AN OPIAL–TYPE INTEGRAL INEQUALITY
AND EXPONENTIALLY CONVEX FUNCTIONS

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Abstract. In this paper a certain class of convex functions in an Opial-type integral inequality is considered. Cauchy type mean value theorems are proved and used in studying Stolarsky type means defined by the observed integral inequality. Also, a method of producing \( n \)-exponentially convex and exponentially convex functions is applied. Some new Opial-type inequalities are given for different types of fractional integrals and fractional derivatives as applications.

1. Introduction and preliminaries

We consider a particular class of convex functions in an Opial-type integral inequality from which we construct functionals \( \Phi_i \) \((i = 1, 2)\). Our object is to give Cauchy type mean value theorems and use them for Stolarsky type means, all defined by the observed integral inequality, and also, to prove the \( n \)-exponential convexity for the functionals. Also we produce new Opial-type inequalities for fractional integrals (of function with respect to an increasing function, the Riemann–Liouville and the Hadamard fractional integrals) and fractional derivatives (the Riemann–Liouville, the Canavati and the Caputo fractional derivatives).

We say that a function \( u : [a, b] \to \mathbb{R} \) belongs to the class \( U(v, K) \) if it admits the representation

\[
  u(x) = \int_a^x K(x, t)v(t)\,dt, \quad (1)
\]

where \( v \) is a continuous function and \( K \) is an arbitrary nonnegative kernel such that \( v(x) > 0 \) implies \( u(x) > 0 \) for every \( x \in [a, b] \). We also assume that all integrals under consideration exist and are finite.

The following inequality is given by Mitroinović and Pečarić in [9] (also see [1, p. 89] and [11, p. 236]).

**Theorem 1.1.** Let \( u_1 \in U(v_1, K), u_2 \in U(v_2, K) \) and \( v_2(x) > 0 \) for every \( x \in [a, b] \). Further, let \( \phi(u) \) and \( f(u) \) be convex and increasing for \( u \geq 0 \) and \( f(0) = 0 \). If

\[
  \int_a^b \phi_{\frac{u_1}{u_2}} \, dt \leq \int_a^b \phi_{\frac{\phi(u_1)}{\phi(u_2)}} \, dt,
\]

where \( \phi_{\frac{u_1}{u_2}} \) and \( \phi_{\frac{\phi(u_1)}{\phi(u_2)}} \) are defined in the sense of Stolarsky means.


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f is a differentiable function and $M = \max K(x, t)$, then

$$
M \int_a^b v_2(t) \phi \left( \frac{v_1(t)}{v_2(t)} \right) f' \left( u_2(t) \phi \left( \frac{u_1(t)}{u_2(t)} \right) \right) dt \\
\leq f \left( M \int_a^b v_2(t) \phi \left( \frac{v_1(t)}{v_2(t)} \right) dt \right).
$$

(2)

The rest of the paper is organized in the following way: in Section 2 we give some new generalized Opial-type inequalities from which we construct functionals and prove Cauchy type mean value theorems. Next, in Section 3, we prove some new Opial-type inequalities for fractional integrals and fractional derivatives as an application of our main results. In Section 4 we produce the $n$-exponentially convex functions by applying an elegant method of exponential convexity. At the end of the paper, we use Cauchy mean value theorems for Stolarsky type means defined by the observed functionals to give the related examples (see Section 5).

2. Main results

In the following theorem we give the generalization of the inequality (2).

**Theorem 2.1.** Let $u_1 \in U(v_1, K)$, $u_2 \in U(v_2, K)$ and $v_2(x) > 0$ for every $x \in [a, b]$. Further, let $\phi(u)$ be convex, nonnegative and increasing for $u \geq 0$, $f(u)$ be convex for $u \geq 0$, and $f(0) = 0$. If $f$ is a differentiable function and $M = \max K(x, t)$, then these inequalities are valid:

$$
M \int_a^b v_2(t) \phi \left( \frac{v_1(t)}{v_2(t)} \right) f' \left( u_2(t) \phi \left( \frac{u_1(t)}{u_2(t)} \right) \right) dt \\
\leq f \left( M \int_a^b v_2(t) \phi \left( \frac{v_1(t)}{v_2(t)} \right) dt \right) \\
\leq \frac{1}{b - a} \int_a^b f \left( M(b - a)v_2(t) \phi \left( \frac{v_1(t)}{v_2(t)} \right) \right) dt.
$$

(3)

(4)

**Proof.** On the right hand side of the inequality (3), if we multiply and divide by the factor $(b - a)$ inside and outside the integral and use Jensen’s inequality for the function $f$, then we obtain the inequality (4).

The condition of Theorem 1.1 that the function $f$ is increasing is actually unneeded. From the proof of the theorem [11, p. 236] one can see that this property is never used, therefore we omit it here. Also, a condition that is missing in Theorem 1.1 is that $\phi$ has to be nonnegative, which we add. □
Motivated by the inequalities given in Theorem 2.1, we define two functionals as:

$$
\Phi_1(f) = f \left( M \int_a^b v_2(t) \phi \left( \left| \frac{v_1(t)}{v_2(t)} \right| \right) dt \right)
- M \int_a^b v_2(t) \phi \left( \left| \frac{v_1(t)}{v_2(t)} \right| \right) f' \left( u_2(t) \phi \left( \left| \frac{u_1(t)}{u_2(t)} \right| \right) \right) dt
$$

(5)

$$
\Phi_2(f) = \frac{1}{b-a} \int_a^b f \left( M(b-a)v_2(t) \phi \left( \left| \frac{v_1(t)}{v_2(t)} \right| \right) \right) dt
-f \left( M \int_a^b v_2(t) \phi \left( \left| \frac{v_1(t)}{v_2(t)} \right| \right) dt \right),
$$

(6)

where \( f \) is a differentiable function with \( f(0) = 0 \), and \( M, \phi, u_i, v_i \) \( (i = 1, 2) \) are as in Theorem 1.1.

If \( f \) is a convex function, then Theorem 2.1 implies that \( \Phi_i(f) \geq 0 \) \( (i = 1, 2) \).

Now, we give mean value theorems for the functionals \( \Phi_i \) \( (i = 1, 2) \).

Let \( 0 < m_2 \leq v_2 \leq M_2 \), \( 0 \leq |v_1| \leq M_1 \) and \( \phi \geq 0 \). Then \( 0 \leq |v_1|/v_2 | \leq M_1/m_2 \). It follows

$$
m_2 M(b-a) \min_{[0, M_1/m_2]} \phi \leq M \int_a^b v_2(t) \phi \left( \left| \frac{v_1(t)}{v_2(t)} \right| \right) dt \leq M_2 M(b-a) \max_{[0, M_1/m_2]} \phi.
$$

Also

$$
0 \leq \left| \frac{u_1(t)}{u_2(t)} \right| \leq \frac{M_1 f_a \int_a^b K(x, \tau) d\tau}{m_2 f_a \int_a^b K(x, \tau) d\tau} = \frac{M_1}{m_2}.
$$

Since obviously \( |u_2(t)| \leq MM_2(b-a) \), we have

$$
0 \leq u_2(t) \phi \left( \left| \frac{u_1(t)}{u_2(t)} \right| \right) \leq MM_2(b-a) \max_{[0, M_1/m_2]} \phi.
$$

Hence, from now on let \( f : I \rightarrow \mathbb{R} \) where

$$
I = \left[ 0, MM_2(b-a) \max_{[0, M_1/m_2]} \phi \right].
$$

(7)

**Theorem 2.2.** Let \( u_1 \in U(v_1, K) \), \( u_2 \in U(v_2, K) \) and \( v_2(x) > 0 \) for every \( x \in [a, b] \). Further, let \( \phi(u) \) be convex, nonnegative and increasing for \( u \geq 0 \). Let \( f \in C^2(I) \) and \( f(0) = 0 \). Then there exists \( \xi \in I \) such that

$$
\Phi_i(f) = \frac{f''(\xi)}{2} \Phi_i(f_0), \quad (i = 1, 2),
$$

(8)

where \( f_0(x) = x^2 \).
Proof. Since \( f \in C^2(I) \), there exist real numbers \( m = \min_{x \in I} f''(x) \) and \( M = \max_{x \in I} f''(x) \). Hence, the functions \( f_1 \) and \( f_2 \) defined by

\[
f_1(x) = \frac{M}{2} x^2 - f(x),
\]
\[
f_2(x) = f(x) - \frac{m}{2} x^2
\]

are convex. Therefore \( \Phi_i(f_1) \geq 0, \Phi_i(f_2) \geq 0 \) \( (i = 1, 2) \), and we get

\[
\frac{m}{2} \Phi_i(f_0) \leq \Phi_i(f) \leq \frac{M}{2} \Phi_i(f_0).
\]

If \( \Phi_i(f_0) = 0 \), then there is nothing to prove. Suppose \( \Phi_i(f_0) > 0 \). We have

\[
m \leq \frac{2 \Phi_i(f)}{\Phi_i(f_0)} \leq M.
\]

Hence, there exists \( \xi \in I \) such that

\[
\Phi_i(f) = \frac{f''(\xi)}{2} \Phi_i(f_0), \quad (i = 1, 2).
\]

This completes the proof. \( \Box \)

Theorem 2.3. Let \( u_1 \in U(v_1, K), u_2 \in U(v_2, K) \) and \( v_2(x) > 0 \) for every \( x \in [a, b] \). Further, let \( \phi(u) \) be convex, nonnegative and increasing for \( u \geq 0 \). Let \( f, g \in C^2(I) \) and \( f(0) = g(0) = 0 \). Then there exists \( \xi \in I \) such that

\[
\frac{\Phi_i(f)}{\Phi_i(g)} = \frac{f''(\xi)}{g''(\xi)}, \quad (i = 1, 2),
\]

provided that the denominators are non-zero.

Proof. Define \( h \in C^2(I) \) by \( h = c_1 f - c_2 g \), where

\[
c_1 = \Phi_i(g), \quad c_2 = \Phi_i(f), \quad (i = 1, 2).
\]

Now using Theorem 2.2 with \( f = h \) there exists \( \xi \in I \) such that

\[
\left( c_1 \frac{f''(\xi)}{2} - c_2 \frac{g''(\xi)}{2} \right) \Phi_i(f_0) = 0, \quad (i = 1, 2).
\]

Since \( \Phi_i(f_0) \neq 0 \) (otherwise we have a contradiction with \( \Phi_i(g) \neq 0 \) by Theorem 2.2), we get

\[
\frac{\Phi_i(f)}{\Phi_i(g)} = \frac{f''(\xi)}{g''(\xi)}, \quad (i = 1, 2).
\]

This completes the proof. \( \Box \)
3. Opial-type inequalities for fractional integrals and fractional derivatives

Here we present some new Opial-type inequalities involving fractional integrals and fractional derivatives. For more details on the fractional integrals of a function with respect to another function, the Riemann–Liouville and the Hadamard fractional integrals, see e.g. [8, Section 2.1, 2.5 and 2.7].

Let \((a, b), \ -\infty \leq a < b \leq \infty,\) be a finite or infinite interval of the real line \(\mathbb{R}\) and \(\alpha > 0.\) Also let \(g\) be an increasing function on \((a, b)\) and \(g'\) be a continuous function on \((a, b)\). The left-sided and right-sided fractional integrals of a function \(f\) with respect to another function \(g\) in \([a, b]\) are given by

\[
I_{a+;g}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t)f(t)dt}{[g(x) - g(t)]^{1-\alpha}}, \quad x > a,
\]

\[
I_{b-;g}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t)f(t)dt}{[g(t) - g(x)]^{1-\alpha}}, \quad x < b,
\]

respectively. Here \(\Gamma\) is the gamma function \(\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt.\)

**Theorem 3.1.** Let \(\alpha \geq 1, \ \phi(u)\) be convex, nonnegative and increasing for \(u \geq 0\) and let \(f(u)\) be convex for \(u \geq 0\) with \(f(0) = 0.\) If \(f\) is a differentiable function, then these inequalities are valid:

\[
\frac{(g(b) - g(a))^{\alpha-1}}{\Gamma(\alpha)} \max_{x \in [a, b]} g'(x) \int_a^b u_2(t) \phi \left( \frac{u_1(t)}{u_2(t)} \right) f' \left( \frac{I_{a+;g}^\alpha u_2(t)}{I_{a+;g}^\alpha u_1(t)} \right) \frac{g'(t)}{[g(x) - g(t)]^{1-\alpha}} dt \\
\leq f \left( \frac{(g(b) - g(a))^{\alpha-1}}{\Gamma(\alpha)} \max_{x \in [a, b]} g'(x) \right) \int_a^b u_2(t) \phi \left( \frac{u_1(t)}{u_2(t)} \right) dt \\
\leq \frac{1}{b-a} \int_a^b f \left( \frac{(b-a)(g(b) - g(a))^{\alpha-1}}{\Gamma(\alpha)} \max_{x \in [a, b]} g'(x) u_2(t) \phi \left( \frac{u_1(t)}{u_2(t)} \right) \right) dt.
\]

**Proof.** We use Theorem 2.1 with the following kernel,

\[
K(x, t) = \left\{ \begin{array}{ll}
\frac{g'(t)}{\Gamma(\alpha)(g(x) - g(t))^{1-\alpha}}, & a < t \leq x; \\
0, & x < t \leq b.
\end{array} \right.
\]

For \(\alpha \geq 1,\) we get

\[
M = \max K(x, t) = \frac{(g(b) - g(a))^{\alpha-1}}{\Gamma(\alpha)} \max_{x \in [a, b]} g'(x).
\]
If we replace \( u_i \) by \( I_{a^+}^\alpha u_i \) and \( v_i \) by \( u_i \) \( (i = 1, 2) \) in inequalities given in (3) an (4), then the inequality (10) follows. □

Let \([a, b], -\infty < a < b < \infty\), be a finite interval on the real axis \( \mathbb{R} \). For \( f \in L_1[a, b] \) the left-sided and right-sided Riemann–Liouville fractional integrals of order \( \alpha > 0 \) are defined by

\[
I_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) \, dt, \quad x > a,
\]

\[
I_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) \, dt, \quad x < b.
\]

If \( g(x) = x \), then \( I_{a^+}^\alpha f(x) \) reduces to \( I_{a^+}^\alpha f(x) \), i.e. left-sided Riemann–Liouville fractional integral. Same follows for the right-sided fractional integral. This gives us the next result.

**Corollary 3.2.** Let \( \alpha \geq 1 \), \( \phi(u) \) be convex, nonnegative and increasing for \( u \geq 0 \) and let \( f(u) \) be convex for \( u \geq 0 \) with \( f(0) = 0 \). If \( f \) is a differentiable function, then these inequalities are valid:

\[
\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} \int_a^b u_2(t) \phi \left( \frac{|u_1(t)|}{u_2(t)} \right) f' \left( I_{a^+}^\alpha u_2(t) \phi \left( \frac{|I_{a^+}^\alpha u_1(t)|}{I_{a^+}^\alpha u_2(t)} \right) \right) \, dt \\
\leq f \left( \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} \int_a^b u_2(t) \phi \left( \frac{|u_1(t)|}{u_2(t)} \right) \, dt \right) \\
\leq \frac{1}{b-a} \int_a^b f \left( \frac{(b-a)^\alpha}{\Gamma(\alpha)} u_2(t) \phi \left( \frac{|u_1(t)|}{u_2(t)} \right) \right) \, dt.
\]

Let \((a, b)\) be a finite or infinite interval of \( \mathbb{R}^+ \) and \( \alpha > 0 \). The left-sided and right-sided Hadamard fractional integrals of order \( \alpha > 0 \) are given by

\[
J_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left( \log \frac{x}{t} \right)^{\alpha-1} f(t) \, dt, \quad x > a,
\]

\[
J_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left( \log \frac{t}{x} \right)^{\alpha-1} f(t) \, dt, \quad x < b.
\]

Notice that Hadamard fractional integrals of order \( \alpha \) are special cases of the left-sided and right-sided fractional integrals of a function \( f \) with respect to a function \( g(x) = \log x \) in \((a, b)\), where \( 0 \leq a < b \leq \infty \).
Corollary 3.3. Let \( \alpha \geq 1 \), \( \phi(u) \) be convex, nonnegative and increasing for \( u \geq 0 \) and let \( f(u) \) be convex for \( u \geq 0 \) with \( f(0) = 0 \). If \( f \) is a differentiable function, then these inequalities are valid:

\[
\frac{(\log b - \log a)^{\alpha-1}}{a \Gamma(\alpha)} \int_a^b u_2(t) \phi \left( \left| \frac{u_1(t)}{u_2(t)} \right| \right) f' \left( J_{a+}^\alpha u_2(t) \phi \left( \left| J_{a+}^\alpha u_1(t) \right| \right) \right) dt \\
\leq f \left( \frac{1}{a \Gamma(\alpha)} (\log b - \log a)^{\alpha-1} \int_a^b u_2(t) \phi \left( \left| \frac{u_1(t)}{u_2(t)} \right| \right) dt \right) \\
\leq \frac{1}{b-a} \int_a^b f \left( \frac{(b-a)(\log b - \log a)^{\alpha-1}}{a \Gamma(\alpha)} u_2(t) \phi \left( \left| \frac{u_1(t)}{u_2(t)} \right| \right) \right) dt.
\]

(12)

Remark 3.4. Similarly we can give analogous results for the right-sided fractional integrals but here we omit the details.

Next we follow with definitions of the Riemann–Liouville, Canavati and Caputo fractional derivatives (see [8, Section 2.1 and 2.4] and [6]) and obtain results.

Let \( \alpha > 0 \) and \( n = [\alpha] + 1 \) ([ ] is the integral part). For \( f : [a,b] \to \mathbb{R} \) the left-sided and right-sided Riemann–Liouville fractional derivatives of order \( \alpha \) are defined by

\[
D_{a+}^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t) dt = \frac{d^n}{dx^n} l_{a+}^{n-\alpha} f(x),
\]

\[
D_{b-}^\alpha f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b (t-x)^{n-\alpha-1} f(t) dt = (-1)^n \frac{d^n}{dx^n} l_{b-}^{n-\alpha} f(x).
\]

The following lemma summarizes conditions in the composition identity for the left-sided Riemann–Liouville fractional derivative. For details see [2, Section 3].

Lemma 3.5. Let \( \beta > \alpha \geq 0 \), \( n = [\beta] + 1 \), \( m = [\alpha] + 1 \). The composition identity

\[
D_{a+}^\alpha f(x) = \frac{1}{\Gamma(\beta-\alpha)} \int_a^x (x-t)^{\beta-\alpha-1} D_{a+}^\beta f(t) dt, \quad x \in [a,b],
\]

(13)
is valid if one of the following conditions holds:

(i) \( f \in l_{a+}^\beta (L_1(a,b)) \).

(ii) \( l_{a+}^{n-k} f \in AC^n[a,b] \) and \( D_{a+}^{\alpha-k} f(a) = 0 \) for \( k = 1, \ldots n \).

(iii) \( D_{a+}^{\beta-k} f \in C[a,b] \) for \( k = 1, \ldots, n \), \( D_{a+}^{\beta-1} f \in AC[a,b] \) and \( D_{a+}^{\beta-k} f(a) = 0 \) for \( k = 1, \ldots, n \).

(iv) \( f \in AC^n[a,b] \), \( D_{a+}^{\beta} f \in L_1(a,b) \), \( D_{a+}^\alpha f \in L_1(a,b) \), \( \beta - \alpha \notin \mathbb{N} \), \( D_{a+}^{\beta-k} f(a) = 0 \) for \( k = 1, \ldots, n \) and \( D_{a+}^{\alpha-k} f(a) = 0 \) for \( k = 1, \ldots, m \).
(v) $f \in AC^n[a,b], D_{a^+}^\beta f \in L_1(a,b), D_{a^+}^\alpha f \in L_1(a,b)$, $\beta - \alpha = l \in \mathbb{N}$, $D_{a^+}^{\beta - k} f(a) = 0$ for $k = 1, \ldots, l$.

(vi) $f \in AC^n[a,b], D_{a^+}^\beta f \in L_1(a,b), D_{a^+}^\alpha f \in L_1(a,b)$ and $f(a) = f'(a) = \cdots = f^{(n-2)}(a) = 0$.

(vii) $f \in AC^n[a,b], D_{a^+}^\beta f \in L_1(a,b), D_{a^+}^\alpha f \in L_1(a,b)$, $\beta \notin \mathbb{N}$ and $D_{a^+}^{\beta - 1} f$ is bounded in a neighborhood of $t = a$.

**Theorem 3.6.** Let $\beta > \alpha + 1$ and let the assumptions of Lemma 3.5 be satisfied. Let $D_{a^+}^\beta u_2(x) > 0$ for every $x \in [a,b]$. Further, let $\phi(u)$ be convex, nonnegative and increasing for $u \geq 0$ and let $f(u)$ be convex for $u \geq 0$ with $f(0) = 0$. Then these inequalities are valid:

$$
\frac{(b-a)^{\beta - \alpha - 1}}{\Gamma(\beta - \alpha)} \int_a^b D_{a^+}^\beta u_2(t) \phi \left( \frac{D_{a^+}^\beta u_1(t)}{D_{a^+}^\alpha u_2(t)} \right) f' \left( D_{a^+}^\alpha u_2(t) \phi \left( \frac{D_{a^+}^\alpha u_1(t)}{D_{a^+}^\alpha u_2(t)} \right) \right) dt \\
\leq f \left( \frac{(b-a)^{\beta - \alpha - 1}}{\Gamma(\beta - \alpha)} \int_a^b D_{a^+}^\beta u_2(t) \phi \left( \frac{D_{a^+}^\beta u_1(t)}{D_{a^+}^\beta u_2(t)} \right) dt \right) \\
\leq \frac{1}{b-a} \int_a^b f \left( \frac{(b-a)^{\beta - \alpha}}{\Gamma(\beta - \alpha)} D_{a^+}^\beta u_2(t) \phi \left( \frac{D_{a^+}^\beta u_1(t)}{D_{a^+}^\beta u_2(t)} \right) \right) dt.
$$

(14)

**Proof.** We use Theorem 2.1 with the following kernel,

$$
K(x,t) = \begin{cases}
(x-t)^{\beta - \alpha - 1} / \Gamma(\beta - \alpha), & a < t \leq x; \\
0, & x < t \leq b.
\end{cases}
$$

For $\beta > \alpha + 1$, we get

$$
M = \max K(x,t) = \frac{(b-a)^{\beta - \alpha - 1}}{\Gamma(\beta - \alpha)}.
$$

If we replace $u_i$ by $D_{a^+}^\alpha u_i$ and $v_i$ by $D_{a^+}^\beta u_i$ ($i = 1, 2$) in inequalities given in (3) and (4), then the inequality (14) follows. $\square$

Next we consider subspaces $C_{a^+}^\alpha[a,b]$ and $C_{b^-}^\alpha[a,b]$ of $C^{n-1}[a,b]$ defined by

$$
C_{a^+}^\alpha[a,b] = \left\{ f \in C^{n-1}[a,b] : I_{a^+}^{n-\alpha} f^{(n-1)} \in C^1[a,b] \right\},
$$

$$
C_{b^-}^\alpha[a,b] = \left\{ f \in C^{n-1}[a,b] : I_{b^-}^{n-\alpha} f^{(n-1)} \in C^1[a,b] \right\}.
$$
For \( f \in C^\alpha_{a^+}[a,b] \) and \( g \in C^\alpha_{b^-}[a,b] \) the left-sided and right-sided Canavati fractional derivatives of order \( \alpha \) are defined by

\[
C^\alpha_{a^+} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{n-\alpha-1} f^{(n-1)}(t) \, dt = \frac{d}{dx} I_{a^+}^{n-\alpha} f^{(n)}(x),
\]

\[
C^\alpha_{b^-} g(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d}{dx} \int_a^x (t-x)^{n-\alpha-1} g^{(n-1)}(t) \, dt = (-1)^n \frac{d}{dx} I_{b^-}^{n-\alpha} g^{(n)}(x).
\]

The following lemma gives conditions in the composition rule for the left-sided Canavati fractional derivative. For details see [4, Section 2].

**Lemma 3.7.** Let \( \beta > \alpha > 0 \), \( n = [\beta] + 1 \), \( m = [\alpha] + 1 \). Let \( f \in C^\beta_{a^+}[a,b] \) be such that \( f^{(i)}(a) = 0 \) for \( i = m-1, \ldots, n-2 \). Then \( f \in C^\alpha_{a^+}[a,b] \) and

\[
C^\alpha_{a^+} f(x) = \frac{1}{\Gamma(\beta-\alpha)} \int_a^x (x-t)^{\beta-\alpha-1} C^\beta_{a^+} f(t) \, dt, \quad x \in [a,b].
\]

**Theorem 3.8.** Let \( \beta > \alpha + 1 \) and let the assumptions of Lemma 3.7 be satisfied. Let \( C^\beta_{a^+} u_2(x) > 0 \) for every \( x \in [a,b] \). Further, let \( \phi(u) \) be convex, nonnegative and increasing for \( u \geq 0 \) and let \( f(u) \) be convex for \( u \geq 0 \) with \( f(0) = 0 \). Then these inequalities are valid:

\[
\frac{(b-a)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} \int_a^b C^\beta_{a^+} u_2(t) \phi \left( \frac{C^\beta_{a^+} u_1(t)}{C^\beta_{a^+} u_2(t)} \right) f' \left( \frac{C^\alpha_{a^+} u_1(t)}{C^\alpha_{a^+} u_2(t)} \right) dt
\]

\[
\leq f \left( \frac{(b-a)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} \int_a^b C^\beta_{a^+} u_2(t) \phi \left( \frac{C^\beta_{a^+} u_1(t)}{C^\beta_{a^+} u_2(t)} \right) dt \right)
\]

\[
\leq \frac{1}{b-a} \int_a^b f \left( \frac{(b-a)^{\beta-\alpha}}{\Gamma(\beta-\alpha)} C^\beta_{a^+} u_2(t) \phi \left( \frac{C^\beta_{a^+} u_1(t)}{C^\beta_{a^+} u_2(t)} \right) dt \right).
\]

(15)

**Proof.** The proof of this theorem is a similar to the proof of Theorem 3.6. \( \square \)

Let \( \alpha \geq 0 \) and \( n = [\alpha] + 1 \). For \( f \in AC^n[a,b] \) the left-sided and right-sided Caputo fractional derivatives of order \( \alpha \) are defined by

\[
\tilde{C}^\alpha_{a^+} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t) \, dt = I_{a^+}^{n-\alpha} f^{(n)}(x),
\]

\[
\tilde{C}^\alpha_{b^-} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_a^x (t-x)^{n-\alpha-1} f^{(n)}(t) \, dt = (-1)^n I_{b^-}^{n-\alpha} f^{(n)}(x).
\]

The final composition identity for the left-sided Caputo fractional derivative is given in [3, Section 2].
Lemma 3.9. Let $\beta > \alpha \geq 0$, $n = [\beta] + 1$, $m = [\alpha] + 1$, $f \in AC^n[a,b]$ and $\overline{c}D^\beta_{a+}f$, $\overline{c}D^\alpha_{a+}f \in L_1[a,b]$. Suppose that one of the following conditions hold:

(i) $\beta, \alpha \not\in N_0$ and $f^{(i)}(a) = 0$ for $i = m, \ldots, n - 1$.

(ii) $\beta \in N_0$, $\alpha \not\in N_0$ and $f^{(i)}(a) = 0$ for $i = m, \ldots, n - 2$.

(iii) $\beta \not\in N_0$, $\alpha \in N_0$ and $f^{(i)}(a) = 0$ for $i = m - 1, \ldots, n - 1$.

(iv) $\beta \in N_0$, $\alpha \in N_0$ and $f^{(i)}(a) = 0$ for $i = m - 1, \ldots, n - 2$.

Then

$$\overline{c}D^\alpha_{a+}f(x) = \frac{1}{\Gamma(\beta - \alpha)} \int_a^x (x-t)^{\beta-\alpha-1} \overline{c}D^\beta_{a+}f(t) \, dt, \, x \in [a,b].$$

Theorem 3.10. Let $\beta > \alpha + 1$ and let the assumptions of Lemma 3.9 be satisfied. Let $\overline{c}D^\beta_{a+}u_2(x) > 0$ for every $x \in [a,b]$. Further, let $\phi(u)$ be convex, nonnegative and increasing for $u \geq 0$ and let $f(u)$ be convex for $u \geq 0$ with $f(0) = 0$. Then these inequalities are valid:

$$\frac{(b-a)^{\beta-\alpha-1}}{\Gamma(\beta - \alpha)} \int_a^b \overline{c}D^\beta_{a+}u_2(t) \phi \left( \frac{\overline{c}D^\beta_{a+}u_1(t)}{\overline{c}D^\alpha_{a+}u_2(t)} \right) f' \left( \overline{c}D^\alpha_{a+}u_2(t) \phi \left( \frac{\overline{c}D^\alpha_{a+}u_1(t)}{\overline{c}D^\alpha_{a+}u_2(t)} \right) \right) \, dt$$

$$\leq f \left( \frac{(b-a)^{\beta-\alpha-1}}{\Gamma(\beta - \alpha)} \int_a^b \overline{c}D^\beta_{a+}u_2(t) \phi \left( \frac{\overline{c}D^\beta_{a+}u_1(t)}{\overline{c}D^\alpha_{a+}u_2(t)} \right) \, dt \right)$$

$$\leq \frac{1}{b-a} \int_a^b f \left( \frac{(b-a)^{\beta-\alpha}}{\Gamma(\beta - \alpha)} \overline{c}D^\beta_{a+}u_2(t) \phi \left( \frac{\overline{c}D^\alpha_{a+}u_1(t)}{\overline{c}D^\alpha_{a+}u_2(t)} \right) \right) \, dt. \quad (16)$$

Proof. The proof of this theorem is similar to the proof of Theorem 3.6. □

Remark 3.11. As we define the functionals $\Phi_1$ and $\Phi_2$ from the inequalities given in (3) and (4), similarly we can define the functionals from the inequalities given in (10), (11), (12), (14), (15) and (16) and analogously we obtain the mean value theorems for our defined functionals.

4. Method of exponential convexity

Following definitions and properties of exponentially convex functions comes from [5], also [10]. Let $I$ be an interval in $\mathbb{R}$.

Definition 4.1. A function $\psi : I \to \mathbb{R}$ is $n$-exponentially convex in the Jensen sense on $I$ if

$$\sum_{i,j=1}^n \xi_i \xi_j \psi \left( \frac{x_i + x_j}{2} \right) \geq 0$$
holds for all choices $\xi_i \in \mathbb{R}$ and $x_i \in I$, $i = 1, \ldots, n$.

A function $\psi: I \rightarrow \mathbb{R}$ is $n$-exponentially convex if it is $n$-exponentially convex in the Jensen sense and continuous on $I$.

**Remark 4.2.** It is clear from the definition that $1$-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, $n$-exponentially convex functions in the Jensen sense are $k$-exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leq n$.

By definition of positive semi-definite matrices and some basic linear algebra we have the following proposition.

**Proposition 4.3.** If $\psi$ is an $n$-exponentially convex in the Jensen sense, then the matrix

$$
\left[ \psi \left( \frac{x_i + x_j}{2} \right)^k \right]_{i,j=1}^n
$$

is a positive semi-definite matrix for all $k \in \mathbb{N}$, $k \leq n$.

Particularly, $\det \left[ \psi \left( \frac{x_i + x_j}{2} \right)^k \right]_{i,j=1}^n \geq 0$ for all $k \in \mathbb{N}$, $k \leq n$.

**Definition 4.4.** A function $\psi: I \rightarrow \mathbb{R}$ is exponentially convex in the Jensen sense on $I$ if it is $n$-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

A function $\psi: I \rightarrow \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

**Remark 4.5.** It is known (and easy to show) that $\psi: I \rightarrow (0, \infty)$ is log-convex in the Jensen sense if and only if

$$
\alpha^2 \psi(x) + 2\alpha \beta \psi \left( \frac{x+y}{2} \right) + \beta^2 \psi(y) \geq 0
$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a function is log-convex in the Jensen-sense if and only if it is $2$-exponentially convex in the Jensen sense.

Also, using basic convexity theory it follows that a function is log-convex if and only if it is $2$-exponentially convex.

We will also need following results (see for example [11]).

**Proposition 4.6.** If $x_1, x_2, x_3 \in I$ are such that $x_1 < x_2 < x_3$, then the function $f: I \rightarrow \mathbb{R}$ is convex if and only if the inequality

$$
(x_3 - x_2) f(x_1) + (x_1 - x_3) f(x_2) + (x_2 - x_1) f(x_3) \geq 0
$$

holds.

**Proposition 4.7.** If $f$ is a convex function on an interval $I$ and if $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid

$$
\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(y_2) - f(y_1)}{y_2 - y_1}.
$$
If the function \( f \) is concave, then the inequality reverses.

Next we need divided differences, commonly used when dealing with functions that have different degree of smoothness.

**Definition 4.8.** The second order divided difference of a function \( f : I \to \mathbb{R} \) at mutually different points \( y_0, y_1, y_2 \in I \) is defined recursively by

\[
[y_i; f] = f(y_i), \quad i = 0, 1, 2
\]

\[
[y_i, y_{i+1}; f] = \frac{f(y_{i+1}) - f(y_i)}{y_{i+1} - y_i}, \quad i = 0, 1
\]

\[
[y_0, y_1, y_2; f] = \frac{[y_1, y_2; f] - [y_0, y_1; f]}{y_2 - y_0}.
\]

(17)

**Remark 4.9.** The value \([y_0, y_1, y_2; f]\) is independent of the order of the points \( y_0, y_1 \) and \( y_2 \). This definition may be extended to include the case in which some or all the points coincide. Namely, taking the limit \( y_1 \to y_0 \) in (17), we get

\[
\lim_{y_1 \to y_0} [y_0, y_1, y_2; f] = [y_0, y_0, y_2; f] = \frac{f(y_2) - f(y_0) - f'(y_0)(y_2 - y_0)}{(y_2 - y_0)^2}, \quad y_2 \neq y_0
\]

provided that \( f' \) exists, and furthermore, taking the limits \( y_i \to y_0, i = 1, 2 \) in (17), we get

\[
\lim_{y_2 \to y_0} \lim_{y_1 \to y_0} [y_0, y_1, y_2; f] = [y_0, y_0, y_0; f] = \frac{f''(y_0)}{2}
\]

provided that \( f'' \) exists.

An elegant method of producing \( n \)-exponentially convex and exponentially convex functions is given in [7]. We use this to prove the \( n \)-exponential convexity for the functionals \( \Phi_i \) \( (i = 1, 2) \). The next theorem is analogous to the one given in [10, Theorem 3.9] and we give a proof for the reader’s convenience.

Note here that for the functionals \( \Phi_i \) \( (i = 1, 2) \) interval \( I \) is defined by (7).

**Theorem 4.10.** Let \( Y = \{f_s : s \in J\} \), where \( J \) is an interval in \( \mathbb{R} \), be a family of functions defined on an interval \( I \) in \( \mathbb{R} \), such that the function \( s \mapsto [y_0, y_1, y_2; f_s] \) is \( n \)-exponentially convex in the Jensen sense on \( J \) for every three mutually different points \( y_0, y_1, y_2 \in I \). Let \( \Phi_i \) \( (i = 1, 2) \) be linear functionals defined as in (5) and (6). Then \( s \mapsto \Phi_i(f_s) \) is \( n \)-exponentially convex function in the Jensen sense on \( J \). If the function \( s \mapsto \Phi_i(f_s) \) is also continuous on \( J \), then it is \( n \)-exponentially convex on \( J \).

**Proof.** For \( \xi_i \in \mathbb{R}, s_i \in J, i = 1, \ldots, n \), we define the function

\[
g(y) = \sum_{i,j=1}^{n} \xi_i \xi_j f_{s_i + s_j}(y).
\]
Using the assumption that the function \( s \mapsto [y_0, y_1, y_2; f_s] \) is \( n \)-exponentially convex in the Jensen sense, we have

\[
[y_0, y_1, y_2; g] = \sum_{i,j=1}^{n} \xi_i \xi_j [y_0, y_1, y_2; f_{y_i+y_j}] \geq 0,
\]

which in turn implies that \( g \) is a convex function on \( I \). Therefore we have \( \Phi_i(g) \geq 0 \) \( (i = 1,2) \). Hence

\[
\sum_{i,j=1}^{n} \xi_i \xi_j \Phi_i(f_{y_i+y_j}) \geq 0.
\]

We conclude that the function \( s \mapsto \Phi_i(f_s) \) is \( n \)-exponentially convex on \( J \) in the Jensen sense. If the function \( s \mapsto \Phi_i(f_s) \) is also continuous on \( J \), then \( s \mapsto \Phi_i(f_s) \) is \( n \)-exponentially convex by definition. \( \square \)

**Corollary 4.11.** Let \( \gamma = \{ f_s : s \in J \} \), where \( J \) is an interval in \( \mathbb{R} \), be a family of functions defined on an interval \( I \) in \( \mathbb{R} \), such that the function \( s \mapsto [y_0, y_1, y_2; f_s] \) is exponentially convex in the Jensen sense on \( J \) for every three mutually different points \( y_0, y_1, y_2 \in I \). Let \( \Phi_i \) \( (i = 1,2) \) be linear functionals defined as in (5) and (6). Then \( s \mapsto \Phi_i(f_s) \) is exponentially convex function in the Jensen sense on \( J \). If the function \( s \mapsto \Phi_i(f_s) \) is continuous on \( J \), then it is exponentially convex on \( J \).

Let us denote a mean for \( f_s, f_q \in \Omega \) by

\[
\mu_{s,q}(\Phi_i, \Omega) = \begin{cases} 
\frac{\Phi_i(\Phi^{-1}(q))}{\Phi_i(f_s)}^{\frac{1}{s-q}}, & s \neq q, \\
\exp\left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right), & s = q.
\end{cases}
\] (18)

**Theorem 4.12.** Let \( \Omega = \{ f_s : s \in J \} \), where \( J \) is an interval in \( \mathbb{R} \), be a family of functions defined on an interval \( I \) in \( \mathbb{R} \), such that the function \( s \mapsto [y_0, y_1, y_2; f_s] \) is \( 2 \)-exponentially convex in the Jensen sense on \( J \) for every three mutually different points \( y_0, y_1, y_2 \in I \). Let \( \Phi_i \) \( (i = 1,2) \) be linear functionals defined as in (5) and (6). Then the following statements hold:

(i) If the function \( s \mapsto \Phi_i(f_s) \) is continuous on \( J \), then it is \( 2 \)-exponentially convex function on \( J \). If the function \( s \mapsto \Phi_i(f_s) \) is additionally positive, then it is also log-convex on \( J \), and for \( r, s, t \in J \) such that \( r < s < t \), we have

\[
(\Phi_i(f_s))^{t-r} \leq (\Phi_i(f_r))^{t-s} (\Phi_i(f_t))^{s-r}, \quad i = 1,2.
\] (19)

(ii) If the function \( s \mapsto \Phi_i(f_s) \) is strictly positive and differentiable on \( J \), then for every \( s, q, r, t \in J \), such that \( s \leq r \) and \( q \leq t \), we have

\[
\mu_{s,q}(\Phi_i, \Omega) \leq \mu_{r,t}(\Phi_i, \Omega), \quad i = 1,2.
\] (20)
Proof. (i) The first part is an immediate consequence of Theorem 4.10 and in second part log-convexity on $J$ follows from Remark 4.5. Since $s \mapsto \Phi_i(f_s)$ is positive, for $r, s, t \in J$ such that $r < s < t$, with $f(s) = \log \Phi_i(f_s)$ in Proposition 4.6, we have

$$(t - s) \log \Phi_i(f_r) + (r - t) \log \Phi_i(f_s) + (s - r) \log \Phi_i(f_t) \geq 0.$$ 

This is equivalent to inequality (19).

(ii) The function $s \mapsto \Phi_i(f_s)$ is log-convex on $J$ by (i), that is, the function $s \mapsto \log \Phi_i(f_s)$ is convex on $J$. Applying Proposition 4.7 we get

$$\log \Phi_i(f_s) - \log \Phi_i(f_q) \leq \log \Phi_i(f_r) - \log \Phi_i(f_t)$$

for $s \leq r, q \leq t, s \neq q, r \neq t$, and therefore we have

$$\mu_{s,q}(\Phi_i, \Omega) \leq \mu_{r,t}(\Phi_i, \Omega).$$

Cases $s = q$ and $r = t$ follows from (21) as limit cases. $\square$

Remark 4.13. The results from Theorem 4.10, Corollary 4.11 and Theorem 4.12 still hold when two of the points $y_0, y_1, y_2 \in I$ coincide, for a family of differentiable functions $f_s$ such that the function $s \mapsto \max_{s \leq r \leq t} f_s$ is $n$-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense). Furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs can be obtained by recalling Remark 4.9 and suitable characterization of convexity.

Remark 4.14. As we prove the $n$-exponential convexity of the functionals $\Phi_1$ and $\Phi_2$ obtained from the inequalities given in (3) and (4), similarly we can define the functionals from the inequalities given in (10), (11), (12), (14), (15) and (16) and prove the $n$-exponential convexity of our defined functionals but here we omit the details.

5. Applications to Stolarsky type means

In this section, we use Cauchy type mean value Theorem 2.2 and Theorem 2.3 for Stolarsky type means, defined by the functional $\Phi_i$ ($i = 1, 2$). Several families of functions which fulfil the conditions of Theorem 4.10, Corollary 4.11 and Theorem 4.12 (and Remark 4.13) that we present here, enable us to construct large families of functions which are exponentially convex.

Example 5.1. Consider a family of functions

$$\Omega_1 = \{f_s : \mathbb{R} \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$f_s(x) = \begin{cases} \frac{e^{sx} - 1}{s^2}, & s \neq 0, \\ \frac{x^2}{2}, & s = 0. \end{cases}$$
Since \( \frac{d^2f_s}{dx^2}(x) = e^{sx} > 0 \), then \( f_s \) is convex on \( \mathbb{R} \) for every \( s \in \mathbb{R} \), and \( s \mapsto \frac{d^2f_s}{dx^2}(x) \) is exponentially convex by definition.

Analogously as in the proof of Theorem 4.10 we conclude that \( s \mapsto [y_0, y_1, y_2; f_s] \) is exponentially convex (and so exponentially convex in the Jensen sense).

Notice that \( f_s(0) = 0 \). By Corollary 4.11 we have that \( s \mapsto \Phi_i(f_s) \) \( (i = 1, 2) \) is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although mapping \( s \mapsto f_s \) is not continuous for \( s = 0 \)), so it is exponentially convex.

For this family of functions, \( \mu_{s,q}(\Phi_i, \Omega_1) \) \( (i = 1, 2) \) from (18) is equal to

\[
\mu_{s,q}(\Phi_i, \Omega_1) = \begin{cases} 
\frac{1}{s-q}, & s \neq q, \\
\exp\left(\frac{\Phi_i(id \cdot f_s) - 2}{s}\right), & s = q \neq 0, \\
\exp\left(\frac{\Phi_i(id \cdot f_0)}{3\Phi_i(f_0)}\right), & s = q = 0,
\end{cases}
\]

and using (20) it is a monotonous in parameters \( s \) and \( q \).

If \( \Phi_i \) is positive, \((i = 1, 2)\), then Theorem 2.3 applied for \( f = f_s \in \Omega_1 \) and \( g = f_q \in \Omega_1 \) yields that there exists \( \xi \in I = \left[0, MM_2(b-a) \max_{[0, M_1]} \phi\right] \) such that

\[
e^{(s-q)\xi} = \frac{\Phi_i(f_s)}{\Phi_i(f_q)}.
\]

It follows that

\[
M_{s,q}(\Phi_i, \Omega_1) = \log \mu_{s,q}(\Phi_i, \Omega_1)
\]

satisfy \( 0 \leq M_{s,q}(\Phi_i, \Omega_1) \leq MM_2(b-a) \max_{[0, M_1]} \phi \), which shows that \( M_{s,q}(\Phi_i, \Omega_1) \) is a mean, and by (20) it is a monotonous mean, \( i = 1, 2 \).

**Example 5.2.** Consider a family of functions

\[
\Omega_2 = \{ g_s : [0, \infty) \to \mathbb{R} : s \in \mathbb{R} \}
\]

defined by

\[
g_s(x) = \begin{cases} 
\frac{(x+1)^{s-1}}{s(s-1)}, & s \neq 0, 1, \\
-\log(x+1), & s = 0, \\
(x+1)\log(x+1), & s = 1.
\end{cases}
\]

Here, \( \frac{d^2g_s}{dx^2}(x) = (x+1)^{s-2} = e^{(s-2)\log(x+1)} > 0 \) which shows that \( g_s \) is convex for \( x > 0 \) and \( s \mapsto \frac{d^2g_s}{dx^2}(x) \) is exponentially convex by definition. Also, \( g_s(0) = 0 \). Arguing as in Example 5.1 we get that the mapping \( s \mapsto \Phi_i(g_s) \) is exponentially convex and also log-convex.
For this family of functions, \( \mu_{s,q}(\Phi_i, \Omega_2) \) from (18) is equal to

\[
\mu_{s,q}(\Phi_i, \Omega_2) = \begin{cases} 
\left( \frac{\Phi_i(g_s)}{\Phi_i(g_q)} \right)^{\frac{1}{s-q}}, & s \neq q, \\
\exp \left( \frac{1 - 2s}{s(s-1)} - \frac{\Phi_i(g_0 g_s)}{\Phi_i(g_s)} - \frac{1}{s(s-1)} \frac{\Phi_i(g_0)}{\Phi_i(g_s)} \right), & s = q \neq 0, 1, \\
\exp \left( 1 - \frac{\Phi_i(g_0^2)}{2\Phi_i(g_0)} \right), & s = q = 0, \\
\exp \left( -1 - \frac{\Phi_i(g_0 g_1)}{2\Phi_i(g_1)} \right), & s = q = 1,
\end{cases}
\]

and by (20) it is monotonous in parameters \( s \) and \( q \).

Using Theorem 2.3 it follows that there exists \( \xi \in I \) such that

\[(\xi + 1)^{s-q} = \frac{\Phi_i(g_s)}{\Phi_i(g_q)}.
\]

Since the function \( \xi \mapsto (\xi + 1)^{s-q} \) is invertible for \( s \neq q \), we have

\[0 \leq \left( \frac{\Phi_i(g_s)}{\Phi_i(g_q)} \right)^{\frac{1}{s-q}} \leq MM_2(b-a) \max \phi \left[ 0, \frac{M_1}{M_2} \right]
\]

which together with the fact that \( \mu_{s,q}(\Phi_i, \Omega_2) \) is continuous, symmetric and monotonous, shows that \( \mu_{s,q}(\Phi_i, \Omega_2) \) is a mean, \( i = 1, 2 \).

**Example 5.3.** Consider a family of functions

\[\Omega_3 = \{ h_s : [0, \infty) \rightarrow \mathbb{R} : s > 0 \}\]

defined by

\[h_s(x) = \begin{cases} 
x^{-x} \frac{x-1}{\log x}, & s \neq 1, \\
x^2, & s = 1.
\end{cases}
\]

Since \( s \mapsto \frac{d^2 h_s}{dx^2}(x) = s^{-x} \) is the Laplace transform of a non-negative function ([12]), that is \( s^{-x} = \frac{1}{\Gamma(x)} \int_0^\infty e^{-xt} t^{x-1} dt \), it is exponentially convex on \((0, \infty)\). Obviously \( h_s \) are convex functions for every \( s > 0 \) and \( h_s(0) = 0 \).

For this family of functions, \( \mu_{s,q}(\Phi_i, \Omega_3) \) from (18) is equal to

\[
\mu_{s,q}(\Phi_i, \Omega_3) = \begin{cases} 
\left( \frac{\Phi_i(h_s)}{\Phi_i(h_q)} \right)^{\frac{s}{s-q}}, & s \neq q, \\
\exp \left( -\frac{\Phi_i(id-h_s)}{3\Phi_i(h_s)} - \frac{2}{s \log s} \right), & s = q \neq 1, \\
\exp \left( -\frac{\Phi_i(id-h_1)}{3\Phi_i(h_1)} \right), & s = q = 1,
\end{cases}
\]
and it is monotonous in parameters $s$ and $q$ by (20).

Using Theorem 2.3 it follows that there exists $\xi \in I$ such that

$$\left(\frac{s}{q}\right)^{-\xi} = \frac{\Phi_i(h_s)}{\Phi_i(h_q)}.$$ 

Hence,

$$M_{s,q}(\Phi_i, \Omega_3) = -L(s,q) \log \mu_{s,q}(\Phi_i, \Omega_3),$$

satisfies $0 \leq M_{s,q}(\Phi_i, \Omega_3) \leq MM_2 (b-a) \max_{[0, \frac{M_i}{m_2}]} \phi$, which shows that $M_{s,q}(\Phi_i, \Omega_3)$ is a mean, $i = 1, 2$.

$L(s,q)$ is the logarithmic mean defined by

$$L(s,q) = \begin{cases} \frac{s-q}{\log s - \log q}, & s \neq q, \\ \frac{s}{q}, & s = q. \end{cases}$$

**Example 5.4.** Consider a family of functions $\Omega_4 = \{k_s : [0, \infty) \to \mathbb{R} : s > 0\}$ defined by

$$k_s(x) = \frac{e^{-x\sqrt{s}} - 1}{s}.$$ 

Again we conclude, since $s \mapsto \frac{d^2 k_s}{dx^2}(x) = e^{-x\sqrt{s}}$ is the Laplace transform of a non-negative function ([12]), that is $e^{-x\sqrt{s}} = \frac{s}{2\sqrt{\pi}} \int_{0}^{\infty} e^{-st} e^{-x^2/4t} t \sqrt{t} dt$ it is exponentially convex on $(0, \infty)$. For every $s > 0$, $k_s$ are convex functions and $k_s(0) = 0$.

For this family of functions, $\mu_{s,q}(\Phi_i, \Omega_4)$ from (18) is equal to

$$\mu_{s,q}(\Phi_i, \Omega_4) = \begin{cases} \left(\frac{\Phi_i(k_s)}{\Phi_i(k_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi_i(id\cdot k_s)}{2\sqrt{s \Phi_i(k_s)}} - \frac{1}{s}\right), & s = q, \end{cases}$$

and by (20) it is monotonous in parameters $s$ and $q$.

Using Theorem 2.3 it follows that there exists $\xi \in I$ such that

$$e^{-\xi(\sqrt{s}-\sqrt{q})} = \frac{\Phi_i(h_s)}{\Phi_i(h_q)}.$$ 

Hence,

$$M_{s,q}(\Phi_i, \Omega_4) = -(\sqrt{s} + \sqrt{q}) \log \mu_{s,q}(\Phi_i, \Omega_4)$$

satisfies $0 \leq M_{s,q}(\Phi_i, \Omega_4) \leq MM_2 (b-a) \max_{[0, \frac{M_i}{m_2}]} \phi$, which shows that $M_{s,q}(\Phi_i, \Omega_4)$ is a mean, $i = 1, 2$.

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REFERENCES