Fractional
Differential
Calculus
Volume 5, Number 1 (2015), 25–42
doi:10.7153/fdc-05-03

AN OPIAL–TYPE INTEGRAL INEQUALITY AND EXPONENTIALLY CONVEX FUNCTIONS

MAJA ANDRIĆ, ANA BARBIR, SAJID IQBAL AND JOSIP PEČARIĆ

Abstract. In this paper a certain class of convex functions in an Opial-type integral inequality is considered. Cauchy type mean value theorems are proved and used in studying Stolarsky type means defined by the observed integral inequality. Also, a method of producing n-exponentially convex and exponentially convex functions is applied. Some new Opial-type inequalities are given for different types of fractional integrals and fractional derivatives as applications.

1. Introduction and preliminaries

We consider a particular class of convex functions in an Opial-type integral inequality from which we construct functionals Φ_i $(i=1,2)$. Our object is to give Cauchy type mean value theorems and use them for Stolarsky type means, all defined by the observed integral inequality, and also, to prove the n-exponential convexity for the functionals. Also we produce new Opial-type inequalities for fractional integrals (of function with respect to an increasing function, the Riemann–Liouville and the Hadamard fractional integrals) and fractional derivatives (the Riemann–Liouville, the Canavati and the Caputo fractional derivatives).

We say that a function $u:[a,b]\to\mathbb{R}$ belongs to the class $U(v,K)$ if it admits the representation

$$u(x)=\int_a^x K(x,t)\,v(t)\,dt\,, \tag{1}$$

where v is a continuous function and K is an arbitrary nonnegative kernel such that $v(x)>0$ implies $u(x)>0$ for every $x\in[a,b]$. We also assume that all integrals under consideration exist and are finite.

The following inequality is given by Mitrinović and Pečarić in [9] (also see [1, p. 89] and [11, p. 236]).

THEOREM 1.1. *Let $u_1\in U(v_1,K)$, $u_2\in U(v_2,K)$ and $v_2(x)>0$ for every $x\in[a,b]$. Further, let $\phi(u)$ and $f(u)$ be convex and increasing for $u\geqslant 0$ and $f(0)=0$. If*

Mathematics subject classification (2010): 26D10, 26D15, 26A33.
Keywords and phrases: Cauchy mean value theorems, exponential convexity, Stolarsky means, fractional integral, fractional derivative.

Paper FDC-05-03

f is a differentiable function and $M = \max K(x,t)$, then

$$M \int_a^b v_2(t)\,\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right) f'\left(u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right)\right) dt$$
$$\leqslant f\left(M \int_a^b v_2(t)\,\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right) dt\right). \tag{2}$$

The rest of the paper is organized in the following way: in Section 2 we give some new generalized Opial-type inequalities from which we construct functionals and prove Cauchy type mean value theorems. Next, in Section 3, we prove some new Opial-type inequalities for fractional integrals and fractional derivatives as an application of our main results. In Section 4 we produce the n-exponentially convex functions by applying an elegant method of exponential convexity. At the end of the paper, we use Cauchy mean value theorems for Stolarsky type means defined by the observed functionals to give the related examples (see Section 5).

2. Main results

In the following theorem we give the generalization of the inequality (2).

THEOREM 2.1. *Let $u_1 \in U(v_1,K)$, $u_2 \in U(v_2,K)$ and $v_2(x) > 0$ for every $x \in [a,b]$. Further, let $\phi(u)$ be convex, nonnegative and increasing for $u \geqslant 0$, $f(u)$ be convex for $u \geqslant 0$, and $f(0) = 0$. If f is a differentiable function and $M = \max K(x,t)$, then these inequalities are valid:*

$$M \int_a^b v_2(t)\,\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right) f'\left(u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right)\right) dt$$
$$\leqslant f\left(M \int_a^b v_2(t)\,\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right) dt\right) \tag{3}$$
$$\leqslant \frac{1}{b-a} \int_a^b f\left(M(b-a)v_2(t)\,\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right)\right) dt. \tag{4}$$

Proof. On the right hand side of the inequality (3), if we multiply and divide by the factor $(b-a)$ inside and outside the integral and use Jensen's inequality for the function f, then we obtain the inequality (4).

The condition of Theorem 1.1 that the function f is increasing is actually unneeded. From the proof of the theorem [11, p. 236] one can see that this property is never used, therefore we omit it here. Also, a condition that is missing in Theorem 1.1 is that ϕ has to be nonnegative, which we add. □

Motivated by the inequalities given in Theorem 2.1, we define two functionals as:

$$\begin{aligned}\Phi_1(f) &= f\left(M\int_a^b v_2(t)\,\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right)dt\right)\\ &\quad - M\int_a^b v_2(t)\,\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right) f'\left(u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right)\right)dt \qquad (5)\end{aligned}$$

$$\begin{aligned}\Phi_2(f) &= \frac{1}{b-a}\int_a^b f\left(M(b-a)v_2(t)\,\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right)\right)dt\\ &\quad - f\left(M\int_a^b v_2(t)\,\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right)dt\right), \qquad (6)\end{aligned}$$

where f is a differentiable function with $f(0)=0$, and M, ϕ, u_i, v_i $(i=1,2)$ are as in Theorem 1.1.

If f is a convex function, then Theorem 2.1 implies that $\Phi_i(f)\geqslant 0$ $(i=1,2)$.

Now, we give mean value theorems for the functionals Φ_i $(i=1,2)$.

Let $0<m_2\leqslant v_2\leqslant M_2$, $0\leqslant |v_1|\leqslant M_1$ and $\phi\geqslant 0$. Then $0\leqslant\left|\frac{v_1}{v_2}\right|\leqslant\frac{M_1}{m_2}$. It follows

$$m_2M(b-a)\min_{\left[0,\frac{M_1}{m_2}\right]}\phi\leqslant M\int_a^b v_2(t)\,\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right)dt\leqslant M_2M(b-a)\max_{\left[0,\frac{M_1}{m_2}\right]}\phi.$$

Also

$$0\leqslant\left|\frac{u_1(t)}{u_2(t)}\right|\leqslant\frac{M_1\int_a^t K(x,\tau)\,d\tau}{m_2\int_a^t K(x,\tau)\,d\tau}=\frac{M_1}{m_2}.$$

Since obviously $|u_2(t)|\leqslant MM_2(b-a)$, we have

$$0\leqslant u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right)\leqslant MM_2(b-a)\max_{\left[0,\frac{M_1}{m_2}\right]}\phi.$$

Hence, from now on let $f: I\to\mathbb{R}$ where

$$I=\left[0,MM_2(b-a)\max_{\left[0,\frac{M_1}{m_2}\right]}\phi\right]. \qquad (7)$$

THEOREM 2.2. *Let $u_1\in U(v_1,K)$, $u_2\in U(v_2,K)$ and $v_2(x)>0$ for every $x\in[a,b]$. Further, let $\phi(u)$ be convex, nonnegative and increasing for $u\geqslant 0$. Let $f\in C^2(I)$ and $f(0)=0$. Then there exists $\xi\in I$ such that*

$$\Phi_i(f)=\frac{f''(\xi)}{2}\Phi_i(f_0),\quad (i=1,2), \qquad (8)$$

where $f_0(x)=x^2$.

Proof. Since $f \in C^2(I)$, there exist real numbers $m = \min_{x\in I} f''(x)$ and $M = \max_{x\in I} f''(x)$. Hence, the functions f_1 and f_2 defined by

$$f_1(x) = \frac{M}{2}x^2 - f(x),$$
$$f_2(x) = f(x) - \frac{m}{2}x^2$$

are convex. Therefore $\Phi_i(f_1) \geqslant 0$, $\Phi_i(f_2) \geqslant 0$ $(i = 1,2)$, and we get

$$\frac{m}{2}\Phi_i(f_0) \leqslant \Phi_i(f) \leqslant \frac{M}{2}\Phi_i(f_0).$$

If $\Phi_i(f_0) = 0$, then there is nothing to prove. Suppose $\Phi_i(f_0) > 0$. We have

$$m \leqslant \frac{2\Phi_i(f)}{\Phi_i(f_0)} \leqslant M.$$

Hence, there exists $\xi \in I$ such that

$$\Phi_i(f) = \frac{f''(\xi)}{2}\Phi_i(f_0), \quad (i = 1,2).$$

This completes the proof. □

THEOREM 2.3. *Let $u_1 \in U(v_1,K)$, $u_2 \in U(v_2,K)$ and $v_2(x) > 0$ for every $x \in [a,b]$. Further, let $\phi(u)$ be convex, nonnegative and increasing for $u \geqslant 0$. Let $f,g \in C^2(I)$ and $f(0) = g(0) = 0$. Then there exists $\xi \in I$ such that*

$$\frac{\Phi_i(f)}{\Phi_i(g)} = \frac{f''(\xi)}{g''(\xi)}, \quad (i = 1,2), \tag{9}$$

provided that the denominators are non-zero.

Proof. Define $h \in C^2(I)$ by $h = c_1 f - c_2 g$, where

$$c_1 = \Phi_i(g), \; c_2 = \Phi_i(f), \quad (i = 1,2).$$

Now using Theorem 2.2 with $f = h$ there exists $\xi \in I$ such that

$$\left(c_1\frac{f''(\xi)}{2} - c_2\frac{g''(\xi)}{2}\right)\Phi_i(f_0) = 0, \quad (i = 1,2).$$

Since $\Phi_i(f_0) \neq 0$ (otherwise we have a contradiction with $\Phi_i(g) \neq 0$ by Theorem 2.2), we get

$$\frac{\Phi_i(f)}{\Phi_i(g)} = \frac{f''(\xi)}{g''(\xi)}, \quad (i = 1,2).$$

This completes the proof. □

3. Opial-type inequalities for fractional integrals and fractional derivatives

Here we present some new Opial-type inequalities involving fractional integrals and fractional derivatives. For more details on the fractional integrals of a function with respect to another function, the Riemann–Liouville and the Hadamard fractional integrals, see e.g. [8, Section 2.1, 2.5 and 2.7].

Let (a,b), $-\infty \leqslant a < b \leqslant \infty$, be a finite or infinite interval of the real line $\mathbb{R}$ and $\alpha > 0$. Also let g be an increasing function on (a,b) and g' be a continuous function on (a,b). The *left-sided and right-sided fractional integrals of a function* f *with respect to another function* g in $[a,b]$ are given by

$$I^{\alpha}_{a+;g}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t)f(t)dt}{[g(x)-g(t)]^{1-\alpha}}, \quad x > a,$$

$$I^{\alpha}_{b-;g}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t)f(t)dt}{[g(t)-g(x)]^{1-\alpha}}, \quad x < b,$$

respectively. Here Γ is the gamma function $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}\,dt$.

THEOREM 3.1. *Let* $\alpha \geqslant 1$, $\phi(u)$ *be convex, nonnegative and increasing for* $u \geqslant 0$ *and let* $f(u)$ *be convex for* $u \geqslant 0$ *with* $f(0) = 0$. *If* f *is a differentiable function, then these inequalities are valid:*

$$\begin{aligned}
&\frac{(g(b)-g(a))^{\alpha-1}\max\limits_{x\in[a,b]} g'(x)}{\Gamma(\alpha)} \int_a^b u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right) f'\left(I^{\alpha}_{a+;g}u_2(t)\,\phi\left(\left|\frac{I^{\alpha}_{a+;g}u_1(t)}{I^{\alpha}_{a+;g}u_2(t)}\right|\right)\right) dt\\
&\leqslant f\left(\frac{(g(b)-g(a))^{\alpha-1}\max\limits_{x\in[a,b]} g'(x)}{\Gamma(\alpha)} \int_a^b u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right) dt\right)\\
&\leqslant \frac{1}{b-a}\int_a^b f\left(\frac{(b-a)(g(b)-g(a))^{\alpha-1}\max\limits_{x\in[a,b]} g'(x)}{\Gamma(\alpha)} u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right)\right) dt.
\end{aligned} \tag{10}$$

Proof. We use Theorem 2.1 with the following kernel,

$$K(x,t) = \begin{cases} \dfrac{g'(t)}{\Gamma(\alpha)(g(x)-g(t))^{1-\alpha}}, & a < t \leqslant x; \\ 0, & x < t \leqslant b. \end{cases}$$

For $\alpha \geqslant 1$, we get

$$M = \max K(x,t) = \frac{(g(b)-g(a))^{\alpha-1}\max\limits_{x\in[a,b]} g'(x)}{\Gamma(\alpha)}.$$

If we replace u_i by $I^{\alpha}_{a+;g}u_i$ and v_i by u_i $(i=1,2)$ in inequalities given in (3) an (4), then the inequality (10) follows. $\square$

Let $[a,b]$, $-\infty<a<b<\infty$, be a finite interval on the real axis $\mathbb{R}$. For $f\in L_1[a,b]$ the *left-sided and right-sided Riemann–Liouville fractional integrals* of order $\alpha>0$ are defined by

$$I^{\alpha}_{a+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt,\ \ x>a,$$

$$I^{\alpha}_{b-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\,dt,\ \ x<b.$$

If $g(x)=x$, then $I^{\alpha}_{a+;x}f(x)$ reduces to $I^{\alpha}_{a+}f(x)$, i.e. left-sided Riemann–Liouville fractional integral. Same follows for the right-sided fractional integral. This gives us the next result.

COROLLARY 3.2. *Let $\alpha\geqslant 1$, $\phi(u)$ be convex, nonnegative and increasing for $u\geqslant 0$ and let $f(u)$ be convex for $u\geqslant 0$ with $f(0)=0$. If f is a differentiable function, then these inequalities are valid:*

$$\begin{aligned}
&\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}\int_a^b u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right) f'\left(I^{\alpha}_{a+}u_2(t)\,\phi\left(\left|\frac{I^{\alpha}_{a+}u_1(t)}{I^{\alpha}_{a+}u_2(t)}\right|\right)\right)dt\\
&\quad\leqslant f\left(\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}\int_a^b u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right)dt\right)\\
&\quad\leqslant \frac{1}{b-a}\int_a^b f\left(\frac{(b-a)^{\alpha}}{\Gamma(\alpha)}u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right)\right)dt.
\end{aligned}\tag{11}$$

Let (a,b) be a finite or infinite interval of $\mathbb{R}^+$ and $\alpha>0$. The *left-sided and right-sided Hadamard fractional integrals* of order $\alpha>0$ are given by

$$J^{\alpha}_{a+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x \left(\log\frac{x}{t}\right)^{\alpha-1}\frac{f(t)dt}{t},\ \ x>a,$$

$$J^{\alpha}_{b-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b \left(\log\frac{t}{x}\right)^{\alpha-1}\frac{f(t)dt}{t},\ \ x<b.$$

Notice that Hadamard fractional integrals of order α are special cases of the left-sided and right-sided fractional integrals of a function f with respect to a function $g(x)=\log x$ in (a,b), where $0\leqslant a<b\leqslant\infty$.

COROLLARY 3.3. *Let $\alpha \geqslant 1$, $\phi(u)$ be convex, nonnegative and increasing for $u \geqslant 0$ and let $f(u)$ be convex for $u \geqslant 0$ with $f(0)=0$. If f is a differentiable function, then these inequalities are valid:*

$$\begin{aligned}
&\frac{(\log b-\log a)^{\alpha-1}}{a\Gamma(\alpha)}\int_a^b u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right) f'\left(J_{a+}^{\alpha}u_2(t)\,\phi\left(\left|\frac{J_{a+}^{\alpha}u_1(t)}{J_{a+}^{\alpha}u_2(t)}\right|\right)\right)dt\\
&\leqslant f\left(\frac{1}{a\Gamma(\alpha)}(\log b-\log a)^{\alpha-1}\int_a^b u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right)dt\right)\\
&\leqslant \frac{1}{b-a}\int_a^b f\left(\frac{(b-a)(\log b-\log a)^{\alpha-1}}{a\Gamma(\alpha)}u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right)\right)dt. \qquad (12)
\end{aligned}$$

REMARK 3.4. Similarly we can give analogous results for the right-sided fractional integrals but here we omit the details.

Next we follow with definitions of the Riemann–Liouville, Canavati and Caputo fractional derivatives (see [8, Section 2.1 and 2.4] and [6]) and obtain results.

Let $\alpha > 0$ and $n=[\alpha]+1$ ($[\cdot]$ is the integral part). For $f:[a,b]\to\mathbb{R}$ the *left-sided and right-sided Riemann–Liouville fractional derivatives* of order α are defined by

$$D_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\alpha-1}f(t)\,dt=\frac{d^n}{dx^n}I_{a+}^{n-\alpha}f(x),$$

$$D_{b-}^{\alpha}f(x)=\frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^b (t-x)^{n-\alpha-1}f(t)\,dt=(-1)^n\frac{d^n}{dx^n}I_{b-}^{n-\alpha}f(x).$$

The following lemma summarizes conditions in the composition identity for the left-sided Riemann–Liouville fractional derivative. For details see [2, Section 3].

LEMMA 3.5. *Let $\beta > \alpha \geqslant 0$, $n=[\beta]+1$, $m=[\alpha]+1$. The composition identity*

$$D_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\beta-\alpha)}\int_a^x (x-t)^{\beta-\alpha-1}D_{a+}^{\beta}f(t)\,dt, \quad x\in[a,b], \qquad (13)$$

is valid if one of the following conditions holds:

(i) $f\in I_{a+}^{\beta}(L_1(a,b))$.

(ii) $I_{a+}^{n-\beta}f\in AC^n[a,b]$ *and* $D_{a+}^{\beta-k}f(a)=0$ *for* $k=1,\dots n$.

(iii) $D_{a+}^{\beta-k}f\in C[a,b]$ *for* $k=1,\dots,n$, $D_{a+}^{\beta-1}f\in AC[a,b]$ *and* $D_{a+}^{\beta-k}f(a)=0$ *for* $k=1,\dots n$.

(iv) $f\in AC^n[a,b]$, $D_{a+}^{\beta}f\in L_1(a,b)$, $D_{a+}^{\alpha}f\in L_1(a,b)$, $\beta-\alpha\notin\mathbb{N}$, $D_{a+}^{\beta-k}f(a)=0$ *for* $k=1,\dots,n$ *and* $D_{a+}^{\alpha-k}f(a)=0$ *for* $k=1,\dots,m$.

(v) $f \in AC^n[a,b]$, $D^{\beta}_{a+}f \in L_1(a,b)$, $D^{\alpha}_{a+}f \in L_1(a,b)$, $\beta - \alpha = l \in \mathbb{N}$, $D^{\beta-k}_{a+}f(a) = 0$ *for* $k = 1,\dots,l$.

(vi) $f \in AC^n[a,b]$, $D^{\beta}_{a+}f \in L_1(a,b)$, $D^{\alpha}_{a+}f \in L_1(a,b)$ *and* $f(a) = f'(a) = \cdots = f^{(n-2)}(a) = 0$.

(vii) $f \in AC^n[a,b]$, $D^{\beta}_{a+}f \in L_1(a,b)$, $D^{\alpha}_{a+}f \in L_1(a,b)$, $\beta \notin \mathbb{N}$ *and* $D^{\beta-1}_{a+}f$ *is bounded in a neighborhood of* $t = a$.

THEOREM 3.6. *Let* $\beta > \alpha + 1$ *and let the assumptions of Lemma 3.5 be satisfied. Let* $D^{\beta}_{a+}u_2(x) > 0$ *for every* $x \in [a,b]$. *Further, let* $\phi(u)$ *be convex, nonnegative and increasing for* $u \geqslant 0$ *and let* $f(u)$ *be convex for* $u \geqslant 0$ *with* $f(0) = 0$. *Then these inequalities are valid:*

$$\begin{aligned}
&\frac{(b-a)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} \int_a^b D^{\beta}_{a+}u_2(t)\,\phi\left(\left|\frac{D^{\beta}_{a+}u_1(t)}{D^{\beta}_{a+}u_2(t)}\right|\right) f'\left(D^{\alpha}_{a+}u_2(t)\,\phi\left(\left|\frac{D^{\alpha}_{a+}u_1(t)}{D^{\alpha}_{a+}u_2(t)}\right|\right)\right) dt \\
&\quad \leqslant f\left(\frac{(b-a)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} \int_a^b D^{\beta}_{a+}u_2(t)\,\phi\left(\left|\frac{D^{\beta}_{a+}u_1(t)}{D^{\beta}_{a+}u_2(t)}\right|\right) dt\right) \\
&\quad \leqslant \frac{1}{b-a} \int_a^b f\left(\frac{(b-a)^{\beta-\alpha}}{\Gamma(\beta-\alpha)} D^{\beta}_{a+}u_2(t)\,\phi\left(\left|\frac{D^{\beta}_{a+}u_1(t)}{D^{\beta}_{a+}u_2(t)}\right|\right)\right) dt. \qquad (14)
\end{aligned}$$

Proof. We use Theorem 2.1 with the following kernel,

$$K(x,t) = \begin{cases} \dfrac{(x-t)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}, & a < t \leqslant x; \\ 0, & x < t \leqslant b. \end{cases}$$

For $\beta > \alpha + 1$, we get

$$M = \max K(x,t) = \frac{(b-a)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}.$$

If we replace u_i by $D^{\alpha}_{a+}u_i$ and v_i by $D^{\beta}_{a+}u_i$ $(i = 1,2)$ in inequalities given in (3) and (4), then the inequality (14) follows. □

Next we consider subspaces $C^{\alpha}_{a+}[a,b]$ and $C^{\alpha}_{b-}[a,b]$ of $C^{n-1}[a,b]$ defined by

$$C^{\alpha}_{a+}[a,b] = \left\{ f \in C^{n-1}[a,b] : I^{n-\alpha}_{a+} f^{(n-1)} \in C^1[a,b] \right\},$$

$$C^{\alpha}_{b-}[a,b] = \left\{ f \in C^{n-1}[a,b] : I^{n-\alpha}_{b-} f^{(n-1)} \in C^1[a,b] \right\}.$$

For $f \in C^{\alpha}_{a+}[a,b]$ and $g \in C^{\alpha}_{b-}[a,b]$ the *left-sided and right-sided Canavati fractional derivatives* of order α are defined by

$$^{C}D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d}{dx}\int_a^x (x-t)^{n-\alpha-1} f^{(n-1)}(t)\,dt = \frac{d}{dx}I^{n-\alpha}_{a+}f^{(n-1)}(x),$$

$$^{C}D^{\alpha}_{b-}g(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d}{dx}\int_x^b (t-x)^{n-\alpha-1} g^{(n-1)}(t)\,dt = (-1)^n\frac{d}{dx}I^{n-\alpha}_{b-}g^{(n-1)}(x).$$

The following lemma gives conditions in the composition rule for the left-sided Canavati fractional derivative. For details see [4, Section 2].

LEMMA 3.7. *Let $\beta > \alpha > 0$, $n = [\beta]+1$, $m = [\alpha]+1$. Let $f \in C^{\beta}_{a+}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1,\dots,n-2$. Then $f \in C^{\alpha}_{a+}[a,b]$ and*

$$^{C}D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\beta-\alpha)}\int_a^x (x-t)^{\beta-\alpha-1}\,{}^{C}D^{\beta}_{a+}f(t)\,dt\,,\ x\in[a,b].$$

THEOREM 3.8. *Let $\beta > \alpha+1$ and let the assumptions of Lemma 3.7 be satisfied. Let ${}^{C}D^{\beta}_{a+}u_2(x) > 0$ for every $x \in [a,b]$. Further, let $\phi(u)$ be convex, nonnegative and increasing for $u \geqslant 0$ and let $f(u)$ be convex for $u \geqslant 0$ with $f(0) = 0$. Then these inequalities are valid:*

$$\begin{aligned}
&\frac{(b-a)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}\int_a^b {}^{C}D^{\beta}_{a+}u_2(t)\,\phi\left(\left|\frac{{}^{C}D^{\beta}_{a+}u_1(t)}{{}^{C}D^{\beta}_{a+}u_2(t)}\right|\right) f'\left({}^{C}D^{\alpha}_{a+}u_2(t)\,\phi\left(\left|\frac{{}^{C}D^{\alpha}_{a+}u_1(t)}{{}^{C}D^{\alpha}_{a+}u_2(t)}\right|\right)\right)dt\\
&\qquad\leqslant f\left(\frac{(b-a)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}\int_a^b {}^{C}D^{\beta}_{a+}u_2(t)\,\phi\left(\left|\frac{{}^{C}D^{\beta}_{a+}u_1(t)}{{}^{C}D^{\beta}_{a+}u_2(t)}\right|\right)dt\right)\\
&\qquad\leqslant \frac{1}{b-a}\int_a^b f\left(\frac{(b-a)^{\beta-\alpha}}{\Gamma(\beta-\alpha)}\,{}^{C}D^{\beta}_{a+}u_2(t)\,\phi\left(\left|\frac{{}^{C}D^{\beta}_{a+}u_1(t)}{{}^{C}D^{\beta}_{a+}u_2(t)}\right|\right)\right)dt. \qquad (15)
\end{aligned}$$

Proof. The proof of this theorem is a similar to the proof of Theorem 3.6. □

Let $\alpha \geqslant 0$ and $n = [\alpha]+1$. For $f \in AC^n[a,b]$ the *left-sided and right-sided Caputo fractional derivatives* of order α are defined by

$$^{\bar{C}}D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\,dt = I^{n-\alpha}_{a+}f^{(n)}(x),$$

$$^{\bar{C}}D^{\alpha}_{b-}f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b (t-x)^{n-\alpha-1} f^{(n)}(t)\,dt = (-1)^n I^{n-\alpha}_{b-}f^{(n)}(x).$$

The final composition identity for the left-sided Caputo fractional derivative is given in [3, Section 2].

LEMMA 3.9. *Let $\beta > \alpha \geqslant 0$, $n = [\beta]+1$, $m = [\alpha]+1$, $f \in AC^n[a,b]$ and ${}^{\bar{C}}D^{\beta}_{a+}f$, ${}^{\bar{C}}D^{\alpha}_{a+}f \in L_1[a,b]$. Suppose that one of the following conditions hold:*

(i) *$\beta, \alpha \notin \mathbb{N}_0$ and $f^{(i)}(a) = 0$ for $i = m, \ldots, n-1$.*

(ii) *$\beta \in \mathbb{N}_0$, $\alpha \notin \mathbb{N}_0$ and $f^{(i)}(a) = 0$ for $i = m, \ldots, n-2$.*

(iii) *$\beta \notin \mathbb{N}_0$, $\alpha \in \mathbb{N}_0$ and $f^{(i)}(a) = 0$ for $i = m-1, \ldots, n-1$.*

(iv) *$\beta \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0$ and $f^{(i)}(a) = 0$ for $i = m-1, \ldots, n-2$.*

Then

$$
{}^{\bar{C}}D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\beta-\alpha)} \int_a^x (x-t)^{\beta-\alpha-1}\, {}^{\bar{C}}D^{\beta}_{a+}f(t)\,dt\,,\ x \in [a,b].
$$

THEOREM 3.10. *Let $\beta > \alpha + 1$ and let the assumptions of Lemma 3.9 be satisfied. Let ${}^{\bar{C}}D^{\beta}_{a+}u_2(x) > 0$ for every $x \in [a,b]$. Further, let $\phi(u)$ be convex, nonnegative and increasing for $u \geqslant 0$ and let $f(u)$ be convex for $u \geqslant 0$ with $f(0) = 0$. Then these inequalities are valid:*

$$
\begin{aligned}
&\frac{(b-a)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} \int_a^b {}^{\bar{C}}D^{\beta}_{a+}u_2(t)\,\phi\left(\left|\frac{{}^{\bar{C}}D^{\beta}_{a+}u_1(t)}{{}^{\bar{C}}D^{\beta}_{a+}u_2(t)}\right|\right) f'\left({}^{\bar{C}}D^{\alpha}_{a+}u_2(t)\,\phi\left(\left|\frac{{}^{\bar{C}}D^{\alpha}_{a+}u_1(t)}{{}^{\bar{C}}D^{\alpha}_{a+}u_2(t)}\right|\right)\right) dt \\
&\leqslant f\left(\frac{(b-a)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} \int_a^b {}^{\bar{C}}D^{\beta}_{a+}u_2(t)\,\phi\left(\left|\frac{{}^{\bar{C}}D^{\beta}_{a+}u_1(t)}{{}^{\bar{C}}D^{\beta}_{a+}u_2(t)}\right|\right) dt\right) \\
&\leqslant \frac{1}{b-a}\int_a^b f\left(\frac{(b-a)^{\beta-\alpha}}{\Gamma(\beta-\alpha)}\, {}^{\bar{C}}D^{\beta}_{a+}u_2(t)\,\phi\left(\left|\frac{{}^{\bar{C}}D^{\beta}_{a+}u_1(t)}{{}^{\bar{C}}D^{\beta}_{a+}u_2(t)}\right|\right)\right) dt. \qquad (16)
\end{aligned}
$$

Proof. The proof of this theorem is similar to the proof of Theorem 3.6. □

REMARK 3.11. As we define the functionals Φ_1 and Φ_2 from the inequalities given in (3) and (4), similarly we can define the functionals from the inequalities given in (10), (11), (12), (14), (15) and (16) and analogously we obtain the mean value theorems for our defined functionals.

4. Method of exponential convexity

Following definitions and properties of exponentially convex functions comes from [5], also [10]. Let I be an interval in $\mathbb{R}$.

DEFINITION 4.1. A function $\psi: I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$
\sum_{i,j=1}^{n} \xi_i \xi_j \psi\left(\frac{x_i + x_j}{2}\right) \geqslant 0
$$

holds for all choices $\xi_i \in \mathbb{R}$ and $x_i \in I$, $i = 1, \dots, n$.

A function $\psi\colon I \to \mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

REMARK 4.2. It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leqslant n$.

By definition of positive semi-definite matrices and some basic linear algebra we have the following proposition.

PROPOSITION 4.3. *If ψ is an n-exponentially convex in the Jensen sense, then the matrix* $\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k}$ *is a positive semi-definite matrix for all $k \in \mathbb{N}$, $k \leqslant n$. Particularly,* $\det\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k} \geqslant 0$ *for all $k \in \mathbb{N}$, $k \leqslant n$.*

DEFINITION 4.4. A function $\psi\colon I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

A function $\psi\colon I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

REMARK 4.5. It is known (and easy to show) that $\psi : I \to (0,\infty)$ is log-convex in the Jensen sense if and only if

$$\alpha^2\psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2\psi(y) \geqslant 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a function is log-convex in the Jensen-sense if and only if it is 2-exponentially convex in the Jensen sense.

Also, using basic convexity theory it follows that a function is log-convex if and only if it is 2-exponentially convex.

We will also need following results (see for example [11]).

PROPOSITION 4.6. *If $x_1, x_2, x_3 \in I$ are such that $x_1 < x_2 < x_3$, then the function $f : I \to \mathbb{R}$ is convex if and only if the inequality*

$$(x_3 - x_2)f(x_1) + (x_1 - x_3)f(x_2) + (x_2 - x_1)f(x_3) \geqslant 0$$

holds.

PROPOSITION 4.7. *If f is a convex function on an interval I and if $x_1 \leqslant y_1$, $x_2 \leqslant y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leqslant \frac{f(y_2) - f(y_1)}{y_2 - y_1}.$$

If the function f is concave, then the inequality reverses.

Next we need divided differences, commonly used when dealing with functions that have different degree of smoothness.

DEFINITION 4.8. The second order divided difference of a function $f\colon I \to \mathbb{R}$ at mutually different points $y_0, y_1, y_2 \in I$ is defined recursively by

$$\begin{aligned} [y_i; f] &= f(y_i), \quad i = 0,1,2 \\ [y_i, y_{i+1}; f] &= \frac{f(y_{i+1}) - f(y_i)}{y_{i+1} - y_i}, \quad i = 0,1 \\ [y_0, y_1, y_2; f] &= \frac{[y_1, y_2; f] - [y_0, y_1; f]}{y_2 - y_0}. \end{aligned} \tag{17}$$

REMARK 4.9. The value $[y_0, y_1, y_2; f]$ is independent of the order of the points y_0, y_1 and y_2. This definition may be extended to include the case in which some or all the points coincide. Namely, taking the limit $y_1 \to y_0$ in (17), we get

$$\lim_{y_1 \to y_0} [y_0, y_1, y_2; f] = [y_0, y_0, y_2; f] = \frac{f(y_2) - f(y_0) - f'(y_0)(y_2 - y_0)}{(y_2 - y_0)^2}, \quad y_2 \neq y_0$$

provided that f' exists, and furthermore, taking the limits $y_i \to y_0$, $i = 1,2$ in (17), we get

$$\lim_{y_2 \to y_0} \lim_{y_1 \to y_0} [y_0, y_1, y_2; f] = [y_0, y_0, y_0; f] = \frac{f''(y_0)}{2}$$

provided that f'' exists.

An elegant method of producing n-exponentially convex and exponentially convex functions is given in [7]. We use this to prove the n-exponential convexity for the functionals Φ_i $(i = 1,2)$. The next theorem is analogous to the one given in [10, Theorem 3.9] and we give a proof for the reader's convenience.

Note here that for the functionals Φ_i $(i = 1,2)$ interval I is defined by (7).

THEOREM 4.10. *Let $\Upsilon = \{f_s : s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is n-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Let Φ_i $(i = 1,2)$ be linear functionals defined as in (5) and (6). Then $s \mapsto \Phi_i(f_s)$ is n-exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Phi_i(f_s)$ is also continuous on J, then it is n-exponentially convex on J.*

Proof. For $\xi_i \in \mathbb{R}$, $s_i \in J$, $i = 1, \ldots, n$, we define the function

$$g(y) = \sum_{i,j=1}^{n} \xi_i \xi_j f_{\frac{s_i + s_j}{2}}(y).$$

Using the assumption that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is n-exponentially convex in the Jensen sense, we have

$$[y_0, y_1, y_2; g] = \sum_{i,j=1}^{n} \xi_i \xi_j [y_0, y_1, y_2; f_{\frac{s_i+s_j}{2}}] \geqslant 0,$$

which in turn implies that g is a convex function on I. Therefore we have $\Phi_i(g) \geqslant 0$ $(i = 1, 2)$. Hence

$$\sum_{i,j=1}^{n} \xi_i \xi_j \Phi_i(f_{\frac{s_i+s_j}{2}}) \geqslant 0.$$

We conclude that the function $s \mapsto \Phi_i(f_s)$ is n-exponentially convex on J in the Jensen sense. If the function $s \mapsto \Phi_i(f_s)$ is also continuous on J, then $s \mapsto \Phi_i(f_s)$ is n-exponentially convex by definition. □

COROLLARY 4.11. *Let $\Upsilon = \{f_s : s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Let Φ_i $(i = 1, 2)$ be linear functionals defined as in (5) and (6). Then $s \mapsto \Phi_i(f_s)$ is exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Phi_i(f_s)$ is continuous on J, then it is exponentially convex on J.*

Let us denote a mean for $f_s, f_q \in \Omega$ by

$$\mu_{s,q}(\Phi_i, \Omega) = \begin{cases} \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{\frac{d}{ds}\Phi_i(f_s)}{\Phi_i(f_s)}\right), & s = q. \end{cases} \tag{18}$$

THEOREM 4.12. *Let $\Omega = \{f_s : s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is 2-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Let Φ_i $(i = 1, 2)$ be linear functionals defined as in (5) and (6). Then the following statements hold:*

(*i*) *If the function $s \mapsto \Phi_i(f_s)$ is continuous on J, then it is 2-exponentially convex function on J. If the function $s \mapsto \Phi_i(f_s)$ is additionally positive, then it is also log-convex on J, and for $r, s, t \in J$ such that $r < s < t$, we have*

$$(\Phi_i(f_s))^{t-r} \leqslant (\Phi_i(f_r))^{t-s} (\Phi_i(f_t))^{s-r}, \quad i = 1, 2. \tag{19}$$

(*ii*) *If the function $s \mapsto \Phi_i(f_s)$ is strictly positive and differentiable on J, then for every $s, q, r, t \in J$, such that $s \leqslant r$ and $q \leqslant t$, we have*

$$\mu_{s,q}(\Phi_i, \Omega) \leqslant \mu_{r,t}(\Phi_i, \Omega), \quad i = 1, 2. \tag{20}$$

Proof. (i) The first part is an immediate consequence of Theorem 4.10 and in second part log-convexity on J follows from Remark 4.5. Since $s \mapsto \Phi_i(f_s)$ is positive, for $r,s,t \in J$ such that $r < s < t$, with $f(s) = \log\Phi_i(f_s)$ in Proposition 4.6, we have

$$(t-s)\log\Phi_i(f_r) + (r-t)\log\Phi_i(f_s) + (s-r)\log\Phi_i(f_t) \geqslant 0.$$

This is equivalent to inequality (19).

(ii) The function $s \mapsto \Phi_i(f_s)$ is log-convex on J by (i), that is, the function $s \mapsto \log\Phi_i(f_s)$ is convex on J. Applying Proposition 4.7 we get

$$\frac{\log\Phi_i(f_s) - \log\Phi_i(f_q)}{s-q} \leqslant \frac{\log\Phi_i(f_r) - \log\Phi_i(f_t)}{r-t} \tag{21}$$

for $s \leqslant r$, $q \leqslant t$, $s \neq q$, $r \neq t$, and therefore we have

$$\mu_{s,q}(\Phi_i,\Omega) \leqslant \mu_{r,t}(\Phi_i,\Omega).$$

Cases $s = q$ and $r = t$ follows from (21) as limit cases. □

REMARK 4.13. The results from Theorem 4.10, Corollary 4.11 and Theorem 4.12 still hold when two of the points $y_0, y_1, y_2 \in I$ coincide, for a family of differentiable functions f_s such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense). Furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs can be obtained by recalling Remark 4.9 and suitable characterization of convexity.

REMARK 4.14. As we prove the n-exponential convexity of the functionals Φ_1 and Φ_2 obtained from the inequalities given in (3) and (4), similarly we can define the functionals from the inequalities given in (10), (11), (12), (14), (15) and (16) and prove the n-exponential convexity of our defined functionals but here we omit the details.

5. Applications to Stolarsky type means

In this section, we use Cauchy type mean value Theorem 2.2 and Theorem 2.3 for Stolarsky type means, defined by the functional Φ_i $(i = 1,2)$. Several families of functions which fulfil the conditions of Theorem 4.10, Corollary 4.11 and Theorem 4.12 (and Remark 4.13) that we present here, enable us to construct large families of functions which are exponentially convex.

EXAMPLE 5.1. Consider a family of functions

$$\Omega_1 = \{f_s : \mathbb{R} \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$f_s(x) = \begin{cases} \frac{e^{sx}-1}{s^2}, & s \neq 0, \\ \frac{x^2}{2}, & s = 0. \end{cases}$$

Since $\frac{d^2 f_s}{dx^2}(x) = e^{sx} > 0$, then f_s is convex on $\mathbb{R}$ for every $s \in \mathbb{R}$, and $s \mapsto \frac{d^2 f_s}{dx^2}(x)$ is exponentially convex by definition.

Analogously as in the proof of Theorem 4.10 we conclude that $s \mapsto [y_0, y_1, y_2; f_s]$ is exponentially convex (and so exponentially convex in the Jensen sense).

Notice that $f_s(0) = 0$. By Corollary 4.11 we have that $s \mapsto \Phi_i(f_s)$ $(i = 1,2)$ is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although mapping $s \mapsto f_s$ is not continuous for $s = 0$), so it is exponentially convex.

For this family of functions, $\mu_{s,q}(\Phi_i, \Omega_1)$ $(i = 1,2)$ from (18) is equal to

$$\mu_{s,q}(\Phi_i,\Omega_1) = \begin{cases} \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{\Phi_i(id \cdot f_s)}{\Phi_i(f_s)} - \frac{2}{s}\right), & s = q \neq 0, \\ \exp\left(\frac{\Phi_i(id \cdot f_0)}{3\Phi_i(f_0)}\right), & s = q = 0, \end{cases}$$

and using (20) it is a monotonous in parameters s and q.

If Φ_i is positive, $(i = 1,2)$, then Theorem 2.3 applied for $f = f_s \in \Omega_1$ and $g = f_q \in \Omega_1$ yields that there exists $\xi \in I = \left[0, MM_2\,(b-a)\max_{\left[0,\frac{M_1}{m_2}\right]} \phi\right]$ such that

$$e^{(s-q)\xi} = \frac{\Phi_i(f_s)}{\Phi_i(f_q)}.$$

It follows that

$$M_{s,q}(\Phi_i,\Omega_1) = \log \mu_{s,q}(\Phi_i,\Omega_1)$$

satisfy $0 \leqslant M_{s,q}(\Phi_i,\Omega_1) \leqslant MM_2\,(b-a)\max_{\left[0,\frac{M_1}{m_2}\right]} \phi$, which shows that $M_{s,q}(\Phi_i,\Omega_1)$ is a mean, and by (20) it is a monotonous mean, $i = 1,2$.

EXAMPLE 5.2. Consider a family of functions

$$\Omega_2 = \{g_s : [0,\infty) \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$g_s(x) = \begin{cases} \frac{(x+1)^s - 1}{s(s-1)}, & s \neq 0,1, \\ -\log(x+1), & s = 0, \\ (x+1)\log(x+1), & s = 1. \end{cases}$$

Here, $\frac{d^2 g_s}{dx^2}(x) = (x+1)^{s-2} = e^{(s-2)\log(x+1)} > 0$ which shows that g_s is convex for $x > 0$ and $s \mapsto \frac{d^2 g_s}{dx^2}(x)$ is exponentially convex by definition. Also, $g_s(0) = 0$. Arguing as in Example 5.1 we get that the mapping $s \mapsto \Phi_i(g_s)$ is exponentially convex and also log-convex.

For this family of functions, $\mu_{s,q}(\Phi_i,\Omega_2)$ from (18) is equal to

$$\mu_{s,q}(\Phi_i,\Omega_2) = \begin{cases} \left(\frac{\Phi_i(g_s)}{\Phi_i(g_q)}\right)^{\frac{1}{s-q}}, & s\neq q\,, \\ \exp\left(\frac{1-2s}{s(s-1)} - \frac{\Phi_i(g_0g_s)}{\Phi_i(g_s)} - \frac{1}{s(s-1)}\frac{\Phi_i(g_0)}{\Phi_i(g_s)}\right), & s=q\neq 0,1\,, \\ \exp\left(1-\frac{\Phi_i(g_0^2)}{2\Phi_i(g_0)}\right), & s=q=0\,, \\ \exp\left(-1-\frac{\Phi_i(g_0g_1)}{2\Phi_i(g_1)}\right), & s=q=1\,, \end{cases}$$

and by (20) it is monotonous in parameters s and q.

Using Theorem 2.3 it follows that there exists $\xi\in I$ such that

$$(\xi+1)^{s-q} = \frac{\Phi_i(g_s)}{\Phi_i(g_q)}\,.$$

Since the function $\xi\mapsto(\xi+1)^{s-q}$ is invertible for $s\neq q$, we have

$$0\leqslant\left(\frac{\Phi_i(g_s)}{\Phi_i(g_q)}\right)^{\frac{1}{s-q}}\leqslant MM_2\,(b-a)\max_{\left[0,\frac{M_1}{m_2}\right]}\phi$$

which together with the fact that $\mu_{s,q}(\Phi_i,\Omega_2)$ is continuous, symmetric and monotonous, shows that $\mu_{s,q}(\Phi_i,\Omega_2)$ is a mean, $i=1,2$.

EXAMPLE 5.3. Consider a family of functions

$$\Omega_3 = \{h_s : [0,\infty)\to\mathbb{R} : s>0\}$$

defined by

$$h_s(x) = \begin{cases} \frac{s^{-x}-1}{\log^2 s}\,, & s\neq 1\,, \\ \frac{x^2}{2}\,, & s=1\,. \end{cases}$$

Since $s\mapsto\frac{d^2h_s}{dx^2}(x) = s^{-x}$ is the Laplace transform of a non-negative function ([12]), that is $s^{-x} = \dfrac{1}{\Gamma(x)}\displaystyle\int_0^\infty e^{-st}t^{x-1}\,dt$, it is exponentially convex on $(0,\infty)$. Obviously h_s are convex functions for every $s>0$ and $h_s(0)=0$.

For this family of functions, $\mu_{s,q}(\Phi_i,\Omega_3)$ from (18) is equal to

$$\mu_{s,q}(\Phi_i,\Omega_3) = \begin{cases} \left(\frac{\Phi_i(h_s)}{\Phi_i(h_q)}\right)^{\frac{1}{s-q}}, & s\neq q\,, \\ \exp\left(-\frac{\Phi_i(id\cdot h_s)}{s\Phi_i(h_s)} - \frac{2}{s\log s}\right), & s=q\neq 1\,, \\ \exp\left(-\frac{\Phi_i(id\cdot h_1)}{3\Phi_i(h_1)}\right), & s=q=1\,, \end{cases}$$

and it is monotonous in parameters s and q by (20).

Using Theorem 2.3 it follows that there exists $\xi \in I$ such that

$$\left(\frac{s}{q}\right)^{-\xi} = \frac{\Phi_i(h_s)}{\Phi_i(h_q)}.$$

Hence,

$$M_{s,q}(\Phi_i,\Omega_3) = -L(s,q)\log \mu_{s,q}(\Phi_i,\Omega_3),$$

satisfies $0 \leqslant M_{s,q}(\Phi_i,\Omega_3) \leqslant MM_2\,(b-a)\max_{\left[0,\frac{M_1}{m_2}\right]}\phi$, which shows that $M_{s,q}(\Phi_i,\Omega_3)$ is a mean, $i=1,2$.

$L(s,q)$ is the logarithmic mean defined by

$$L(s,q) = \begin{cases} \frac{s-q}{\log s - \log q}\,, & s \neq q\,,\\ s\,, & s=q\,. \end{cases}$$

EXAMPLE 5.4. Consider a family of functions

$$\Omega_4 = \{k_s : [0,\infty) \to \mathbb{R} : s>0\}$$

defined by

$$k_s(x) = \frac{e^{-x\sqrt{s}}-1}{s}.$$

Again we conclude, since $s \mapsto \frac{d^2k_s}{dx^2}(x) = e^{-x\sqrt{s}}$ is the Laplace transform of a non-negative function ([12]), that is $e^{-x\sqrt{s}} = \frac{s}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-st}e^{-x^2/4t}}{t\sqrt{t}}\,dt$ it is exponentially convex on $(0,\infty)$. For every $s>0$, k_s are convex functions and $k_s(0)=0$.

For this family of functions, $\mu_{s,q}(\Phi_i,\Omega_4)$ from (18) is equal to

$$\mu_{s,q}(\Phi_i,\Omega_4) = \begin{cases} \left(\frac{\Phi_i(k_s)}{\Phi_i(k_q)}\right)^{\frac{1}{s-q}}, & s\neq q,\\ \exp\left(-\frac{\Phi_i(id\cdot k_s)}{2\sqrt{s}\Phi_i(k_s)} - \frac{1}{s}\right), & s=q, \end{cases}$$

and by (20) it is monotonous in parameters s and q.

Using Theorem 2.3 it follows that there exists $\xi \in I$ such that

$$e^{-\xi\left(\sqrt{s}-\sqrt{q}\right)} = \frac{\Phi_i(k_s)}{\Phi_i(k_q)}.$$

Hence,

$$M_{s,q}(\Phi_i,\Omega_4) = -\left(\sqrt{s}+\sqrt{q}\right)\log \mu_{s,q}(\Phi_i,\Omega_4)$$

satisfies $0 \leqslant M_{s,q}(\Phi_i,\Omega_4) \leqslant MM_2\,(b-a)\max_{\left[0,\frac{M_1}{m_2}\right]}\phi$, which shows that $M_{s,q}(\Phi_i,\Omega_4)$ is a mean, $i=1,2$.

Acknowledgements. The authors would like to express their gratitude to professor Ivan Perić for his help and suggestions. This work has been supported in part by Croatian Science Foundation under the project 5435.

REFERENCES

[1] R. P. AGARWAL AND P. Y. H. PANG, *Opial Inequalities with Applications in Differential and Difference Equations*, Kluwer Academic Publishers, Dordrecht, Boston, London (1995).

[2] M. ANDRIĆ, J. PEČARIĆ AND I. PERIĆ, *A multiple Opial type inequality for the Riemann–Liouville fractional derivatives*, J. Math. Inequal., **7** (1) (2013), 139–150.

[3] M. ANDRIĆ, J. PEČARIĆ AND I. PERIĆ, *Composition identities for the Caputo fractional derivatives*, Math. Inequal. Appl., **16** (3) (2013), 657–670.

[4] M. ANDRIĆ, J. PEČARIĆ AND I. PERIĆ, *Improvements of composition rule for Canavati fractional derivative and applications to Opial-type inequalities*, Dynam. Systems Appl., **20** (2011), 383–394.

[5] M. ANWAR, J. JAKŠETIĆ, J. PEČARIĆ AND ATIQ UR REHMAN, *Exponential convexity, positive semi-definite matrices and fundamental inequalities*, J. Math. Inequal., **4** (2) (2010), 171–189.

[6] J. A. CANAVATI, *The Riemann–Liouville Integral*, Nieuw Archief Voor Wiskunde **5** (1) (1987), 53–75.

[7] J. JAKŠETIĆ AND J. PEČARIĆ, *Exponential Convexity Method*, J. Convex Anal., **20** (1) (2013), 181–197.

[8] A. A. KILBAS, H. M. SRIVASTAVA AND J. J. TRUJILLO, *Theory and Application of Fractinal Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier, New York-London, 2006.

[9] D. S. MITRINOVIĆ AND J. E. PEČARIĆ, *Generalizations of two inequalities of Godunova and Levin*, Bull. Polish Acad. Sci. Math., **36** (1988), 645–648.

[10] J. PEČARIĆ AND J. PERIĆ, *Improvements of the Giaccardi and Petrović inequality and related Stolarsky type means*, An. Univ. Craiova Ser. Mat. Inform., **39** (1) (2012), 65–75.

[11] J. E. PEČARIĆ, F. PROSCHAN AND Y. L. TONG, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Inc. (1992).

[12] D. V. WIDDER, *The Laplace Transform*, Princeton University Press (1941).

(Received March 29, 2013)

Maja Andrić
Faculty of Civil Engineering, Architecture and Geodesy
University of Split
Matice hrvatske 15, 21000 Split, Croatia
e-mail: maja.andric@gradst.hr

Ana Barbir
Faculty of Civil Engineering, Architecture and Geodesy
University of Split
Matice hrvatske 15, 21000 Split, Croatia
e-mail: abarbir@gradst.hr

Sajid Iqbal
Department of Mathematics, University of Sargodha
Sub-Campus Bhakkar, Bhakkar, Pakistan
e-mail: sajid_uos2000@yahoo.com

Josip Pečarić
Faculty of Textile Technology, University of Zagreb
Prilaz baruna Filipovića 28a, 10000 Zagreb, Croatia
e-mail: pecaric@element.hr