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A NEW SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS ASSOCIATED WITH FRACTIONAL CALCULUS OPERATOR

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Abstract. The purpose of the present paper is to study a new subclass of harmonic univalent functions associated with fractional calculus operator. We obtain coefficient conditions, distortion bounds and extreme points for this class and discuss a class preserving integral operator. We also show that the class studied in this paper is closed under convolution and convex combination. The results obtained for the class reduce to the corresponding results for several known classes in the literature are briefly indicated.

1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply-connected domain D is said to be harmonic in D if both u and v are harmonic in D. In any simply-connected domain D we can write $f = h + \bar{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$, $z \in D$. See Clunie and Sheil-Small [2].

Denote by S_H the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense-preserving in the open unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1}$$

The class S_H reduces to class S of normalized analytic univalent functions if the co-analytic part of f i.e. $g \equiv 0$. For this class $f(z)$ may be expressed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{2}$$

For more basic results on harmonic univalent functions one may refer to the following introductory text book by Duren [6], (see also [1], [9]).

The following definitions of fractional derivatives are due to Owa [7] and Srivastava and Owa [12].

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DEFINITION 1. The fractional derivative of order λ is defined for a function $f(z)$ of the form (2) by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}} d\xi,$$

where $0 \leqslant \lambda < 1$, $f(z)$ is an analytic functions in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi) > 0$.

DEFINITION 2. Under the hypothesis of Definition 1 the fractional derivative of order $n+\lambda$ is defined for a function $f(z)$ by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z), \tag{3}$$

where $0 \leqslant \lambda < 1$ and $n \in N_0 = \{0, 1, 2 \ldots\}$.

Using the Definition 1 and its known extension involving fractional derivatives Owa and Srivastava [8] introduced the operator $\Omega^{\lambda} : A \to A$ as follows

$$\Omega^{\lambda} f(z) = \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z), \ (\lambda \neq 2, 3, 4, \ldots), \tag{4}$$

where A denote the class of functions of form (2) which are analytic in U.

Recently, Porwal and Dixit [10] (see also [5]), defined the subclass $M_H(\beta) \subset S_H$ consisting of harmonic univalent functions $f(z)$ of the form (1) satisfying the following condition

$$M_H(\beta) = \left\{ f \in S_H : \Re\left(\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} \right) < \beta \right\}, \left(1 < \beta \leqslant 4/3\right) \text{ and } z \in U. \tag{5}$$

For $g \equiv 0$ the class of $M_H(\beta)$ reduce to the class $M(\beta)$ studied by Uralegaddi *et al.* [13].

Generalizing the class $M_H(\beta)$, we let $M_H(\lambda, \beta, t)$, denote the family of functions $f = h + \overline{g}$ of form (1) which satisfy the condition

$$\Re\left\{ \frac{z\left(\Omega^{\lambda} h(z)\right)' - \overline{z\left(\Omega^{\lambda} g(z)\right)'}}{(1-t)z + t\left(\Omega^{\lambda} h(z) + \overline{\Omega^{\lambda} g(z)}\right)} \right\} < \beta, \tag{6}$$

where $1 < \beta \leqslant 4/3$, $0 \leqslant \lambda < 1$ and $0 \leqslant t \leqslant 1$.

Further, let $VM_H(\lambda, \beta, t)$ be the subclass of $M_H(\lambda, \beta, t)$ consisting of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} |a_k| \ z^k - \sum_{k=1}^{\infty} |b_k| \ \overline{z}^k. \tag{7}$$

In this paper, we give a sufficient coefficient condition for $f = h + \overline{g}$, given by (1) to be in $M_H(\lambda, \beta, t)$ and it is shown that this condition is also necessary for functions of the form (7) in $VM_H(\lambda, \beta, t)$. We then obtain distortion theorem, extreme points for this class. We also show that the class $VM_H(\lambda, \beta, t)$ is closed under convolution and convex combination. Finally, we discuss a class preserving integral operator for functions in $VM_H(\lambda, \beta, t)$.

2. Main results

First, we give a sufficient coefficient bound for the class $M_H(\lambda,\beta,t)$.

THEOREM 1. *If* $f = h + \overline{g} \in S_H$ *be given by* (1). *If*

$$\sum_{k=2}^{\infty} \frac{k-\beta t}{\beta-1}\phi(k,\lambda)|a_k| + \sum_{k=1}^{\infty} \frac{k+\beta t}{\beta-1}\phi(k,\lambda)|b_k| \leqslant 1, \tag{8}$$

where

$$1 < \beta \leqslant 4/3,\ \ 0 \leqslant \lambda < 1,\ \ 0 \leqslant t \leqslant 1 \text{ and } \phi(k,\lambda) = \frac{\Gamma(k+1)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)},$$

then $f \in M_H(\lambda,\beta,t)$.

Proof. Let $\sum_{k=2}^{\infty} \frac{k-\beta t}{\beta-1}\phi(k,\lambda)|a_k| + \sum_{k=1}^{\infty} \frac{k+\beta t}{\beta-1}\phi(k,\lambda)|b_k| \leqslant 1$.
It suffices to show that

$$\left| \frac{\frac{z(\Omega^\lambda h(z))' - \overline{z(\Omega^\lambda g(z))'}}{(1-t)z + t\left(\Omega^\lambda h(z) + \overline{\Omega^\lambda g(z)}\right)} - 1}{\frac{z(\Omega^\lambda h(z))' - \overline{z(\Omega^\lambda g(z))'}}{(1-t)z + t\left(\Omega^\lambda h(z) + \overline{\Omega^\lambda g(z)}\right)} - (2\beta-1)} \right| < 1, \quad z \in U.$$

We have

$$\left| \frac{\frac{z(\Omega^\lambda h(z))' - \overline{z(\Omega^\lambda g(z))'}}{(1-t)z + t\left(\Omega^\lambda h(z) + \overline{\Omega^\lambda g(z)}\right)} - 1}{\frac{z(\Omega^\lambda h(z))' - \overline{z(\Omega^\lambda g(z))'}}{(1-t)z + t\left(\Omega^\lambda h(z) + \overline{\Omega^\lambda g(z)}\right)} - (2\beta-1)} \right|$$

$$\leqslant \frac{\sum_{k=2}^{\infty}(k-t)\phi(k,\lambda)|a_k||z|^{k-1} + \sum_{k=1}^{\infty}(k+t)\phi(k,\lambda)|b_k||z|^{k-1}}{2(\beta-1) - \sum_{k=2}^{\infty}(k-2\beta t+t)\phi(k,\lambda)|a_k||z|^{k-1} - \sum_{k=1}^{\infty}(k+2\beta t-t)\phi(k,\lambda)|b_k||z|^{k-1}}$$

$$\leqslant \frac{\sum_{k=2}^{\infty}(k-t)\phi(k,\lambda)|a_k| + \sum_{k=1}^{\infty}(k+t)\phi(k,\lambda)|b_k|}{2(\beta-1) - \sum_{k=2}^{\infty}(k-2\beta t+t)\phi(k,\lambda)|a_k| - \sum_{k=1}^{\infty}(k+2\beta t-t)\phi(k,\lambda)|b_k|}.$$

The last expression is bounded above by 1, if

$$\sum_{k=2}^{\infty}(k-t)\phi(k,\lambda)|a_k| + \sum_{k=1}^{\infty}(k+t)\phi(k,\lambda)|b_k|$$
$$\leqslant 2(\beta-1) - \sum_{k=2}^{\infty}(k-2\beta t+t)\phi(k,\lambda)|a_k| - \sum_{k=1}^{\infty}(k+2\beta t-t)\phi(k,\lambda)|b_k|,$$

which is equivalent to

$$\sum_{k=2}^{\infty}\frac{(k-\beta t)}{(\beta-1)}\phi(k,\lambda)\,|a_k|+\sum_{k=1}^{\infty}\frac{(k+\beta t)}{(\beta-1)}\phi(k,\lambda)\,|b_k|\leqslant 1. \tag{9}$$

But (9) is true by hypothesis.

Hence $\left|\dfrac{\frac{z\left(\Omega^{\lambda}h(z)\right)'-\overline{z\left(\Omega^{\lambda}g(z)\right)'}}{(1-t)z+t\left(\Omega^{\lambda}h(z)+\overline{\Omega^{\lambda}g(z)}\right)}-1}{\frac{z\left(\Omega^{\lambda}h(z)\right)'-\overline{z\left(\Omega^{\lambda}g(z)\right)'}}{(1-t)z+t\left(\Omega^{\lambda}h(z)+\overline{\Omega^{\lambda}g(z)}\right)}-(2\beta-1)}\right|<1,\ z\in U$, and the theorem is proved.

The harmonic univalent functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}\frac{(\beta-1)}{(k-\beta t)\,\phi(k,\lambda)}x_kz^k+\sum_{k=1}^{\infty}\frac{(\beta-1)}{(k+\beta t)\,\phi(k,\lambda)}\overline{y_kz^k}, \tag{10}$$

where $1<\beta\leqslant 4/3,\ 0\leqslant\lambda<1,\ 0\leqslant t\leqslant 1$ and $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$, show that the coefficient bound given by (8) is sharp. It is worthy to note that the function of the form (10) belongs to the class $R_H(n,\beta,\lambda)$ for all $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|\leqslant 1$ because coefficient inequality (8) holds. □

In the following theorem, it is proved that the condition (8) is also necessary for functions $f=h+\overline{g}\in VM_H(\lambda,\beta,t)$ to be given by (7).

THEOREM 2. *A function $f(z)$ of the form (7) is in $VM_H(\lambda,\beta,t)$, if and only if*

$$\sum_{k=2}^{\infty}(k-\beta t)\,\phi(k,\lambda)\,|a_k|+\sum_{k=1}^{\infty}(k+\beta t)\,\phi(k,\lambda)\,|b_k|\leqslant(\beta-1). \tag{11}$$

Proof. Since $VM_H(\lambda,\beta,t)\subset M_H(\lambda,\beta,t)$, we only need to prove the "only if" part of the theorem. For this we show that $f\notin VM_H(\lambda,\beta,t)$ if the condition (11) does not hold.

Note that a necessary and sufficient condition for $f=h+\overline{g}$ given by (7) is in $VM_H(\lambda,\beta,t)$ if

$$\Re\left\{\frac{z\left(\Omega^{\lambda}h(z)\right)'-\overline{z\left(\Omega^{\lambda}g(z)\right)'}}{(1-t)z+t\left(\Omega^{\lambda}h(z)+\overline{\Omega^{\lambda}g(z)}\right)}\right\}<\beta,$$

which is equivalent to

$$\Re\left\{\frac{(\beta-1)z-\sum_{k=2}^{\infty}(k-\beta t)\,\phi(k,\lambda)\,|a_k|\,z^k-\sum_{k=1}^{\infty}(k+\beta t)\,\phi(k,\lambda)\,|b_k|\,\overline{z}^k}{z+\sum_{k=2}^{\infty}\phi(k,\lambda)t\,|a_k|\,z^k-\sum_{k=1}^{\infty}\phi(k,\lambda)t\,|b_k|\,\overline{z}^k}\right\}\geqslant 0.$$

The above condition must hold for all values of z, $|z| = r < 1$. Upon choosing the values of z on the positive real axis where $0 \leqslant z = r < 1$, we must have

$$\frac{(\beta-1) - \sum_{k=2}^{\infty}(k-\beta t)\,\phi(k,\lambda)\,|a_k|\,r^{k-1} - \sum_{k=1}^{\infty}(k+\beta t)\,\phi(k,\lambda)\,|b_k|\,r^{k-1}}{1+\sum_{k=2}^{\infty}\phi(k,\lambda)t\,|a_k|\,r^{k-1} - \sum_{k=1}^{\infty}\phi(k,\lambda)t\,|b_k|\,r^{k-1}} \geqslant 0. \tag{12}$$

If the condition (11) does not hold then the numerator of (12) is negative for r sufficiently close to 1. Thus there exists a $z_0 = r_0$ in (0,1) for which the quotient in (12) is negative. This contradicts the required condition for $f \in VM_H(\lambda,\beta,t)$ and so the proof is complete.

The harmonic univalent functions of the form

$$f(z) = z + \sum_{k=2}^{\infty}\frac{(\beta-1)}{(k-\beta t)\,\phi(k,\lambda)}x_k z^k + \sum_{k=1}^{\infty}\frac{(\beta-1)}{(k+\beta t)\,\phi(k,\lambda)}y_k \overline{z^k}, \tag{13}$$

where $1 < \beta \leqslant 4/3$, $0 \leqslant \lambda < 1$, $0 \leqslant t \leqslant 1$, $x_k \geqslant 0$, $y_k \geqslant 0$ and $\sum_{k=2}^{\infty} x_k + \sum_{k=1}^{\infty} y_k \leqslant 1$ belongs to the class $VM_H(\lambda,\beta,t)$. □

Next, we determine the extreme points of the closed convex hulls of $VM_H(\lambda,\beta,t)$, denoted by $clcoVM_H(\lambda,\beta,t)$. For this we recall the definition of extreme points.

DEFINITION 3. A function f in a family G is said to be an extreme point of G if f can not be expressed as a proper convex combination of two distinct functions in G.

THEOREM 3. *$f \in clcoVM_H(\lambda,\beta,t)$, if and only if*

$$f(z) = \sum_{k=1}^{\infty}(x_k h_k(z) + y_k g_k(z)), \tag{14}$$

where

$$\begin{aligned} h_1(z) &= z, \\ h_k(z) &= z + \frac{(\beta-1)}{(k-\beta t)\,\phi(k,\lambda)}z^k, \quad k = (2,3,\ldots), \\ g_k(z) &= z - \frac{(\beta-1)}{(k+\beta t)\,\phi(k,\lambda)}\overline{z}^k, \quad k = (1,2,3,\ldots), \qquad \sum_{k=1}^{\infty}(x_k+y_k) = 1, \end{aligned}$$

$x_k \geqslant 0$ and $y_k \geqslant 0$. In particular the extreme points of $VM_H(\lambda,\beta,t)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. For functions f of the form (14), we have

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty}(x_k h_k(z) + y_k g_k(z)) \\ &= z + \sum_{k=2}^{\infty}\frac{(\beta-1)}{(k-\beta t)\phi(k,\lambda)}x_k z^k - \sum_{k=1}^{\infty}\frac{(\beta-1)}{(k+\beta t)\phi(k,\lambda)}y_k \overline{z}^k. \end{aligned}$$

Then

$$\sum_{k=2}^{\infty}\frac{(k-\beta t)}{(\beta-1)}\phi(k,\lambda)\left(\frac{(\beta-1)}{(k-\beta t)\phi(k,\lambda)}x_k\right)+\sum_{k=1}^{\infty}\frac{(k+\beta t)}{(\beta-1)}\phi(k,\lambda)\left(\frac{(\beta-1)}{(k+\beta t)\phi(k,\lambda)}y_k\right)$$

$$=\sum_{k=2}^{\infty}x_k+\sum_{k=1}^{\infty}y_k$$

$$=1-x_1\leqslant 1, \text{ and so } f\in \operatorname{clco}VM_H(\lambda,\beta,t).$$

Conversely, suppose that $f\in \operatorname{clco}VM_H(\lambda,\beta,t)$. Set $x_k=\frac{(k-\beta t)\phi(k,\lambda)}{(\beta-1)}|a_k|$, $(k=2,3,4\ldots)$ and $y_k=\frac{(k+\beta t)\phi(k,\lambda)}{(\beta-1)}|b_k|$, $(k=1,2,3\ldots)$. Then note that by Theorem 2, $0\leqslant x_k\leqslant 1$, $(k=2,3,4\ldots)$ and $0\leqslant y_k\leqslant 1$, $(k=1,2,3\ldots)$. We define $x_1=1-\sum_{k=2}^{\infty}x_k-\sum_{k=1}^{\infty}y_k$, and by Theorem 2, $x_1\geqslant 0$.

Consequently, we obtain $f(z)=\sum_{k=1}^{\infty}(x_kh_k(z)+y_kg_k(z))$ as required. □

In next theorem, we obtain maximum and minimum value of $|f(z)|$ in $|z|=r<1$.

THEOREM 4. *If* $f\in VM_H(\lambda,\beta,t)$ *then*

$$|f(z)|\leqslant(1+|b_1|)r+\frac{2-\lambda}{2}\left(\frac{(\beta-1)}{(2-\beta t)}-\frac{(\beta t+1)}{(2-\beta t)}|b_1|\right)r^2,\qquad |z|=r<1$$

and

$$|f(z)|\geqslant(1-|b_1|)r-\frac{2-\lambda}{2}\left(\frac{(\beta-1)}{(2-\beta t)}-\frac{(\beta t+1)}{(2-\beta t)}|b_1|\right)r^2,\qquad |z|=r<1.$$

Proof. We only prove the right hand inequality. The proof for left hand inequality is similar and will be omitted. Let $f\in VM_H(\lambda,\beta,t)$. Taking the absolute value of f, we have

$$\begin{aligned}
|f(z)|&\leqslant(1+|b_1|)r+\sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k\\
&\leqslant(1+|b_1|)r+\sum_{k=2}^{\infty}(|a_k|+|b_k|)r^2\\
&=(1+|b_1|)r+\frac{(\beta-1)}{(2-\beta t)\phi(2,\lambda)}\sum_{k=2}^{\infty}\left(\frac{(2-\beta t)\phi(2,\lambda)}{(\beta-1)}|a_k|+\frac{(2-\beta t)\phi(2,\lambda)}{(\beta-1)}|b_k|\right)r^2\\
&\leqslant(1+|b_1|)r+\frac{(\beta-1)(2-\lambda)}{2(2-\beta t)}\sum_{k=2}^{\infty}\left(\frac{(k-\beta t)}{(\beta-1)}\phi(k,\lambda)|a_k|+\frac{(k+\beta t)}{(\beta-1)}\phi(k,\lambda)|b_k|\right)r^2\\
&\leqslant(1+|b_1|)r+\frac{(\beta-1)(2-\lambda)}{2(2-\beta t)}\left(1-\frac{(\beta t+1)}{(\beta-1)}|b_1|\right)r^2\\
&=(1+|b_1|)r+\frac{(2-\lambda)}{2}\left(\frac{(\beta-1)}{(2-\beta t)}-\frac{(\beta t+1)}{(2-\beta t)}|b_1|\right)r^2.
\end{aligned}$$

The result is sharp for the function

$$f(z)=z-|b_1|\overline{z}+\frac{(2-\lambda)}{2}\left(\frac{(\beta-1)}{(2-\beta t)}-\frac{(\beta t+1)}{(2-\beta t)}|b_1|\right)z^2. \quad \square$$

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}|a_k|\,z^k-\sum_{k=1}^{\infty}|b_k|\,\overline{z}^k$$

and

$$F(z)=z+\sum_{k=2}^{\infty}|A_k|\,z^k-\sum_{k=1}^{\infty}|B_k|\,\overline{z}^k$$

we define the convolution of two harmonic functions f and F as

$$(f*F)(z)=f(z)*F(z)=z+\sum_{k=2}^{\infty}|a_kA_k|\,z^k-\sum_{k=1}^{\infty}|b_kB_k|\,\overline{z}^k. \tag{15}$$

The computation of (15) arises from the formula (see [11])

$$\begin{aligned}(f*F)(r^2e^{i\theta})&=\frac{1}{2\pi}\int_0^{2\pi}h(re^{i(\theta-t)})H(re^{it})dt-\overline{\frac{1}{2\pi}\int_0^{2\pi}g(re^{i(\theta-t)})G(re^{it})dt}\\&=\frac{1}{2\pi}\int_0^{2\pi}\left\{\sum_{k=1}^{\infty}|a_k|r^ke^{ik(\theta-t)}\right\}\left\{\sum_{m=1}^{\infty}|A_m|r^me^{imt}\right\}dt\\&\quad-\overline{\frac{1}{2\pi}\int_0^{2\pi}\left\{\sum_{k=1}^{\infty}|b_k|r^ke^{ik(\theta-t)}\right\}\left\{\sum_{m=1}^{\infty}|B_m|r^me^{imt}\right\}dt}\\&\quad\text{where } a_1=A_1=1\\&=\frac{1}{2\pi}\left[\int_0^{2\pi}\left\{\sum_{k=1}^{\infty}|a_kA_k|r^{2k}e^{ik\theta}+\sum_{\substack{k,m=1\\k\neq m}}^{\infty}|a_kA_m|r^{k+m}e^{i(k\theta-kt+mt)}\right\}dt\right]\\&\quad-\overline{\frac{1}{2\pi}\left[\int_0^{2\pi}\left\{\sum_{k=1}^{\infty}|b_kB_k|r^{2k}e^{ik\theta}+\sum_{\substack{k,m=1\\k\neq m}}^{\infty}|b_kB_m|r^{k+m}e^{i(k\theta-kt+mt)}\right\}dt\right]}\\&=\sum_{k=1}^{\infty}|a_kA_k|r^{2k}e^{ik\theta}-\overline{\sum_{k=1}^{\infty}|b_kB_k|r^{2k}e^{ik\theta}}.\end{aligned}$$

Equivalently

$$(f_1*f_2)(z)=z+\sum_{k=2}^{\infty}|a_kA_k|z^k-\sum_{k=1}^{\infty}|b_kB_k|\overline{z^k},\ \ |z|<R=r^2.$$

Using this definition, we show that the class $VM_H(\lambda,\beta,t)$ is closed under convolution.

THEOREM 5. *For* $1<\alpha\leqslant\beta\leqslant 4/3$, *let* $f\in VM_H(\lambda,\alpha,t)$ *and* $F\in VM_H(\lambda,\beta,t)$. *Then*

$$(f*F)(z)\in VM_H(\lambda,\alpha,t)\subseteq VM_H(\lambda,\beta,t).$$

Proof. Let $f(z)=z+\sum_{k=2}^{\infty}|a_k|\,z^k-\sum_{k=1}^{\infty}|b_k|\,\bar{z}^k$ be in $VM_H(\lambda,\alpha,t)$ and $F(z)=z+\sum_{k=2}^{\infty}|A_k|\,z^k-\sum_{k=1}^{\infty}|B_k|\,\bar{z}^k$ be in $VM_H(\lambda,\beta,t)$.

Then the convolution $(f*F)(z)$ is given by (15). We wish to show that the coefficients of $(f*F)(z)$ satisfy the required condition given in Theorem 2. For $F(z)\in VM_H(\lambda,\beta,t)$ we note that $|A_k|\leqslant 1$ and $|B_k|\leqslant 1$. Now, for the convolution function $(f*F)(z)$, we obtain

$$\begin{aligned}&\sum_{k=2}^{\infty}\frac{(k-\alpha t)}{(\alpha-1)}\phi(k,\lambda)|a_kA_k|+\sum_{k=1}^{\infty}\frac{(k+\alpha t)}{(\alpha-1)}\phi(k,\lambda)|b_kB_k|\\ &\leqslant\sum_{k=2}^{\infty}\frac{(k-\alpha t)}{(\alpha-1)}\phi(k,\lambda)|a_k|+\sum_{k=1}^{\infty}\frac{(k+\alpha t)}{(\alpha-1)}\phi(k,\lambda)|b_k|\\ &\leqslant 1,\quad(\text{since } f\in VM_H(\lambda,\alpha,t)).\end{aligned}$$

Therefore $f*F\in VM_H(\lambda,\alpha,t)\subseteq VM_H(\lambda,\beta,t)$. □

THEOREM 6. *The class* $VM_H(\lambda,\beta,t)$ *is closed under convex combination.*

Proof. For $i=1,2,3\ldots$, let $f_i(z)\in VM_H(\lambda,\beta,t)$, where $f_i(z)$ is given by

$$f_i(z)=z+\sum_{k=2}^{\infty}\left|a_{k_i}\right|\,z^k-\sum_{k=1}^{\infty}\left|b_{k_i}\right|\,\bar{z}^k.$$

Then by Theorem 2,

$$\sum_{k=2}^{\infty}\frac{(k-\beta t)}{(\beta-1)}\phi(k,\lambda)\left|a_{k_i}\right|+\sum_{k=1}^{\infty}\frac{(k+\beta t)}{(\beta-1)}\phi(k,\lambda)\left|b_{k_i}\right|\leqslant 1. \tag{16}$$

For $\sum_{i=1}^{\infty}t_i=1$, $0\leqslant t_i\leqslant 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty}t_if_i(z)=z+\sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty}t_i\left|a_{k_i}\right|\right)z^k-\sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty}t_i\left|b_{k_i}\right|\right)\bar{z}^k.$$

Then by (16)

$$\sum_{k=2}^{\infty}\frac{(k-\beta t)}{(\beta-1)}\phi(k,\lambda)\left(\sum_{i=1}^{\infty}t_i\left|a_{k_i}\right|\right)+\sum_{k=1}^{\infty}\frac{(k+\beta t)}{(\beta-1)}\phi(k,\lambda)\left(\sum_{i=1}^{\infty}t_i\left|b_{k_i}\right|\right)$$
$$=\sum_{i=1}^{\infty}t_i\left(\sum_{k=2}^{\infty}\frac{(k-\beta t)}{(\beta-1)}\phi(k,\lambda)\left|a_{k_i}\right|+\sum_{k=1}^{\infty}\frac{(k+\beta t)}{(\beta-1)}\phi(k,\lambda)\left|b_{k_i}\right|\right)$$
$$\leqslant\sum_{i=1}^{\infty}t_i=1.$$

This is the condition required by Theorem 2 and so $\sum_{i=1}^{\infty}t_if_i(z)\in VM_H(\lambda,\beta,t)$. □

3. A family of class preserving integral operator

Let $f(z)=h(z)+\overline{g(z)}$ be defined by (1), then $F(z)$ defined by the relation

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}h(t)\,dt+\overline{\frac{c+1}{z^c}\int_0^z t^{c-1}g(t)\,dt},(c>-1). \tag{17}$$

THEOREM 7. *Let $f(z)=h(z)+\overline{g(z)}\in S_H$ be given by (7) and $f\in VM_H(\lambda,\beta,t)$ then $F(z)$ be defined by (17) also belong to $VM_H(\lambda,\beta,t)$.*

Proof. Let $f(z)=z+\sum_{k=2}^{\infty}|a_k|z^k-\sum_{k=1}^{\infty}|b_k|\bar{z}^k$ be in $VM_H(\lambda,\beta,t)$, then by Theorem 2, we have

$$\sum_{k=2}^{\infty}\frac{(k-\beta t)}{(\beta-1)}\phi(k,\lambda)|a_k|+\sum_{k=1}^{\infty}\frac{(k+\beta t)}{(\beta-1)}\phi(k,\lambda)|b_k|\leqslant 1.$$

By definition of $F(z)$, we have

$$F(z)=z+\sum_{k=2}^{\infty}\frac{(c+1)}{(c+k)}|a_k|z^k-\sum_{k=1}^{\infty}\frac{(c+1)}{(c+k)}|b_k|\bar{z}^k.$$

Now

$$\sum_{k=2}^{\infty}\frac{(k-\beta t)}{(\beta-1)}\phi(k,\lambda)\left(\frac{(c+1)}{(c+k)}|a_k|\right)+\sum_{k=1}^{\infty}\frac{(k+\beta t)}{(\beta-1)}\phi(k,\lambda)\left(\frac{(c+1)}{(c+k)}|b_k|\right)$$
$$\leqslant\sum_{k=2}^{\infty}\frac{(k-\beta t)}{(\beta-1)}\phi(k,\lambda)|a_k|+\sum_{k=1}^{\infty}\frac{(k+\beta t)}{(\beta-1)}\phi(k,\lambda)|b_k|$$
$$\leqslant 1.$$

Thus $F(z)\in VM_H(\lambda,\beta,t)$. □

REMARK 1. If we put $\lambda = 0$, $t = 1$ in Theorems 1–7, then we obtain corresponding results of Porwal and Dixit [10].

REMARK 2. If we put $g \equiv 0$, $t = 1$ in Theorems 1–7, then we obtain corresponding results of Dixit and Pathak [3].

REMARK 3. If we put $g \equiv 0$, $t = 0$ in Theorems 1–7, then we obtain corresponding results of Dixit and Pathak [4].

REMARK 4. If we put $g \equiv 0$, $t = 1$ with $\lambda = 0$ in Theorems 1–7, then we obtain corresponding results of Uralegaddi *et al.* [13].

REMARK 5. If we put $g \equiv 0$, $t = 0$ with $\lambda = 0$ in Theorems 1–7, then we obtain corresponding results of Uralegaddi *et al.* [14].

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