_X\\
&=\|S_\alpha(t)(\phi(0)-(h(z_{t_1},\cdots,z_{t_p})(0))+\Sigma_{0<t_i<t}S_\alpha(t-t_i)I_i(z(t_i^-))\\
&\quad+\int_0^t T_\alpha(t-s)[f(s,y_{\rho(s,y_s+\bar{z}_s)}+\bar{z}_{\rho(s,y_s+\bar{z}_s)})\\
&\quad+\int_0^s q(s-\xi)g(\xi,y_\xi+\bar{z}_\xi)d\xi]ds\\
&\quad-S_\alpha(t)(\phi^*(0)-(h(z^*_{t_1},\cdots,z^*_{t_p})(0))+\Sigma_{0<t_i<t}S_\alpha(t-t_i)I_i(z^*(t_i^-))\\
&\quad-\int_0^t T_\alpha(t-s)[f(s,y_{\rho(s,y_s+\overline{z^*}_s)}+\overline{z^*}_{\rho(s,y_s+\overline{z^*}_s)})\\
&\quad+\int_0^s q(s-\xi)g(\xi,y_\xi+\overline{z^*}_\xi)d\xi]ds\|\\
&\leqslant\|S_\alpha(t)\|\|\phi(0)-\phi^*(0)\|+\|h(z_{t_1},\cdots,z_{t_p})-h(z^*_{t_1},\cdots,z^*_{t_p})\|\\
&\quad+\Sigma_{0<t_i<t}\|S_\alpha(t-t_i)\|\|I_i(z(t_i^-))-I_i(z^*(t_i^-))\|\\
&\quad+\int_0^t S_\alpha(t-s)\|\|[f(s,y_{\rho(s,y_s+\bar{z}_s)}+\bar{z}_{\rho(s,y_s+\bar{z}_s)})
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^s q(s-\xi)g(\xi,y_\xi+\bar{z}_\xi)d\xi]-[f(s,y_{\rho(s,y_s+\bar{z^*}_s)}+\bar{z^*}_{\rho(s,y_s+\bar{z^*}_s)})\\
&+\int_0^s q(s-\xi)g(\xi,y_\xi+\bar{z^*}_\xi)d\xi]\|ds\\
\leqslant\ & \tilde{M}_S\|\phi(0)-\phi^*(0)\|+(L_h+m\tilde{M}_SL_I+\tilde{M}_T\frac{T^\alpha}{\alpha}[L_fK_b+q*L_gK_b])\|z-z*\|_{\mathbf{B}_h}.
\end{aligned}
$$

Which implies that

$$
\|z(t)-z^*(t)\|_X\leqslant\frac{\tilde{M}_S}{1-(L_h+m\tilde{M}_SL_I+\tilde{M}_T\frac{T^\alpha}{\alpha}[L_fK_b+q*L_gK_b])}\|\phi-\phi^*\|_{\mathbf{B}_h},
$$

which is desired estimates for all $t\in J$. This completes the proof of the theorem. □

4. Application

In this section, we shall consider an application of our abstract results.

$$
\begin{aligned}
\frac{\partial^\alpha u(t,x)}{\partial t^\alpha}&=\frac{\partial^2 u(t,x)}{\partial x^2}+\int_{-\infty}^t a_1(t,x,s-t)Q_1(u(s-\rho_1(t)\rho_2(\|u(t)\|),x))ds\\
&\quad+\int_0^t\cos(t-s)\left[\int_{-\infty}^s a_2(t,x,\xi-s)Q_2(u(\xi,x))d\xi\right]ds, \qquad (4.1)
\end{aligned}
$$

$$u(t,0)=0=u(t,\pi),\ \ t\geqslant 0, \qquad (4.2)$$

$$u(t,x)+\sum_0^p\int_0^\pi k(x,\gamma)u_{t_i}(t,\gamma)d\gamma=\phi(t,x),\ \ t\in(-\infty,0],\ \ x\in[0,\pi], \qquad (4.3)$$

$$\Delta u(t_i,x)=\int_{-\infty}^{t_i}\gamma_i(t_i-s)u(s,x)ds, \qquad (4.4)$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is Caputo's fractional derivative of order $\alpha\in(0,1)$, $0<t_1<t_2<\cdots<t_n<T$ are prefixed numbers and $\phi\in\mathbf{B}_h$. Let $X=L^2[0,\pi]$ and define the operator $A:D(A)\subset X\to X$ by $Aw=w''$ with the domain $D(A):=\{w\in X:\ w,w'$ are absolutely continuous, $w''\in X, w(0)=0=w(\pi)\}$. Then

$$
Aw=\sum_{n=1}^\infty n^2(w,w_n)w_n,\ \ w\in D(A),
$$

where $w_n(x)=\sqrt{\frac{2}{\pi}}\sin(nx)$, $n\in\mathbb{N}$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geqslant 0}$ in X and is given by

$$
T(t)w=\sum_{n=1}^\infty e^{-n^2t}(w,w_n)w_n,\ \ \text{for all}\ \ w\in X,\ \ \text{and every}\ \ t>0.
$$

The subordination principle of solution operator (Theorem 3.1 in [7]) implies that A is the infinitesimal generator of a solution operator $\{S_\alpha(t)\}_{t\geqslant 0}$. Since $S_\alpha(t)$ is strongly continuous on $[0,\infty)$, by uniformly bounded theorem, there exists a constant $M>0$ such that $\|S_\alpha(t)\|_{L(X)} \leqslant M$ for $t\in[0,T]$.

Let $h(s)=e^{2s}$, $s<0$ then $l=\int_{-\infty}^{0} h(s)ds=\frac{1}{2}$ and define

$$\|\phi\|_{\mathbf{B}_h}=\int_{-\infty}^{0} h(s)\sup_{\theta\in[s,0]}|\phi(\theta)\|_{L^2}ds.$$

Hence for $(t,\phi)\in[0,T]\times\mathbf{B}_h$, where $\phi(\theta)(x)=\phi(\theta,x)$, $(\theta,x)\in(-\infty,0]\times[0,\pi]$. Set $u(t)(x)=u(t,x)$,

$$g(t,\phi)(x) = \int_{-\infty}^{0} a_2(t,x,\theta)Q_2(\phi(\theta)(x))d\theta \tag{4.5}$$

$$I_i(\phi)(x) = \int_{-\infty}^{0} \gamma_i(-\theta)\phi(\theta)(x)d\theta \tag{4.6}$$

$$f(t,\phi)(x) = \int_{-\infty}^{0} a_1(t,x,\theta)Q_1(\phi(\theta)(x))d\theta \tag{4.7}$$

$$h(x_{t_1},\cdots,x_{t_p})(t) = \sum_{0}^{p}\int_{0}^{\pi} k(x,\gamma)u_{t_i}(t,\gamma)d\gamma \tag{4.8}$$

$$q(t,\phi)(x) = \cos(t-s) \tag{4.9}$$

$$\rho(t,\phi) = t-\rho_1(t)\rho_2(\|\phi(0)\|) \tag{4.10}$$

Then with these settings the system (4.1)–(4.4) can be written in the abstract form of the system (1.1)–(1.3). To treat this system, we assume that $\rho_i:[0,\infty)\to[0,\infty)$, $i=1,2$, are continuous and the following conditions hold:

1. The functions Q_i, $i=1,2$ are continuous and $u(\theta,x)$, $v(\theta,x)$ are continuous in $(-\infty,0]\times[0,\pi]$,

$$0\leqslant Q_i(u(\theta)(x))-Q_i(v(\theta)(x))\leqslant \int_{-\infty}^{0} e^{2s}\|u(s,.)-v(s,.)\|_{L^2}ds.$$

2. The functions $a_i(t,x,\theta)$, $i=1,2$, are continuous in $[0,T]\times[o,\pi]\times(-\infty,0)$ and satisfy $\int_{-\infty}^{0} a_i^2(t,x,\theta)d\theta<C_i$, $C_i>0$, $i=1,2$.

3. The functions $\gamma_i\in C(\mathbb{R},\mathbb{R})$ and $d_i=(\int_{-\infty}^{0}\frac{\gamma_i^2(-\theta)}{h(\theta)}d\theta)<\infty$ for $i=1,2,\dots,m$.

Now we can see that for $(t,\phi),(t,\psi)\in[0,T]\times\mathbf{B}_h$, we have

$$\begin{aligned}
&\|f(t,\phi)-f(t,\psi)\|_{L^2}\\
&=\left[\int_{0}^{\pi}\left\{\int_{-\infty}^{0} a_1(t,x,\theta)(Q_1(\phi(\theta)(x))-Q_1(\psi(\theta)(x)))d\theta\right\}^2 dx\right]^{1/2}\\
&\leqslant\left[\int_{0}^{\pi}\left\{\int_{-\infty}^{0} a_1(t,x,\theta)\left(\int_{-\infty}^{0} e^{2s}\|\phi(s,.)-\psi(s,.)\|_{L^2}ds\right)d\theta\right\}^2 dx\right]^{1/2}
\end{aligned}$$

$$\leqslant \left[\int_0^\pi \left\{ \int_{-\infty}^0 a_1(t,x,\theta) \left(\int_{-\infty}^0 e^{2s} \sup_{s\in[\theta,0]} \|\phi(s,.) - \psi(s,.)\|_{L^2} ds \right) d\theta \right\}^2 dx \right]^{1/2}$$

$$\leqslant \left[\int_0^\pi \left\{ \int_{-\infty}^0 a_1(t,x,\theta) d\theta \right\}^2 dx \right]^{1/2} \|\phi - \psi\|_{\mathbf{B}_h}$$

$$\leqslant C_1\sqrt{\pi}\|\phi - \psi\|_{\mathbf{B}_h}.$$

Hence the function f satisfies (H1). Similarly we can show that the functions g, h, I_i satisfy (H2), (H3) respectively. All the conditions of Theorem 3.2 have been fulfilled so we deduced that the system (4.1)–(4.4) has a unique mild solution on J.

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