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SUBORDINATION RESULTS ON MULTIVALENT FUNCTIONS RELATED TO THE SAIGO FRACTIONAL DIFFERINTEGRAL OPERATOR

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Abstract. In this paper we consider a class of multivalent analytics functions based on the use of the Saigo operators of fractional calculus, known as Saigo hypergeometric fractional integrals and derivatives. We obtain some useful properties and characteristics using the techniques of differential subordinations. The main results are illustrated by several interesting corollaries and show their relevance with earlier results.

1. Introduction and definitions

Let $\mathscr{A}_p$ denote the class of functions $f(z)$ of the form

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n} \qquad (p\in\mathbb{N}=\{1,2,3,\dots\}), \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}=\{z:\ z\in\mathbb{C} \text{ and } |z|<1\}$. For the functions $f(z)$ and $g(z)$, which are analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and write $f(z)\prec g(z)$, if there exists a function $w(z)$ analytic in $\mathbb{U}$ such that $|w(z)|<1$, $z\in\mathbb{U}$, and $w(0)=0$ with $f(z)=g(w(z))$ in $\mathbb{U}$. In particular, if $f(z)$ is univalent in $\mathbb{U}$, we have the following equivalence:

$$f(z)\prec g(z) \qquad (z\in\mathbb{U})\Longleftrightarrow [f(0)=g(0) \text{ and } f(\mathbb{U})\subset g(\mathbb{U})].$$

An analytic function f is said to be p-valent $(p\in\mathbb{N})$ in a domain D, if it assumes no value more than p times in D and there is some w_0 such that $f(z)=w_0$ has exactly p solutions in D, when roots are counted in accordance with their multiplicities. If $p=1$, then function f is said to be univalent in D. For a given positive integer p, a p-valent is called p-valent starlike function in the open unit disk $\mathbb{D}$, if there exist a $\delta>0$ so that

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>0 \quad \text{for} \quad \delta<|z|<1.$$

Further, for the functions $f_j(z)\in\mathscr{A}_p$, given by

$$f_j(z)=z^p+\sum_{n=1}^{\infty}a_{p+n,j}z^{p+n} \qquad (j=1,2;\ p\in\mathbb{N}),$$

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the Hadamard product (or Convolution) of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) := z^p + \sum_{n=1}^{\infty} a_{p+n,1}a_{p+n,2}z^{p+n} \qquad (p \in \mathbb{N},\ z \in \mathbb{U}).$$

Let $\alpha_j \ \ (j = 1,\ldots,p)$ and $\beta_j \ \ (j = 1,\ldots,q)$ be complex numbers with $\beta \neq 0,-1,-2,\ldots \ (j = 1,\ldots,q)$. Then the generalized hypergeometric function ${}_pF_q$ is defined by (cf. e.g. [2])

$$ {}_pF_q(z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{z^n}{n!} \quad (p,q \in \mathbb{N}_0,\ \ p \leqslant q+1,\ z \in \mathbb{U}), \tag{1.2}$$

here $(x)_0 = 1$ for $x \neq 0$, and $(a)_n$ is the Pochhammer symbol (or shifted factorial function), defined by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)(x+2)\ldots(x+n-1)$$

for $n = 1,2,\cdots$. Here Γ denotes the well known gamma function. We recall here the following definitions of the Saigo hypergeometric fractional integral operator and differential operator, used in this paper.

DEFINITION 1. For real numbers $\lambda > 0$, μ and η, the Saigo hypergeometric fractional integral operator $I_{0,z}^{\lambda,\mu,\eta}$ is defined by

$$I_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{z^{-\lambda-\mu}}{\Gamma(\lambda)} \int_0^z (z-t)^{\lambda-1} {}_2F_1\left(\lambda+\mu,-\eta;\lambda;1-\frac{t}{z}\right) f(t)\,dt, \tag{1.3}$$

where the function $f(z)$ is analytic in a simply-connected region of the complex z-plane containing the origin, with the order

$$f(z) = O(|z|^{\varepsilon}) \qquad (z \to 0,\ \varepsilon > \max\{0,\mu-\eta\}-1),$$

and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

The operator $I_{0,z}^{\lambda,\mu,\eta}$ has been initially introduced by Saigo in a series of papers for studying boundary value problems for partial differential equations, especially for the Euler-Darboux equation (see [17, 18]). Later the Saigo hypergeometric operator and its modifications have been used in many papers, to study various problems of the Univalent function theory (see [6, 12, 13, 15, 19]).

DEFINITION 2. Under the hypotheses of Definition 1, the Saigo hypergeometric fractional derivative operator $J_{0,z}^{\lambda,\mu,\eta}$ is defined by

$$J_{0,z}^{\lambda,\mu,\eta} f(z) = \begin{cases} \dfrac{1}{\Gamma(1-\lambda)} \dfrac{d}{dz} z^{\lambda-\mu} \int_0^z (z-t)^{-\lambda} {}_2F_1\left(\mu-\lambda,1-\eta;1-\lambda;1-\dfrac{t}{z}\right) f(t)\,dt \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (0 \leqslant \lambda < 1); \\ \dfrac{d^n}{dz^n} J_{0,z}^{\lambda,\mu,\eta} f(z) \qquad\qquad (n \leqslant \lambda < n+1;\ \ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \end{cases} \tag{1.4}$$

where the multiplicity of $(z-t)^{-\lambda}$ is removed as in Definition 1.

It may be remarked that

$$I_{0,z}^{\lambda,-\lambda,\eta} f(z) = D_z^{-\lambda} f(z) \ \ (\lambda > 0) \quad \text{and} \quad J_{0,z}^{\lambda,\mu,\eta} f(z) = D_z^{\lambda} f(z) \ \ (0 \leqslant \lambda < 1),$$

where $D_z^{-\lambda}$ denotes the fractional integral operator, D_z^{λ} – resp. the fractional derivative operator in the sense of classical (Riemann-Liouville) fractional calculus, see [16], adopted for analytic functions in complex domain (as used for example by Owa [10], see also [15], [5, Ch. 5] and many other works on this topic).

Recently, Goyal and Prajapat [3, 13] have considered the generalized fractional differintegral operator $\mathscr{S}_{0,z}^{\lambda,\mu,\eta} : \mathscr{A}_p \to \mathscr{A}_p$, defined by

$$\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z) = \begin{cases} \dfrac{\Gamma(1+p-\mu)\Gamma(1+p+\eta-\lambda)}{\Gamma(1+p)\Gamma(1+p+\eta-\mu)} z^{\mu} J_{0,z}^{\lambda,\mu,\eta} f(z) & (0 \leqslant \lambda < \eta+p+1); \\[2ex] \dfrac{\Gamma(1+p-\mu)\Gamma(1+p+\eta-\lambda)}{\Gamma(1+p)\Gamma(1+p+\eta-\mu)} z^{\mu} I_{0,z}^{\lambda,\mu,\eta} f(z) & (-\infty < \lambda < 0), \end{cases} \tag{1.5}$$

$$(z \in \mathbb{U};\ \ \mu, \eta \in \mathbb{R};\ \ \mu < p+1;\ \ \mu-\eta < p+1)$$

as a variant of the "normalized" Saigo operator mapping $\mathscr{A}_1$ into $\mathscr{A}_1$, from Kiryakova [5, Ex. 8, p. 296–297] and [6, operators (23)].

It is easily seen from (1.5) that for a function f of the form (1.1), we have

$$\begin{aligned} \mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z) &= z^p + \sum_{n=1}^{\infty} \frac{(1+p)_n (1+p+\eta-\mu)_n}{(1+p-\mu)_n (1+p+\eta-\lambda)_n} a_{p+n} z^{p+n} \\ &= z^p\, {}_3F_2(1, 1+p, 1+p+\eta-\mu; 1+p-\mu, 1+p+\eta-\lambda; z) * f(z). \end{aligned} \tag{1.6}$$

It is easy to find from (1.6) that, generalized fractional differintegral operator $\mathscr{S}_{0,z}^{\lambda,\mu,\eta}$ satisfies the following recurrence relation:

$$z\big(\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)\big)' = (p+\eta-\lambda)\mathscr{S}_{0,z}^{\lambda+1,\mu,\eta} f(z) - (\eta-\lambda)\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z). \tag{1.7}$$

Note that

$$\mathscr{S}_{0,z}^{0,0,0} f(z) = f(z), \quad \mathscr{S}_{0,z}^{1,1,1} f(z) = \mathscr{S}_{0,z}^{1,0,0} f(z) = \frac{zf'(z)}{p}$$

and

$$\mathscr{S}_{0,z}^{2,1,1} f(z) = \frac{zf'(z) + z^2 f''(z)}{p^2}.$$

We also note that

$$\mathscr{S}_{0,z}^{\lambda,\lambda,\eta} f(z) = \mathscr{S}_{0,z}^{\lambda,\mu,0} f(z) = \Omega_z^{\lambda,p} f(z),$$

where $\Omega_z^{\lambda,p}$ is the extended fractional differintegral operator studied by Patel and Mishra [11] (see also [14]). On the other hand, if we set $\lambda = -\alpha$, $\mu = 0$ and $\eta = \beta - 1$ in

(1.6), we obtain the following p-valent generalization of multiplier transformation operator [4]

$$\mathscr{Q}_{\beta}^{\alpha} f(z) = \binom{p+\alpha+\beta-1}{p+\beta-1} \frac{\alpha}{z^{\beta}} \int_0^z t^{\beta-1}\left(1-\frac{t}{z}\right)^{\alpha-1} f(t)\,dt$$
$$= z^p + \sum_{n=2}^{\infty} \frac{\Gamma(p+\beta+n)\Gamma(p+\alpha+\beta)}{\Gamma(p+\alpha+\beta+n)\Gamma(p+\beta)} a_{n+p} z^{n+p} \qquad (\beta > -p,\ \ \alpha+\beta > -p). \tag{1.8}$$

Furthermore, if we set $\mu = 0$, $\lambda = -1$ and $\eta = \gamma - 1$ in (1.6), we obtain the generalized Bernardi-Libera-Livingston operator [1], defined by

$$\mathscr{F}_{\gamma,p}(f)(z) := \frac{\gamma+p}{z^{\gamma}} \int_0^z t^{\gamma-1} f(t)\,dt$$
$$= z^p + \sum_{n=1}^{\infty} \frac{\gamma+p}{\gamma+p+n} a_{n+p} z^{n+p}, \qquad \gamma > -p. \tag{1.9}$$

Using the generalized fractional differintegral operator $\mathscr{S}_{0,z}^{\lambda,\mu,\eta}$, we now introduce the following subclass of $\mathscr{A}_p$:

DEFINITION 3. For fixed parameters A, B $(-1 \leqslant B < A \leqslant 1)$ and $\alpha > 0$, we say that a function $f(z) \in \mathscr{A}_p$ is in the class $\mathscr{M}^{\lambda,\mu,\eta}(\alpha, A, B)$, if it satisfies the following subordination condition:

$$(1-\alpha)\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p} + \alpha \frac{\mathscr{S}_{0,z}^{\lambda+1,\mu,\eta} f(z)}{z^p} \prec \frac{1+Az}{1+Bz}, \quad z \in \mathbb{U}. \tag{1.10}$$

For simplicity, we put

$$\mathscr{M}^{\lambda,\mu,\eta}\left(\alpha, 1-\frac{2\beta}{p}, -1\right) = \widetilde{\mathscr{M}}^{\lambda,\mu,\eta}(\alpha,\beta),$$

where $\widetilde{\mathscr{M}}^{\lambda,\mu,\eta}(\alpha,\beta)$ denotes the class of functions $f \in \mathscr{A}_p$, which satisfy the inequality

$$\Re\left((1-\alpha)\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p} + \alpha \frac{\mathscr{S}_{0,z}^{\lambda+1,\mu,\eta} f(z)}{z^p}\right) > \frac{\beta}{p} \quad (z \in \mathbb{U},\ \ \alpha > 0,\ \ 0 \leqslant \beta < p). \tag{1.11}$$

In the present paper we derive various useful and interesting properties and characteristics of the function classes $\mathscr{M}^{\lambda,\mu,\eta}(\alpha, A, B)$ and $\widetilde{\mathscr{M}}^{\lambda,\mu,\eta}(\alpha,\beta)$ (defined above) by using the subordination principle. Several corollaries are deduced from the main results and their connections with known results are also pointed out.

2. Preliminaries

We shall require the following lemmas to investigate the function classes $\mathscr{M}^{\lambda,\mu,\eta}(\alpha,A,B)$ and $\widetilde{\mathscr{M}}^{\lambda,\mu,\eta}(\alpha,\beta)$.

LEMMA 1. ([7, p. 71]) *Let $h(z)$ be a convex (univalent) function in $\mathbb{U}$ with $h(0)=1$, and let the function $\phi(z)=1+p_1z+p_2z^2+\dots$ be analytic in $\mathbb{U}$. If*

$$\phi(z)+\frac{z\phi'(z)}{\gamma}\prec h(z) \tag{2.1}$$

for $\gamma\neq 0$ and $\Re(\gamma)\geqslant 0$, then

$$\phi(z)\prec\psi(z):=\frac{\gamma}{z^{\gamma}}\int_0^z t^{\gamma-1}h(t)\,dt\prec h(z) \tag{2.2}$$

and $\psi(z)$ is the best dominant.

LEMMA 2. ([8, Theorem 8]) *Let a function $f(z)$ of the form (1.1) be analytic in $\mathbb{U}$. If there exists a $(p-m+1)$-valent starlike function of the form*

$$g(z)=z^{p-m+1}+\sum_{k=p-m+2}^{\infty}a_kz^k$$

in $\mathbb{U}$ such that

$$\Re\left(\frac{zf^{(m)}(z)}{g(z)}\right)>0,\qquad z\in\mathbb{U},$$

then $f(z)$ is $p-$valent in $\mathbb{U}$.

Below we remind some known formulas for the hypergeometric function ${}_2F_1$ (cf. e.g. [2]), that will be used next in our proofs.

LEMMA 3. *For real or complex numbers a,b and c $(c\neq 0,-1,-2,\dots)$, we have*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a,b;c;z)\quad(\Re(c)>\Re(b)>0), \tag{2.3}$$

$${}_2F_1(a,b;c;z)=(1-z)^{-a}\,{}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right), \tag{2.4}$$

and

$$(b+1)\,{}_2F_1(1,b;b+1;z)=(b+1)+bz\,{}_2F_1(1,b+1;b+2;z). \tag{2.5}$$

3. Main results

Our first result is given by the following theorem:

THEOREM 1. *If* $f(z) \in \mathscr{M}^{\lambda,\mu,\eta}(\alpha,A,B)$, *then*

$$\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p} \prec \mathscr{X}(z) \prec \frac{1+Az}{1+Bz}, \qquad z \in \mathbb{U}, \tag{3.1}$$

where

$$\mathscr{X}(z) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1\left(1,1;\frac{p+\eta-\lambda}{\alpha}+1;\frac{Bz}{Bz+1}\right) & (B \neq 0); \\ 1 + \frac{p+\eta-\lambda}{\alpha+p+\eta-\lambda}Az & (B = 0), \end{cases}$$

and $\mathscr{X}(z)$ *is the best dominant of (3.1). Also,*

$$\Re\left\{\left(\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p}\right)^{\frac{1}{m}}\right\} > \{\mathscr{X}(-1)\}^{\frac{1}{m}}. \tag{3.2}$$

The result (3.2) is sharp.

Proof. Let $f(z) \in \mathscr{M}^{\lambda,\mu,\eta}(\alpha,A,B)$, and assume that

$$\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p} = p(z), \quad z \in \mathbb{U}. \tag{3.3}$$

We may express the function $p(z)$ as

$$p(z) = 1 + c_1 z + c_2 z^2 + \dots \tag{3.4}$$

which is analytic in $\mathbb{U}$ with $p(0) = 1$. Differentiating (3.3) with respect to z, we get

$$\frac{\mathscr{S}_{0,z}^{\lambda+1,\mu,\eta} f(z)}{z^p} = p(z) + \frac{1}{p+\eta-\lambda} z p'(z). \tag{3.5}$$

From (1.10), (3.3) and (3.5), we obtain

$$\begin{aligned}(1-\alpha)\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p} + \alpha\frac{\mathscr{S}_{0,z}^{\lambda+1,\mu,\eta} f(z)}{z^p} &= p(z) + \frac{\alpha}{p+\eta-\lambda} z p'(z) \\ &\prec \frac{1+Az}{1+Bz}, \qquad z \in \mathbb{U}.\end{aligned}$$

Thus applying Lemma 1, we obtain (3.1). Moreover by Lemma 3, we have

$$\begin{aligned} p(z) &\prec \frac{p+\eta-\lambda}{\alpha} z^{-\frac{p+\eta-\lambda}{\alpha}} \int_0^z t^{\frac{p+\eta-\lambda}{\alpha}-1} \frac{1+At}{1+Bt}\, dt \\ &= \begin{cases} \frac{A}{B}+\left(1-\frac{A}{B}\right)(1+Bz)^{-1}\, {}_2F_1\left(1,1;\frac{p+\eta-\lambda}{\alpha}+1;\frac{Bz}{Bz+1}\right) & (B\neq 0); \\ 1+\frac{p+\eta-\lambda}{\alpha+p+\eta-\lambda} Az & (B=0), \end{cases} \\ &= \mathscr{X}(z). \end{aligned} \tag{3.6}$$

Next to prove (3.2), we observe that the subordination relation (3.6) is equivalent to

$$\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p} = \frac{p+\eta-\lambda}{\alpha} \int_0^1 u^{\frac{p+\eta-\lambda}{\alpha}-1} \frac{1+Auw(z)}{1+Buw(z)}\, du,$$

where $w(z)$ is analytic in $\mathbb{U}$ with $w(0)=1$ and $|w(z)|<1$ in $\mathbb{U}$. Hence

$$\begin{aligned} \Re\left(\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p}\right) &= \frac{p+\eta-\lambda}{\alpha} \int_0^1 u^{\frac{p+\eta-\lambda}{\alpha}-1} \Re\left(\frac{1+Auw(z)}{1+Buw(z)}\right) du \\ &> \frac{p+\eta-\lambda}{\alpha} \int_0^1 u^{\frac{p+\eta-\lambda}{\alpha}-1} \frac{1-Au}{1-Bu}\, du. \\ &= \mathscr{X}(-1). \end{aligned} \tag{3.7}$$

Therefore, with the aid of the identity

$$\Re(w^{\frac{1}{m}}) \geqslant \{\Re(w)\}^{\frac{1}{m}} \quad \text{for} \quad \Re(w)>0 \quad \text{and} \quad m\geqslant 1,$$

the identity (3.1) follows directly from (3.7).

To establish sharpness of result (3.2), it is sufficient to show that

$$\inf_{|z|<1} \Re\{\mathscr{X}(z)\} = \mathscr{X}(-1). \tag{3.8}$$

We observe from (3.7) that for $|z|\leqslant r\ (0<r<1)$:

$$\Re\left(\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p}\right) \geqslant \frac{p+\eta-\lambda}{\alpha} \int_0^1 u^{\frac{p+\eta-\lambda}{\alpha}-1} \Re\left(\frac{1+Aur}{1+Bur}\right) du$$

$$\to \mathscr{X}(-1) \ \text{ as } \ r\to 1^-,$$

which establishes (3.8) and this completes the proof of Theorem 1.

REMARK 1. Setting $\eta=0$ in Theorem 1, we get an improved form of a result due to Patel and Mishra [11, p. 115, Theorem 1.8].

COROLLARY 1. *Let* $-1 \leqslant B < A_1 \leqslant 1$ *and* $B \neq 0$. *If* $f(z) \in \mathscr{M}^{\lambda,\mu,\eta}(\alpha, A_1, B)$, *where* A_1 *is given by*

$$A_1 = \frac{B\,{}_2F_1\left(1,1;\dfrac{p+\eta-\lambda}{\alpha}+1;\dfrac{B}{B-1}\right)}{{}_2F_1\left(1,1;\dfrac{p+\eta-\lambda}{\alpha}+1;\dfrac{B}{B-1}\right)+(B-1)},$$

then

$$\Re\left(\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p}\right) > 0 \qquad (z \in \mathbb{U}),$$

hence $\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)$ *is* p*-valent in* $\mathbb{U}$.

Proof. Putting $m = 0$ and replacing A by A_1 in Theorem 1, we get

$$\Re\left(\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p}\right) = \Re\left(\frac{z\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)}{z^{p+1}}\right) > 0, \qquad z \in \mathbb{U},$$

Since z^{p+1} is $(p+1)$-valently starlike function in $\mathbb{U}$, hence in view of Lemma 2, we obtain that $\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)$ is p-valent in $\mathbb{U}$. $\square$

Further, putting

$$\alpha = \frac{p}{p+1}, \quad \lambda = 1 \quad \text{and} \quad \mu = \eta = 1,$$

in Corollary 1, we get the following result:

COROLLARY 2. *If* $f(z) \in \mathscr{A}_p$, *such that*

$$\frac{2f'(z) + zf''(z)}{p(p+1)z^{p-1}} \prec \frac{1+A_1 z}{1+Bz} \qquad (z \in \mathbb{U}, \ -1 \leqslant B < A_2 \leqslant 1; \ B \neq 0),$$

where A_2 *is given by*

$$A_2 = \frac{B\,{}_2F_1\left(1,1;p+2;\dfrac{B}{B-1}\right)}{{}_2F_1\left(1,1;p+2;\dfrac{B}{B-1}\right)+(B-1)},$$

then

$$\Re\left(\frac{f'(z)}{z^{p-1}}\right) > 0, \qquad z \in \mathbb{U},$$

and $f(z)$ *is hence* p*-valent in* $\mathbb{U}$.

THEOREM 2. *Let* $f(z) \in \mathscr{A}_p$, $\alpha > 0$, $\gamma > -p$ *and* $-1 \leqslant B < A \leqslant 1$. *If*

$$(1-\alpha)\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\gamma,p}f(z)}{z^p} + \alpha\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)}{z^p} \prec \frac{1+Az}{1+Bz}, \qquad z \in \mathbb{U}, \tag{3.9}$$

then

$$\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\gamma,p}f(z)}{z^p} \prec \tau(z) \prec \frac{1+Az}{1+Bz}, \tag{3.10}$$

where $\mathscr{F}_{\gamma,p}$ *is defined by (1.9) and the function* $\tau(z)$ *is given by*

$$\tau(z) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1\left(1, 1; \frac{\gamma+p}{\alpha}+1; \frac{Bz}{Bz+1}\right) & (B \neq 0); \\ 1 + \frac{\gamma+p}{\gamma+p+\alpha}Az & (B = 0), \end{cases}$$

and $\tau(z)$ *is the best dominant of (3.10). Furthermore*

$$\Re\left\{\left(\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\gamma,p}f(z)}{z^p}\right)^{\frac{1}{m}}\right\} > \{\tau(-1)\}^{\frac{1}{m}}. \tag{3.11}$$

The result (3.11) is the best possible.

Proof. It is follows from (1.6) and (1.9), that

$$z\left(\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\gamma,p}f(z)\right)' = (\gamma+p)\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z) - \gamma\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\gamma,p}f(z), \quad z \in \mathbb{U}. \tag{3.12}$$

Now we assume that

$$q(z) = \frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\gamma,p}f(z)}{z^p}, \tag{3.13}$$

then $q(z)$ is of the form (3.4) and analytic in $\mathbb{U}$ with $q(0) = 1$. Differentiating (3.13) with respect to z and using (3.12), we get

$$\begin{aligned}(1-\alpha)\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\gamma,p}f(z)}{z^p} + \alpha\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)}{z^p} &= q(z) + \frac{\alpha}{p+\lambda}zq'(z) \\ &\prec \frac{1+Az}{1+Bz}, \quad z \in \mathbb{U}.\end{aligned}$$

Now following the same process as of Theorem 1, we get the required result and hence we omit it. □

Putting $m = \alpha = 1$ in Theorem 2 and observing that

$$\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\gamma,p}f(z) = \frac{\gamma+p}{z^\gamma}\int_0^z t^{\gamma-1}\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(t)\,dt, \qquad (f \in \mathscr{A}_p,\, z \in \mathbb{U}),$$

then we get

COROLLARY 3. *Let $f(z)\in\mathscr{A}_p$, $\gamma>-p$ and $-1\leqslant B<A\leqslant 1$. If*

$$\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)}{z^p}\prec\frac{1+Az}{1+Bz}\qquad(\alpha>0,\ z\in\mathbb{U}),$$

then

$$\frac{\gamma+p}{z^\gamma}\int_0^z t^{\gamma-1}\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(t)\,dt\prec\Xi(z)\prec\frac{1+Az}{1+Bz}\quad(z\in\mathbb{U}),\tag{3.14}$$

where the function $\Xi(z)$ is given by

$$\Xi(z)=\begin{cases}\dfrac{A}{B}+\left(1-\dfrac{A}{B}\right)(1+Bz)^{-1}{}_2F_1\left(1,1;\gamma+p+1;\dfrac{Bz}{Bz+1}\right) & (B\neq 0);\\ 1+\dfrac{\gamma+p}{\gamma+p+1}Az & (B=0),\end{cases}$$

and $\Xi(z)$ is the best dominant of (3.14). Furthermore

$$\Re\left(\frac{\gamma+p}{z^\gamma}\int_0^z t^{\gamma-1}\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(t)\right)>\Xi(-1).\tag{3.15}$$

The result (3.15) is the best possible.

A further special case of Corollary 3, when

$$A=1-2\beta\ \ (0\leqslant\beta<1),\ \ B=-1,\ \ p=1\ \text{ and }\ \lambda=\mu=\eta=0,$$

would immediately yields the following result.

COROLLARY 4. *If $f(z)\in\mathscr{A}$ satisfies*

$$\Re\left(\frac{f(z)}{z}\right)>\beta,\qquad(0\leqslant\beta<1,\ z\in\mathbb{U}),$$

then

$$\Re\left(\frac{\gamma+1}{z^{\gamma+1}}\int_0^z t^{\gamma-1}f(t)dt\right)>\beta+(1-\beta)\left\{{}_2F_1\left(1,1;\gamma+2;\frac{1}{2}\right)-1\right\}.\tag{3.16}$$

The result (3.16) is the best possible.

REMARK 2. In [9], it is proved by Obradović that, if $f(z)\in\mathscr{A}$ and

$$\Re\left(\frac{f(z)}{z}\right)>\beta\qquad(0\leqslant\beta<1,\ z\in\mathbb{U}),$$

then

$$\Re\left(\frac{\gamma+1}{z^{\gamma+1}}\int_0^z t^{\gamma-1}f(t)dt\right)>\beta+\frac{1-\beta}{3+2\gamma}\qquad(\gamma>-1,\ z\in\mathbb{U}),\tag{3.17}$$

this shows that the result of Corollary 4 is an improvement of the result (3.17) given in [11].

THEOREM 3. *Let* $f(z) \in \mathscr{A}_p$,

$$\mathscr{Q}_0^{\lambda,\mu,\eta} f(z) = \frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p}, \qquad z \in \mathbb{U}, \tag{3.18}$$

$$\mathscr{Q}_{\nu,k}^{\lambda,\mu,\eta} f(z) = \frac{\nu+1}{z^{\nu+1}} \int_0^z t^\nu \left(\mathscr{Q}_{\nu,k-1}^{\lambda,\mu,\eta} f(t)\right) dt \qquad (\nu > -1,\ k \in \mathbb{N}_0,\ z \in \mathbb{U}), \tag{3.19}$$

where $\mathscr{Q}_{\nu,0}^{\lambda,\mu,\eta} = \mathscr{Q}_0^{\lambda,\mu,\eta}$. *If* $f(z) \in \mathscr{M}^{\lambda,\mu,\eta}(\alpha,A,B)$, *then for* $|z| = r < 1$,

$$\Re\left(\mathscr{Q}_{\nu,k}^{\lambda,\mu,\eta} f(z)\right) \geqslant \rho_k(r) > \rho_k(1) \qquad (k \in \mathbb{N}), \tag{3.20}$$

where

$$0 < \rho_k(r) = 1 + (B-A)(p+\eta-\lambda)(\nu+1)^k \sum_{n=1}^{\infty} \frac{B^{n-1} r^n}{(\alpha n + p + \eta - \lambda)(n+\nu+1)^k} < 1. \tag{3.21}$$

The estimate (3.20) is sharp.

Proof. We shall prove this theorem by the principle of mathematical induction on n. Let $f(z) \in \mathscr{M}^{\lambda,\mu,\eta}(\alpha,A,B)$, then by the Theorem 1, we get

$$\Re\left(\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p}\right) \geqslant \frac{p+\eta-\lambda}{\alpha} \int_0^z u^{\frac{p+\eta-\lambda}{\alpha}-1} \frac{1-Aur}{1-Bur} du \qquad (|z| = r < 1). \tag{3.22}$$

Using the Lemma 3 in (3.22), we obtain

$$\Re\left(\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p}\right) \geqslant \begin{cases} {}_2F_1\left(1, \frac{p+\eta-\lambda}{\alpha}; \frac{p+\eta-\lambda}{\alpha}+1; Br\right) \\ -\frac{p+\eta-\lambda}{\alpha+p+\eta-\lambda} Ar\, {}_2F_1\left(1, \frac{p+\eta-\lambda}{\alpha}+1; \frac{p+\eta-\lambda}{\alpha}+2; Br\right) \\ \hfill (B \neq 0); \\ 1 - \frac{p+\eta-\lambda}{\alpha+p+\eta-\lambda} Ar \\ \hfill (B = 0). \end{cases}$$

Simplifying right hand side of the above estimate, we deduce that

$$\Re\left(\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p}\right) \geqslant \rho_0(r) = \begin{cases} 1 + (B-A)(p+\eta-\lambda) \sum_{n=1}^{\infty} \frac{B^{n-1} r^n}{(\alpha n + p + \eta - \lambda)} \\ \hfill (B \neq 0); \\ 1 - \frac{p+\eta-\lambda}{\alpha+p+\eta-\lambda} Ar \\ \hfill (B = 0), \end{cases}$$

which implies that (3.20) holds true for $n=0$. Again by letting $t=ue^{i\theta}$, we find that

$$\begin{aligned}
\Re\left(\mathscr{Q}_{\nu,1}^{\lambda,\mu,\eta}f(z)\right) &= \Re\left(\frac{\nu+1}{z^{\nu+1}}\int_0^z t^{\nu}\left(\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(t)}{t^p}\right)dt\right)\\
&= \frac{\nu+1}{r^{\nu+1}}\int_0^r u^{\nu}\Re\Big(\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(ue^{i\theta})}{(ue^{i\theta})^p}\Big)du\\
&\geqslant \frac{\nu+1}{r^{\nu+1}}\int_0^r u^{\nu}\Big(1+(B-A)(p+\eta-\lambda)\sum_{n=1}^{\infty}\frac{B^{n-1}u^n}{\alpha n+p+\eta-\lambda}\Big)du\\
&= 1+\frac{\nu+1}{r^{\nu+1}}(B-A)(p+\eta-\lambda)\int_0^r\Big(\sum_{n=1}^{\infty}\frac{B^{n-1}u^{\nu+n}}{\alpha n+p+\eta-\lambda}\Big)du.
\end{aligned}$$

We note that for $|B|\leqslant 1$, $u<1$ and $\alpha n+p+\eta-\lambda\geqslant \alpha+p+\eta-\lambda$ for all $n\geqslant 1$, the series in the right hand side is uniformly convergent in $\mathbb{U}$, so that it can be integrated term by term. Thus, we have

$$\Re\left(\mathscr{Q}_{\nu,1}^{\lambda,\mu,\eta}f(z)\right)\geqslant \rho_1(r)=1+(B-A)(p+\eta-\lambda)(\mu+1)\sum_{n=1}^{\infty}\frac{B^{n-1}r^n}{(\alpha n+p+\eta-\lambda)(n+\nu+1)}$$

and this shows that (3.20) is also true for $n=1$.

Next, we assume that (3.20) holds true for $n=m$. Then, letting $t=ue^{i\theta}$, we have

$$\begin{aligned}
&\Re\left(\mathscr{Q}_{\nu,m+1}^{\lambda,\mu,\eta}f(z)\right)\\
&= \Re\left(\frac{\nu+1}{z^{\nu+1}}\int_0^z t^{\nu}\left(\mathscr{Q}_{\nu,m}^{\lambda,\mu,\eta}f(t)\right)dt\right)\\
&= \frac{\nu+1}{r^{\nu+1}}\int_0^r u^{\nu}\Re\left(\mathscr{Q}_{\nu,m}^{\lambda,\mu,\eta}f(ue^{i\theta})\right)du\\
&\geqslant \frac{\nu+1}{r^{\nu+1}}\int_0^r u^{\nu}\left(1+(B-A)(p+\eta-\lambda)\sum_{n=1}^{\infty}\frac{(\nu+1)^m B^{n-1}u^n}{(\alpha n+p+\eta-\lambda)(n+\nu+1)^m}\right)du\\
&= 1+\frac{(B-A)(p+\eta-\lambda)(\nu+1)^{m+1}}{r^{\nu+1}}\int_0^r\Big(\sum_{n=1}^{\infty}\frac{B^{n-1}u^{\nu+n}}{(\alpha n+p+\eta-\lambda)(n+\nu+1)^m}\Big)du.
\end{aligned}$$

Noting that the integrand in the right hand side is uniformly convergent in $\mathbb{U}$, we deduce that

$$\begin{aligned}
\Re\left(\mathscr{Q}_{\nu,m+1}^{\lambda,\mu,\eta}f(z)\right) &\geqslant \rho_{m+1}(r)\\
&= 1+(B-A)(p+\eta-\lambda)(\nu+1)^{m+1}\sum_{n=1}^{\infty}\frac{B^{n-1}r^n}{(\alpha n+p+\eta-\lambda)(n+\nu+1)^{m+1}}.
\end{aligned}$$

Therefore, we conclude that

$$\Re\left(\mathscr{Q}_{\nu,k}^{\lambda,\mu,\eta}f(z)\right)\geqslant \rho_n(r),$$

for any integer $k \in \mathbb{N}_0$.

Finally to prove sharpness of the result (3.26), let us consider the function

$$\mathscr{G}_k(r) = 1 + (B-A)(p+\eta-\lambda)(\nu+1)^k \sum_{n=1}^{\infty} \frac{B^{n-1} r^n}{(\alpha n + p + \eta - \lambda)(n+\nu+1)^k}, \quad 0 < r < 1.$$

The series $\mathscr{G}_k(r)$ is absolutely and uniformly convergent for each $k \in \mathbb{N}_0$ and $0 < r < 1$. By suitably rearranging the terms of $\mathscr{G}_k(r)$, it is easy to see that $0 < \mathscr{G}_k(r) < 1$. Further, since $\mathscr{G}_k(r) \leqslant \mathscr{G}_{k-1}(r)$ and

$$r^{\nu+1}\mathscr{G}_k(r) = (\nu+1)\int_0^r u^\nu \mathscr{G}_{k-1}(u)\, du, \quad k \in \mathbb{N},$$

we have that $\mathscr{G}_k'(r) \leqslant 0$ and $\mathscr{G}_k(r)$ decreasing with r as $r \to 1^-$ for fixed k and increases to 1 as $k \to \infty$ for fixed r. This implies that $\mathscr{G}_k(r) > \mathscr{G}_k(1)$. Therefore the estimate in (3.20) is sharp. This completes the proof of theorem. □

Setting

$$A = 1 - \frac{2\beta}{p} \quad (0 \leqslant \beta < p) \qquad \text{and} \qquad B = -1,$$

in Theorem 3, we have the following result.

COROLLARY 5. *If $f(z) \in \widetilde{\mathscr{M}}^{\lambda,\mu,\eta}(\alpha,\beta)$, then*

$$\Re\left(\mathscr{Q}_{\nu,k}^{\lambda,\mu,\eta} f(z)\right) \geqslant \rho_n^*(r) > \rho_n^*(1) \qquad (n \in \mathbb{N}_0, \ |z| = r < 1),$$

where

$$0 < \rho_n^*(r) = 1 - \frac{2(p-\beta)(p+\eta-\lambda)(\nu+1)^k}{p} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} r^n}{(\alpha n + p + \eta - \lambda)(n+\nu+1)^k} < 1.$$

The result is sharp.

Furthermore on taking $\alpha = k = 1$ and $\lambda = \mu = \eta = 1$, in Theorem 3, we have the following result.

COROLLARY 6. *If $f(z) \in \mathscr{A}_p$ such that*

$$\frac{f'(z)}{z^{p-1}} \prec p\frac{1+Az}{1+Bz}, \qquad z \in \mathbb{U},$$

then

$$\Re\left(\frac{\nu+1}{z^{\nu+1}}\int_0^z t^{\nu-p} f(t)\, dt\right) \geqslant \rho, \qquad z \in \mathbb{U},$$

where

$$\rho = 1 + (B-A)(\nu+1)p \sum_{n=1}^{\infty} \frac{(B)^{n-1} r^n}{(p+n)(n+\mu+1)}.$$

The result is sharp.

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