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PARAMETRIC STUDY OF FRACTIONAL BIOHEAT EQUATION IN SKIN TISSUE WITH SINUSOIDAL HEAT FLUX

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Abstract. This paper deals with the study of fractional bioheat equation for heat transfer in skin tissue with sinusoidal heat flux condition on skin surface. Numerical solution is obtained by implicit finite difference method. The effect of anomalous diffusion in skin tissue has been studied with different frequency and blood perfusion respectively, the temperature profile are obtained for different order fractional bioheat equation.

1. Introduction

Heat transfer in biological tissue, is usually expressed as bioheat equation, it involves thermal conduction, convection, perfusion of blood and metabolic heat generation in tissue. Pennes [10] bioheat model is widely used for study the heat transfer in skin tissue due to its simplicity. In human body skin is the largest living organ, temperature distribution in skin tissue is very important for medical application like skin cancer, skin burns etc.

Recently fractional order equations drawn the attention to many researchers and these are focused for many studies due to their frequent appearance in various applications in fluid mechanics, viescoelasticity, biology, physics and engineering etc. Fractals and fractional calculus have been used to improve the modelling accuracy of many phenomena in natural science. The most important advantage of using fractional differential equations in this and other applications is their non-local property. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. These are more realistic and also easy to make the fractional calculus popular [11]. Numerical solution of fractional diffusion equation by finite difference method are studied by many researcher, Meerschaert et al. [7], gave a second order accurate numerical approximation for the fractional diffusion equation, Murio [8] discussed implicit finite difference approximation for time fractional diffusion equation.

Shih et al. [12], Liu and Xu [3] gave analytic solution of pennes bio heat equation with sinusoidal heat flux condition on skin surface, Ng et al. [9] predict the parametric analysis of thermal profiles within heated human skin using boundary element method. Gafiychuk and Lubashevsky [4, 5] discussed Mathematical description of the heat transfer in living tissue. Recently Ahmadikia et al. [1] gave the analytical solution of the perabolic and hyperbolic heat transfer equation with constant and transient

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heat flux condition on skin tissue. Analytical solution of fractional bioheat equation is difficult to find.

In this paper, we give numerical solution of fractional bioheat equation with sinusoidal heat flux condition on skin surface with different frequency and blood perfusion respectively. We consider time fractional derivative of order $\alpha \in (0,1]$, which is in the form of Caputo fractional derivative and applying quadrature formula [8] on it. We use implicit finite difference method to solve the fractional bioheat model. The temperature profiles are obtained for different values of α, for studies the effect of α on temperature profile in skin tissue.

2. Heat transfer model

In this study, we consider fractional Pennes bioheat heat model. Pennes bioheat model [10] is implemented to study the heat transfer in skin tissue. In this model we replaced time derivative by Caputo fractional derivative with sinusoidal heat flux condition on skin surface and Neumann condition at the bottom of the tissue.

$$\rho_t c_t \frac{\partial^\alpha T}{\partial t^\alpha} = k\frac{\partial^2 T_s}{\partial x^2} + \rho_b w_b c_b (T_a - T) + q_{met} + q_{ext}, \qquad 0 < \alpha \leqslant 1, \tag{1}$$

where ρ, c, k, T, t and x represents density, specific heat, thermal conductivity, temperature, time and distance respectively; the subscript t and b are for the tissue and blood respectively. T_a and w_b are artillery temperature and blood perfusion rate respectively. q_{met} and q_{ext} are metabolic heat generation and external heat source in skin tissue respectively.

Used fractional derivative

Fractional derivative is denoted as ${}_aD_t^\alpha f(t)$, the subscript a and t denotes the two limits related to the operation of fractional differentiation, which is called the terminal of fractional differentiation. The negative values for α denoted for fractional integrals of arbitrary order.

***Caputo Fractional Derivative* [11]**

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial t} (t-s)^{-\alpha} d\tau, & \text{for } 0 < \alpha < 1 \\ \frac{\partial u(x,t)}{\partial t}, & \text{for } \alpha = 1 \end{cases} \tag{2}$$

We introduced governing equation for fractional Penns bioheat model,

$$\rho_t c_t \frac{\partial^\alpha T}{\partial t^\alpha} = k\frac{\partial^2 T_s}{\partial x^2} + \rho_b w_b c_b (T_a - T) + q_{met} + q_{ext}, \qquad 0 < \alpha \leqslant 1, \tag{3}$$

In our study we consider constant metabolic heat generation and zero external heat source. On setting $\alpha = 1$ then equation (3) reduces to Pennes bioheat equation [10].

The initial and boundary conditions for the model are considered as,

$$T(x,0) = T_a \tag{4}$$

$$-k\frac{\partial T}{\partial x}_{|x=L}=0, \tag{5}$$

2.1. Sinusoidal heat flux condition

In this section, on taking the cosine heat flux condition on skin surface,

$$-k\frac{\partial T}{\partial x}_{|x=0}=q_0\cos(\omega t), \tag{6}$$

where ω is the heating frequency, On making dimensionless variables, equation (3) to (5) and (6) becomes (7) to (9) in the following form,

$$\zeta=\sqrt{\frac{\omega\rho_t c_t}{k}}x, \qquad \eta=(\omega)^{(1/\alpha)}t, \qquad \theta=\left(\frac{T-T_a}{q_0}\right)\sqrt{k\omega\rho_t c_t},$$

$$C_1=\frac{w_b c_b}{\omega\rho_t c_t}, \qquad \Phi=\frac{q_{met}}{q_0\sqrt{\frac{\omega\rho_b c_b}{k}}}$$

$$\frac{\partial^\alpha\theta}{\partial t^\alpha}=\frac{\partial^2\theta}{\partial\zeta^2}-C_1\theta+\Phi, \qquad 0<\alpha\leqslant 1, \tag{7}$$

$$\theta(\zeta,0)=0 \tag{8}$$

$$\frac{\partial\theta}{\partial\zeta}_{|\zeta=0}=-\cos(\eta), \qquad \frac{\partial\theta}{\partial\zeta}_{|\zeta=\sqrt{\frac{\omega\rho c}{k}}L}=0 \tag{9}$$

3. Implicit finite difference scheme

For implicit finite difference scheme of equation (3) approximated the Caputo fractional derivative by quadrature formula [8] as follows,

$$\frac{\partial^\alpha\theta}{\partial t^\alpha}=\frac{1}{\Gamma(1-\alpha)}\frac{1}{1-\alpha}\frac{1}{k^\alpha}\sum_{j=1}^{n}(\theta_i^j-\theta_i^{j-1})[(n-j+1)^{(1-\alpha)}-(n-j)^{(1-\alpha)}] \tag{10}$$

$$+\frac{1}{\Gamma(1-\alpha)}\frac{1}{1-\alpha}\sum_{j=1}^{n}[(n-j+1)^{(1-\alpha)}-(n-j)^{(1-\alpha)}]O(k)^{2-\alpha}$$

Considering

$$W_j^{(\alpha)}=j^{(1-\alpha)}-(j-1)^{(1-\alpha)}$$

and

$$\sigma_{\alpha,k}=\frac{1}{\Gamma(1-\alpha)}\frac{1}{(1-\alpha)}\frac{1}{(k^\alpha)},$$

the Caputo fractional derivative given by equation(7) become,

$$\left(\frac{\partial^\alpha\theta}{\partial t^\alpha}\right)^n=\sigma_{\alpha,k}\sum_{j=1}^{n}W_j^{(\alpha)}\left(\theta_i^{n-j+1}-C_1\theta_i^{n-j}\right) \tag{11}$$

Now using above approximation and central difference formulae for space derivative in equation (7) we get,

$$\sigma_{\alpha,k}\sum_{j=0}^{n}W_j^{(\alpha)}\left(\theta_i^{n-j+1}-\theta_i^{n-j}\right)=\frac{1}{h^2}\left(\theta_{i-1}^n-2\theta_i^n+\theta_{i-1}^n\right)-C_1\theta_i^n+\Psi \tag{12}$$

On taking $\gamma=\frac{1}{h^2}$ and further simplification, gives

$$-\gamma\theta_{i-1}^n+(\sigma_{\alpha,k}+2\gamma+C_1)\theta_i^n-\gamma\theta_{i-1}^n-\theta_i^n=\sigma_{\alpha,k}\theta_i^{n-1}-\sigma_{\alpha,k}\sum_{j=2}^{n}W_j^{(\alpha)}\left(\theta_i^{n-j+1}-\theta_i^{n-j}\right)+\Psi \tag{13}$$

Initial condition and boundary conditions written as,

$$\theta_i^n|_{\eta=0}=0 \tag{14}$$

$$\theta_2^n-\theta_1^n=-\cos(\eta);\qquad \theta_{m+1}^n-\theta_m^n=0; \tag{15}$$

4. Stability analysis

On making the use of Fourier analysis [2], we can obtain stability of this scheme, which shows that scheme is unconditionally stable. Consider Θ_i^n be the approximate solution of (7) so error is.

$$\varepsilon_i^n=\theta_i^n-\Theta_i^n$$

The round of error of equation (17) is given as,

$$\begin{aligned}&-\gamma\varepsilon_{i-1}^n+(\sigma_{\alpha,k}+2\gamma+C_1)\varepsilon_i^n-\gamma\varepsilon_{i+1}^n\\&=\sigma_{\alpha,k}(1-\varepsilon_2^{\alpha})-\sigma_{\alpha,k}\sum_{j=2}^{n-1}W_j^{(\alpha)}\left(\varepsilon_i^{n-j+1}-\varepsilon_i^{n-j}\right)+W_n^{(\alpha)}\varepsilon_i^0+\Psi\end{aligned} \tag{16}$$

where,

$$\varepsilon^n=\left[\varepsilon_1^n,\varepsilon_2^n,\varepsilon_1^n,----------------,\varepsilon_{m-1}^n\right]$$

and, $\varepsilon_i^n=\xi_n e^{I\varpi_i h}$, where $\varpi=\frac{2\pi I}{L}$, $I=\sqrt{-1}$

$$\begin{aligned}&-\gamma\xi_n e^{I\varpi_{i-1}h}+(\sigma_{\alpha,k}+2\gamma+C_1)\xi_n e^{I\varpi_i h}-\gamma\xi_n e^{I\varpi_{i+1}h}\\&=\sigma_{\alpha,k}(1-W_2^{(\alpha)})\xi_{n-1}e^{I\varpi_i h}-\sigma_{\alpha,k}\sum_{j=2}^{n-1}(W_{j+1}^{(\alpha)}-W_j^{(\alpha)})\xi_{n-j}e^{I\varpi_i h}+W_n^{(\alpha)}\xi_0 e^{I\varpi_i h}\end{aligned} \tag{17}$$

$$\begin{aligned}&(\sigma_{\alpha,k}+C_1+2\gamma(1-\cos(\varpi h)))\xi_n\\&=\sigma_{\alpha,k}(1-W_2^{\alpha})\xi_{n-1}-\sigma_{\alpha,k}\sum_{j=2}^{n-1}(W_{j+1}^{(\alpha)}-W_j^{(\alpha)})\xi_{n-j}+W_n^{(\alpha)}\xi_0\end{aligned}$$

on making $\mu=1+\frac{C_1+2\gamma(1-\cos(\varpi h))}{\sigma_{\alpha,k}}$, the above expression can be written as,

$$\xi_n=\frac{(1-W_2^{\alpha})}{\mu}\xi_{n-1}-\frac{1}{\mu}\sum_{j=2}^{n-1}(W_{j+1}^{(\alpha)}-W_j^{(\alpha)})\xi_{n-j}+\frac{W_n^{(\alpha)}}{\mu}\xi_0$$

On introducing the norm,

$$\| \varepsilon^n \|_2 = \left(\sum_{i=1}^{m-1} h \mid \xi_i^n \mid^2 \right)^{\frac{1}{2}}$$

Now to show the stability of the scheme we can prove $\mid \xi_n \mid \leqslant \mid \xi_0 \mid$ [2]

$$\begin{aligned} \mid \xi_n \mid &\leqslant \frac{(1 - W_2^{\alpha})}{\mu} \mid \xi_{n-1} \mid - \frac{1}{\mu} \sum_{j=2}^{n-1} (W_{j+1}^{(\alpha)} - W_j^{(\alpha)}) \mid \xi_{n-j} \mid + \frac{W_n^{(\alpha)})}{\mu} \mid \xi_0 \mid \\ &\leqslant \frac{1 - W_2^{\alpha} - (W_n^{(\alpha)} - W_2^{(\alpha)}) + W_n^{\alpha}}{\mu} \mid \xi_0 \mid \leqslant \frac{1}{\mu} \mid \xi_0 \mid \leqslant \mid \xi_0 \mid \end{aligned}$$

5. Numerical computation and analysis

In this study we consider the parameter values are $L = 0.02\text{m}$ $\omega = 0.01$ to 0.05, $q_0 = 5000\,\text{W}/\text{m}^2$, $\rho = 1050\,\text{kgm}^{-3}$, $\rho_b = 1000\,\text{kgm}^{-3}$, $T_a = 37°\text{C}$, $q_{met} = 368.1\,\text{Wm}^{-3}$, $\omega = 0.05$, $c_b = 3770\,\text{Jkg}^{-1}\text{C}$, $c = 4180\,\text{Jkg}^{-1}\text{C}$, $K = 0.5\,\text{Wm}^{-1}\text{C}$ and dimensionless blood perfusion $C_1 = 0.001$ to 1. We investigate the effect of anomalous diffusion in this model for different values of α (order of the time derivative).

5.1. Code verification

Figure 1 represent the temperature profile along the distance for different time. This represents the comparison of analytic solution obtained by Shih et al. [12] and numerical solution as well as numerical solution for fractional bioheat equation obtained by us as taking $\alpha \to 1$. We observe that our result is very similar to the analytic solution [12] which verified with the developed Matlab code by us.

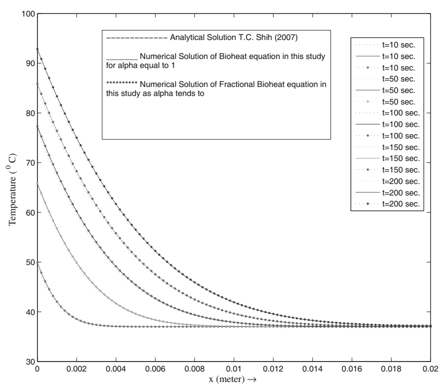

Figure 1: *Comparison of analytic solution with numerical solution*

5.2. Effect of frequency ω

Figure 2, 3 and 4 shows the temperature response on the skin with respect to time with frequency $\omega = 0.01$, $\omega = 0.05$ and $\omega = 0.1$ respectively for different value of α. When $\omega = 0.01$, we see that the amplitude of temperature is lower and response cyclic time is longer for decreasing values of α. For $\omega = 0.05$ and $\omega = 0.1$ the temperature is lower and cyclic time is lower as compare to $\omega = 0.01$. For $\omega = 0.05$ and $\omega = 0.1$ amplitude of temperature is lower and cyclic time is increasing as α decreases.

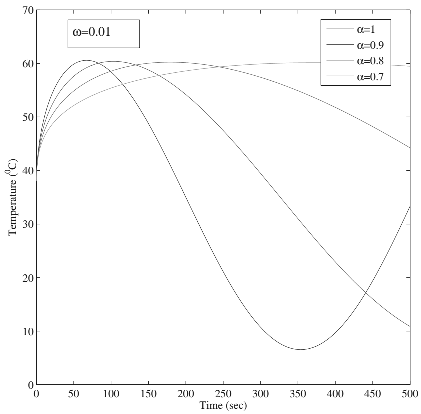

Figure 2: *Temperature profile for* $\omega = 0.01$ *with different* α

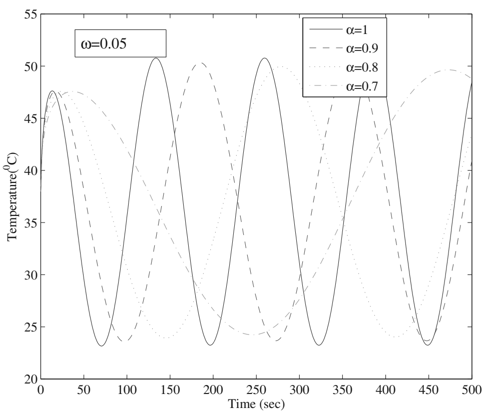

Figure 3: *Temperature profile for* $\omega = 0.05$ *with different* α

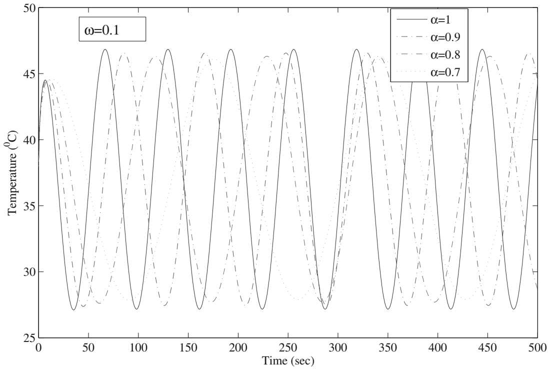

Figure 4: *Temperature profile for* $\omega = 0.1$ *with different* α

5.3. Effect of dimensionless blood perfusion C_1

Figure 5 represents the temperature profile of different α, along time for $C_1 = 0.001$ at different depth $x = 0$ meter, $x = 0.001$ meter and $x = 0.005$ meter. Figure 6 shows the temperature profile of different α, along time for $C_1 = 0.01$ at different depth $x = 0$meter, $x = 0.001$ meter and $x = 0.005$ meter. Figure 7 indicates the temperature profile of different α, along time for $C_1 = 0.05$ at different depth $x = 0$ meter, $x = 0.001$ meter and $x = 0.005$ meter. Figure 8 represents the temperature profile of different α, along time for $C_1 = 1$ at different depth $x = 0$ meter, $x = 0.001$ meter and $x = 0.005$ meter. In fig 5, 6, 7 and 8, we see that the cyclic time is increases as α, decreases and also amplitude is decreases as α decreases. Temperature is decreasing as depth of the tissue increases, here the oscillation at $x = 0$ meter is higher and cyclic time is lower then $x = 0.001$ meter and very law oscillation and cyclic time is higher at $x = 0.005$ meter.

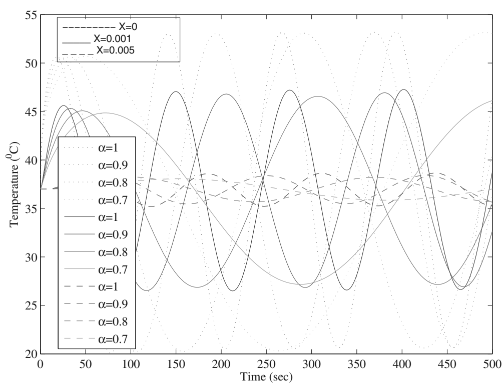

Figure 5: *Temperature profile with* $C_1 = 0.001$ *for different* α *at different depth*

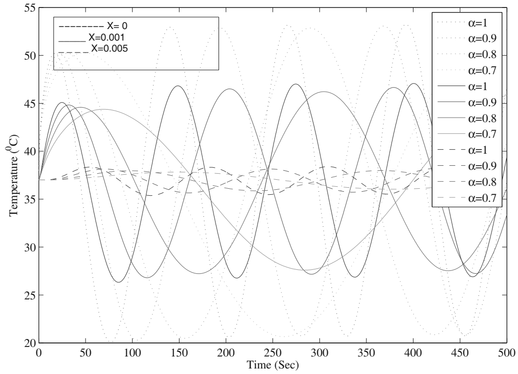

Figure 6: *Temperature profile with* $C_1 = 0.01$ *for different* α *at different depth*

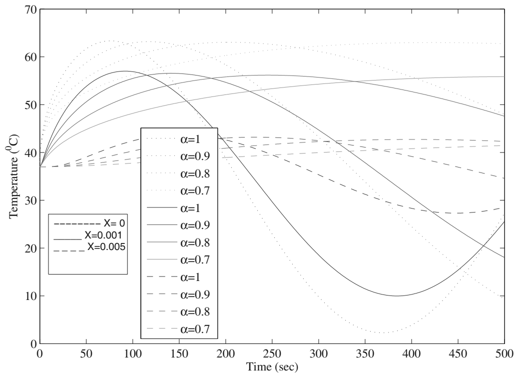

Figure 7: *Temperature profile with* $C_1 = 0.5$ *for different* α *at different depth*

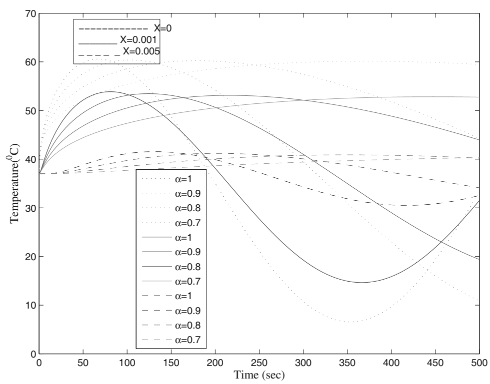

Figure 8: *Temperature profile for* $C_1 = 1$ *for different* α *at different depth*

Figure 9 shows the comparison of temperature profile for different C_1, along time for different values of α at depth $x = 0.001$ meter. As C_1 increases the temperature and cyclic time are increasing but when α decreases the temperature is decreasing but cyclic time increasing. Figure 10 shows the comparison of temperature profile for different C_1, along distance for different values of α at time $t = 100$ seconds. We see that the C_1, decreases temperature is decreasing and temperature is continuously decreasing as length of the tissue increases. The temperature response after $x = 0.012$ meter seems to be independent of the heat flux on the heating skin.

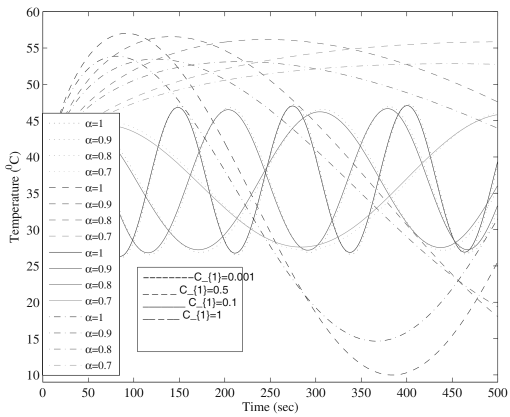

Figure 9: *Temperature profile with different* C_1 *for different* α *at* $x = 0.001$ *meter*

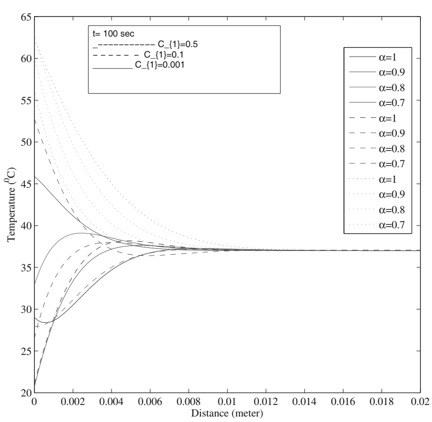

Figure 10: *Temperature profile with different* C_1 *for different* α *at* $t = 100$ *sec*

6. Conclusion

In present paper the temperature distribution of fractional bioheat model has been studied with sinusoidal heat flux on skin surface. The temperature profile has been obtained along distance and time with different frequency and dimensionless blood perfusion C_1. The effect of the anomalous diffusion has been compared with normal diffusion for heat transfer in skin tissue at different depth and time of the tissue with variation in blood perfusion. This is observed that the fractional bioheat model with sinusoidal heat flux condition on skin surface, if frequency ω increases the temperature decreases and also cyclic time decreases it is interesting to note that when α, decreases the temperature is decreasing and cyclic time is increasing. Also observed that if dimensionless blood perfusion C_1 of the tissue increases the temperature increases but temperature is decreases as length of the tissue increases, as α, decreases cyclic time is increases but oscillation decreases as length increases. The obtained numerical solution in this work may be useful to predict the temperature response in fractional bioheat model with sinusoidal heat flux with different frequency and blood perfusion respectively. The obtained solution may also be useful for experimental model to predict the value of α, and applicable for thermal therapy in medical sciences.

Nomenclature

ρ – density $\left(\frac{\text{kg}}{\text{m}^3}\right)$
c – specific heat $\left(\frac{\text{J}}{\text{kg}^\circ\text{C}}\right)$
k – thermal conductivity of the tissue $\left(\frac{\text{W}}{\text{m}^\circ\text{C}}\right)$
W_b – Blood perfusion rate $(\text{Kg}/\text{m}^3\text{s})$
q_0 – volumetric heat generation $\left(\frac{\text{W}}{\text{m}^3}\right)$
ω – heating frequency
T – temperature ($^\circ$C)
t – time (sec)
x – space coordinate (m)
θ – Dimensionless Temperature
ζ – Dimensionless Space
η – Dimensionless Time
C_1 – Dimensionless blood perfusion
$_t$ and $_b$ – tissue and blood respectively

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