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FINITE DIFFERENCE METHOD FOR SOLVING THE SPACE–TIME FRACTIONAL WAVE EQUATION IN THE CAPUTO FORM

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Abstract. In this paper a space-time fractional wave equation on a finite domain is considered. The time and space fractional derivative are described in the Caputo sense. We propose a finite difference scheme to solve the space-time fractional wave equation. We discuss about stability and convergence of the method and prove that the finite difference scheme is unconditionally stable and convergent with (τ^2+h), where τ and h are time and space steps respectively.

1. Introduction

Fractional calculus is becoming a useful and in some cases key tool in the analysis of scientific problems in a broad array of field such as physics, engineering, biology and economics. In particular fractional partial differential equations have turned out to be especially relevant, for example fractional diffusion equations have been successfully used to describe diffusion processes where the diffusion is anomalous, and fractional wave equations have been proposed to deal with viscoelastic problem [4].

A fractional wave equation is obtained from the classical wave equation by replacing the second-order time derivative term by a fractional derivative of order α, $1<\alpha<2$, and the second space derivative by a fractional derivative of order β, $1<\beta<2$.

In this paper, the following space-time fractional wave equation is considered

$$\frac{\partial^\alpha u}{\partial t^\alpha} = {b_c}^2 \frac{\partial^\beta u}{\partial x^\beta} \qquad 1<\alpha\leqslant 2,\ \ 1<\beta\leqslant 2, \tag{1.1}$$

subject to the boundary and initial conditions

$$\begin{array}{ll} u(0,t)=u(L,t)=0 & \quad 0\leqslant t\leqslant T,\\ u(x,0)=f(x) & \quad 0<x<L,\\ \frac{\partial u(x,0)}{\partial t}=g(x) & \quad 0<x\leqslant L. \end{array}$$

Where α and β are parameters describing the order of time and space fractional derivatives respectively, b_c denotes a constant coefficient.

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Fractional order derivatives in (1.1) are Caputo fractional derivatives of order α and β, defined by [9]

$$ {}^{c}D^{\alpha} f(x) = \frac{\partial^{\alpha} f}{\partial x^{\alpha}} = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \tag{1.2} $$

where m is an integer such that $m-1 < \alpha \leqslant m$.

2. The finite difference scheme

For the numerical scheme, suppose $t_k = k\tau,\ k = 0,1,2,\dots,n$ and $x_i = ih,\ i = 0,1,\dots,m$, where $\tau = \frac{T}{n}$ is the time step and $h = \frac{L}{m}$ is the grid step in space. Let u_i^k be the numerical estimate of the value exact solution $u(x,t)$ at the mesh point (x_i,t_k).

For the finite difference scheme, we discretize the Caputo time fractional derivative by L_2 formula [8]

$$ \frac{\partial^{\alpha} u(x_i,t_{k+1})}{\partial t^{\alpha}} = \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)} \sum_{j=0}^{k} b_j(u(x_i,t_{k-j-1}) - 2u(x_i,t_{k-j}) + u(x_i,t_{k-j+1})) + o(\tau^2), \tag{2.1} $$

where

$$ b_j = (j+1)^{2-\alpha} - j^{2-\alpha} \quad j = 0,1,\dots k. \tag{2.2} $$

The Caputo space fractional derivative is discretized by [2]

$$ \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} = ({}_0D_x^{\beta} u)(x,t) - \frac{u(0,t)}{\Gamma(1-\beta)}(x)^{-\beta} - \frac{u'(0,t)}{\Gamma(2-\beta)}(x)^{1-\beta}, \quad (1 < \beta \leqslant 2) \tag{2.3} $$

where $({}_0D_x^{\beta} u)(x,t)$ is the Riemann-Liouville fractional partial derivative of order $1 < \beta \leqslant 2$ defined by [9]

$$ ({}_0D_x^{\beta} u)(x,t) = \frac{1}{\Gamma(2-\beta)} \left(\frac{\partial}{\partial x}\right)^2 \int_0^x \frac{u(\xi,t)}{(x-\xi)^{\beta-1}} d\xi. \tag{2.4} $$

We use the shifted Grunwald formula to discretize the Riemann-Liouville fractional derivative [5]

$$ ({}_0D_x^{\beta} u)(x_i,t_{k+1}) = \frac{1}{h^{\beta}} \sum_{j=0}^{i+1} w_j u(x_{i-j+1},t_{k+1}) + o(h), \tag{2.5} $$

where $w_k = (-1)^k \binom{\beta}{k}$, called the normalized Grunwald weights, $u'(0,t)$ in the (2.3) is discretized using forward difference formula. Therefore the Caputo space fractional derivative is discretized as follows

$$ \frac{\partial^{\beta} u(x_i,t_{k+1})}{\partial x^{\beta}} = \frac{1}{h^{\beta}} \sum_{j=0}^{i+1} w_j u(x_{i-j+1},t_{k+1}) - \frac{u(x_1,t_{k+1})}{h\Gamma(2-\beta)}(x_i^{1-\beta}) + o(h). \tag{2.6} $$

Let $\mu = \frac{b_c^2 \tau^\alpha \Gamma(3-\alpha)}{h^\beta}$, $\mu' = \frac{b_c^2 \tau^\alpha \Gamma(3-\alpha)}{h\Gamma(2-\beta)}$ then using (2.1) and (2.6) we have the following implicit difference method

$$\mu \sum_{j=0}^{i+1} w_j u_{i-j+1}^{k+1} - \mu'(x_i)^{1-\beta} u_1^{k+1} - u_i^{k+1} = \sum_{j=1}^{k} b_j (u_i^{k-i-1} - 2u_i^{k-j} + u_i^{k-j+1}) + u_i^{k-1} - 2u_i^k,$$
$$i = 1,2,\ldots,m, \qquad k = 0,1,2,\ldots,n. \tag{2.7}$$

We know $\frac{\partial u(x,0)}{\partial t} = g(x)$, therefore

$$u_i^{-1} = u_i^1 - 2\tau g(x_i), \quad i = 0,1,\ldots,m,$$

hence for $k = 0$ we have

$$-\frac{\mu}{2} \sum_{j=0, j\neq 1}^{i+1} w_j u_{i-j+1}^1 + \left(1 + \frac{\mu}{2}\beta\right) u_i^1 + \frac{\mu'}{2}(x_i)^{1-\beta} u_1^1 = \tau g(x_i) + u_i^0, \tag{2.8}$$

for $k \neq 0$, equation (2.7) is rewritten as

$$-\mu \sum_{j=0, j\neq 1}^{i+1} w_j u_{i-j+1}^{k+1} + (1+\beta\mu) u_i^{k+1} + \mu'(x_i)^{1-\beta} u_1^{k+1}$$
$$= \sum_{j=1}^{k-1} -(b_{j-1} - 2b_j + b_{j+1}) u_i^{k-j} + (2b_k - b_{k-1}) u_i^0 + (2-b_1) u_i^k - b_k u_i^1 + 2\tau g(x_i) b_k. \tag{2.9}$$

Equation above can be written as

$$\begin{cases} A^{(1)} U^1 = F_1 \\ A^{(2)} U^{k+1} = (2-b_1) U^k + (-b_0 + 2b_1 - b_2) U^{k-1} + \ldots + (2b_k - b_{k-1}) U^0 + F_2 \end{cases}$$

where $U^k = \begin{pmatrix} u_1^k \\ u_2^k \\ \vdots \\ u_{m-1}^k \end{pmatrix}$ and

$$A^{(1)} = (A_{i,j}^{(1)})_{(m-1),(m-1)} = \begin{cases} \frac{-\mu}{2} w_{i-j+1}, & 1 < j \leqslant i-1 \\ \frac{\mu'}{2}(x_i)^{1-\beta} - \frac{\mu}{2} w_i, & j \neq i \text{ and } j = 1 \\ (1 + \beta \frac{\mu}{2}), & j = i \neq 1 \\ (1 + \frac{\mu}{2}\beta) + \frac{\mu'}{2} x_1^{1-\beta}, & j = i = 1 \\ -\frac{\mu}{2} w_0, & j = i+1 \end{cases}$$

and

$$A^{(2)} = (A^{(2)}_{i,j})_{(m-1),(m-1)} = \begin{cases} -\mu w_{i-j+1}, & 1 < j \leqslant i-1 \\ \mu'(x_i)^{1-\beta} - \mu w_i, & j \neq i \text{ and } j = 1 \\ (1+\beta\mu), & j = i \neq 1 \\ (1+\mu\beta) + \mu' x_1^{1-\beta}, & j = i = 1 \\ -\mu w_0, & j = i+1 \end{cases}$$

If $\mu'(x_i)^{1-\beta} < 1$ then the coefficient matrices $A^{(1)}$ and $A^{(2)}$ are strictly diagonally dominant, therefore the matrices $A^{(1)}$ and $A^{(2)}$ are nonsingular, thus they are invertible. Therefore the finite difference scheme for problem (1.1) is unique.

LEMMA 2.1. *For $j = 0,1,2,\ldots$ the coefficients w_j, satisfy*

1) $w_0 = 1,\ w_1 = -\beta$

2) $w_j > 0,\ j = 1,2,\ldots,$

3) $\sum_{j=0,j\neq 1}^{N} w_j \leqslant \beta,$

4) $\sum_{j=0}^{\infty} w_j = 0.$

LEMMA 2.2. *The coefficient $b_j,\ j = 1,2,\ldots$ satisfy*

1) $b_j > 0,\ j = 1,2,\ldots$

2) $b_j > b_{j+1}$

Proof. Using the properties of functions $f(x) = x^{2-\alpha}$ $(x \geqslant 1)$ and $h(x) = (x+1)^{2-\alpha} - x^{2-\alpha}$ $(x \geqslant 0)$, lemma can be obtained. □

3. Stability of finite difference scheme

Let $\tilde{u}_i^k$ and u_i^k $(i = 0,1,2,\ldots,m;\ k = 0,1,2,\ldots,n)$ be the solution of (2.8) and (2.9), the error $\varepsilon_i^k = \tilde{u}_i^k - u_i^k$ $(i = 0,1,2,\ldots,m;\ k = 0,1,2,\ldots,n)$ satisfies

$$-\frac{\mu}{2}\sum_{j=0,j\neq 1}^{i+1} w_j \varepsilon_{i-j+1}^1 + \left(1+\frac{\mu}{2}\beta\right)\varepsilon_i^1 + \frac{\mu'}{2}(x_i)^{1-\beta}\varepsilon_1^1 = \varepsilon_i^0 \tag{3.1}$$

$$\begin{aligned} &-\mu \sum_{j=0,j\neq 1}^{i+1} w_j \varepsilon_{i-j+1}^{k+1} + (1+\beta\mu)\varepsilon_i^{k+1} + \mu'(x_i)^{1-\beta}\varepsilon_1^{k+1} \\ &= \sum_{j=1}^{k-1} -(b_{j-1} - 2b_j + b_{j+1})\varepsilon_i^{k-j} + (2b_k - b_{k-1})\varepsilon_i^0 + (2-b_1)\varepsilon_i^k - b_k\varepsilon_i^1, \\ &\qquad (i = 1,\ldots,m-1) \end{aligned} \tag{3.2}$$

which can be written as

$$\begin{cases} A^{(1)}E^1 = E^0 \\ A^{(2)}E^{k+1} = (2-b_1)E^k + (-b_0+2b_1-b_2)E^{k-1} + \ldots + (2b_k - b_{k-1})E^0 \\ E^0 \quad \text{given} \end{cases}$$

where $E^k = \begin{pmatrix} \varepsilon_1^k \\ \varepsilon_2^k \\ \vdots \\ \varepsilon_{m-1}^k \end{pmatrix}$ and

$$A^{(1)} = (A^{(1)}_{i,j})_{(m-1),(m-1)} = \begin{cases} \frac{-\mu}{2} w_{i-j+1}, & 1 < j \leqslant i-1 \\ \frac{\mu'}{2}(x_i)^{1-\beta} - \frac{\mu}{2} w_i, & j \neq i \text{ and } j = 1 \\ (1+\beta\frac{\mu}{2}), & j = i \neq 1 \\ (1+\frac{\mu}{2}\beta) + \frac{\mu'}{2} x_1^{1-\beta}, & j = i = 1 \\ -\frac{\mu}{2} w_0, & j = i+1 \end{cases}$$

and

$$A^{(2)} = (A^{(2)}_{i,j})_{(m-1),(m-1)} = \begin{cases} -\mu w_{i-j+1}, & 1 < j \leqslant i-1 \\ \mu'(x_i)^{1-\beta} - \mu w_i, & j \neq i \text{ and } j = 1 \\ (1+\beta\mu), & j = i \neq 1 \\ (1+\mu\beta) + \mu' x_1^{1-\beta}, & j = i = 1 \\ -\mu w_0, & j = i+1 \end{cases}$$

Using mathematical induction, we can obtain the following result.

PROPOSITION 3.1. $\|E^k\|_\infty \leqslant C\|E^0\|_\infty$, $(k = 1,2,\ldots,n)$ *where* C *is a positive number independent of* τ *and* h.

Proof. Let $|\varepsilon_l^1| = \max_{1\leqslant i<m-1} |\varepsilon_i^1|$, then

$$\begin{aligned} |\varepsilon_l^1| &\leqslant |\varepsilon_l^1| \left(1+\beta\frac{\mu}{2}\right) - \frac{\mu}{2} \sum_{j=0,j\neq 1}^{l+1} w_j |\varepsilon_{l-j+1}^1| + \frac{\mu'}{2}(x_l)^{1-\beta}|\varepsilon_1^1| \\ &\leqslant \left| -\frac{\mu}{2} \sum_{j=0,j\neq 1}^{l+1} w_j \varepsilon_{l-j+1}^1 + \left(1+\frac{\mu}{2}\beta\right)\varepsilon_l^1 + \frac{\mu'}{2}(x_l)^{1-\beta}\varepsilon_1^1 \right| = |\varepsilon_l^0|, \end{aligned}$$

thus $\|E^1\|_\infty < \|E^0\|_\infty$.

Suppose that $\|E^j\|_\infty \leqslant C\|E^0\|_\infty$, $j = 1,2,\ldots,k$, let $|\varepsilon_l^{k+1}| = \max_{1\leqslant i<m-1} |\varepsilon_i^{k+1}|$. Using lemma (2.2) we have $2b_j - b_{j-1} - b_{j+1} > 0$, $2-b_1 > 0$ and $2b_k - b_{k-1} > 0$, hence

$$
\begin{aligned}
|\varepsilon_l^{k+1}| &\leqslant |\varepsilon_l^{k+1}|(1+\beta\mu) - \mu \sum_{j=0, j\neq 1}^{l+1} w_j |\varepsilon_{l-j+1}^{k+1}| + \mu'(x_l)^{1-\beta} |\varepsilon_1^{k+1}| \\
&\leqslant \left| -\mu \sum_{j=0, j\neq 1}^{l+1} w_j \varepsilon_{l-j+1}^{k+1} + (1+\mu\beta)\varepsilon_l^{k+1} + \mu'(x_l)^{1-\beta}\varepsilon_1^{k+1} \right| \\
&= \left| \sum_{j=1}^{k-1} -(b_{j-1} - 2b_j + b_{j+1})\varepsilon_l^{k-j} + (2b_k - b_{k-1})\varepsilon_l^0 + (2-b_1)\varepsilon_l^k - b_k\varepsilon_l^1 \right| \\
&\leqslant \sum_{j=1}^{k-1} (2b_j - b_{j-1} - b_{j+1})|\varepsilon_l^{k-j}| + (2b_k - b_{k-1})|\varepsilon_l^0| + (2-b_1)|\varepsilon_l^k| + b_k|\varepsilon_l^1| \\
&\leqslant C\left(\sum_{j=1}^{k-1} (2b_j - b_{j-1} - b_{j+1}) + (2-b_1) + (2b_k - b_{k-1}) + b_k \right) |\varepsilon_l^0| \\
&\leqslant 3C|\varepsilon_l^0| = C'|\varepsilon_l^0|,
\end{aligned}
$$

hence $\|E^{k+1}\|_\infty \leqslant C'\|E^0\|_\infty$. □

The following theorem can be obtained by above proposition. [10,11]

THEOREM 3.1. *The finite difference scheme defined by* (2.8) *and* (2.9) *is unconditionally stable.*

4. Convergence analysis

THEOREM 4.1. *Let $u(x_i,t_k)$ be the exact solution of equation* (1.1) *and u_i^k be the solution of finite difference equations* (2.8) *and* (2.9) *at the mesh point (x_i,t_k). There exists the positive constant $\widetilde{C}$ such that*

$$|u_i^k - u(x_i,t_k)| \leqslant \widetilde{C}(\tau^2 + h).$$

Proof. First, the order of the local truncation error T_i^{k+1} is computed. In fact

$$
\begin{aligned}
T_i^{k+1} = &\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)} \sum_{j=0}^{k} b_j (u(x_i,t_{k-j-1}) - 2u(x_i,t_{k-j}) + u(x_i,t_{k-j+1})) \\
&- \frac{b_c}{h^\beta} \sum_{j=0}^{i+1} w_j u(x_{i-j+1},t_{k+1}) - \frac{u(x_1,t_{k+1})}{h\Gamma(2-\beta)},
\end{aligned}
$$

using (2.1) and (2.6) we obtain

$$T_i^{k+1} = \frac{\partial^\alpha u(x_i,t_{k+1})}{\partial t^\alpha} + o(\tau^2) - b_c \frac{\partial^\beta u(x_i,t_{k+1})}{\partial x^\beta} + o(h) = o(\tau^2 + h).$$

Define $e_i^k = u(x_i,t_k) - u_i^k$, $i = 1,2,\dots,m-1$, $k = 1,2,\dots,n$ and $e^k = (e_1^k, e_2^k, \dots, e_{m-1}^k)^T$. Using $e^0 = 0$, substitution into (2.8) and (2.9) leads to

$$\frac{-\mu}{2}\sum_{j=0,j\neq 1}^{i+1} w_j e_{i-j+1}^1 + \left(1+\beta\frac{\mu}{2}\right)e_i^1 + \frac{\mu'}{2}(x_i)^{1-\beta}e_1^1 = \tau^\alpha T_i^1. \tag{4.1}$$

$$\begin{aligned}&-\mu\sum_{j=0,j\neq 1}^{i+1} w_j e_{i-j+1}^{k+1} + (1+\beta\mu)e_i^{k+1} + \mu'(x_i)^{1-\beta}e_1^{k+1}\\ &= \sum_{j=1}^{k-1} -(b_{j-1}-2b_j+b_{j+1})e_i^{k-j} + (2-b_1) - b_k e_1^1 + \tau^\alpha\Gamma(3-\alpha)T_i^{k+1}.\end{aligned} \tag{4.2}$$

Using mathematical induction, we will prove $\|e^k\|_\infty \leqslant C\tau^\alpha b_k^{-1}(\tau^2+h)$.

For $k=0$, let $\|e^1\|_\infty = |e_l^1| = \max_{1\leqslant i\leqslant m-1}|e_i^1|$. Using lemma (2.1) we have $w_1 = -\beta$, $\sum_{j=0,\,j\neq 1}^N w_j \leqslant \beta$, hence

$$\begin{aligned}|e_l^1| &\leqslant |e_l^1|\left(1+\beta\frac{\mu}{2}\right) - \frac{\mu}{2}\sum_{j=0,j\neq 1}^{l+1} w_j|e_{l-j+1}^1| + \frac{\mu'}{2}(x_l)^{1-\beta}|e_1^1|\\ &\leqslant \left|-\frac{\mu}{2}\sum_{j=0,j\neq 1}^{l+1} w_j e_{l-j+1}^1 + \left(1+\frac{\mu}{2}\beta\right)e_l^1 + \frac{\mu'}{2}(x_l)^{1-\beta}e_1^1\right|\\ &= \left|\tau^\alpha T_i^1\right| \leqslant C\tau^\alpha(\tau^2+h) = C\tau^\alpha b_0^{-1}(\tau^2+h),\end{aligned}$$

thus $\|e^1\|_\infty \leqslant C\tau^\alpha b_0^{-1}(\tau^2+h)$.

Suppose that $\|e^j\|_\infty \leqslant C\tau^\alpha b_{j-1}^{-1}(\tau^2+h)$, $j = 1,2,\dots,k$, and let $\|e^{k+1}\|_\infty = |e_l^{k+1}| = \max_{1\leqslant i\leqslant m-1}|e_i^{k+1}|$. We can obtain $\|e^j\|_\infty \leqslant Cb_k^{-1}(\tau^2+h)$, $j = 1,2,\dots,k$, because $b_k^{-1} \geqslant b_j^{-1}$, $j = 0,1,\dots,k$. Using lemma (2.2) we have $2b_j - b_{j-1} - b_{j+1} > 0$ and $2 - b_1 > 0$.

Similarly using $e_l^0 = 0$, we have

$$\begin{aligned}|e_l^{k+1}| &\leqslant |e_l^{k+1}|(1+\beta\mu) - \mu\sum_{j=0,j\neq 1}^{l+1} w_j|e_{l-j+1}^{k+1}| + \mu'(x_l)^{1-\beta}|e_1^{k+1}|\\ &\leqslant \left|-\mu\sum_{j=0,j\neq 1}^{l+1} w_j e_{l-j+1}^{k+1} + (1+\mu\beta)e_l^{k+1} + \mu'(x_l)^{1-\beta}e_1^{k+1}\right|\\ &= \left|\sum_{j=1}^{k-1} -(b_{j-1}-2b_j+b_{j+1})e_l^{k-j} + (2-b_1)e_l^k - b_k e_l^1 + \tau^\alpha\Gamma(3-\alpha)T_i^{k+1}\right|\\ &\leqslant \sum_{j=1}^{k-1}(2b_j-b_{j-1}-b_{j+1})|e_l^{k-j}| + (2-b_1)|e_l^k| + b_k|e_l^1| + C'\tau^\alpha(\tau^2+h)\\ &\leqslant Cb_k^{-1}\tau^\alpha(\tau^2+h)\left(b_k + \sum_{j=1}^{k-1}(2b_j-b_{j-1}-b_{j+1}) + (2-b_1) + 1\right)\\ &\leqslant 3Cb_k^{-1}\tau^\alpha(\tau^2+h) = C_0 b_k^{-1}\tau^\alpha(\tau^2+h),\end{aligned}$$

thus $\|e^{k+1}\|_\infty \leqslant C_0 b_k^{-1}\tau^\alpha(\tau^2+h)$. We can prove that

$$\lim_{k\to\infty}\frac{b_k^{-1}}{k^\alpha}=\lim_{k\to\infty}\frac{k^{-\alpha}}{(k+1)^{2-\alpha}-k^{2-\alpha}}=\lim_{k\to\infty}\frac{k^{-2}}{(1+\frac{1}{k})^{2-\alpha}-1}=\frac{1}{2-\alpha}.$$

Therefore, there is a constant $\bar{C}$ so that

$$\|e^k\|_\infty \leqslant \bar{C}k^\alpha\tau^\alpha(\tau^2+h).$$

Because $k\tau \leqslant T$ is finite, we have

$$|u_i^k-u(x_i,t_k)| \leqslant \widetilde{C}(\tau^2+h),\ \ i=1,2,\dots,m-1,\ \ k=1,2,\dots,n. \quad \square$$

5. Numerical example

In this section we consider the following space-time fractional wave equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^\beta u}{\partial x^\beta},\ \ 1<\alpha\leqslant 2,\ \ 1<\beta\leqslant 2,\ \ 0\leqslant x\leqslant 1,\ \ 0\leqslant t\leqslant 1,$$

subject to the initial conditions

$$u(x,0)=\sin(2\pi x),$$
$$\frac{\partial u(x,0)}{\partial t}=2\pi\sin(2\pi x),$$

and the boundary conditions $u(0,t)=u(1,t)=0$, $0\leqslant t\leqslant 1$. Following figure shows the evolution results using the finite difference scheme with $h=\frac{1}{20}$, $\tau=0.01$, $\alpha=1.9$ and $\beta=1.4$.

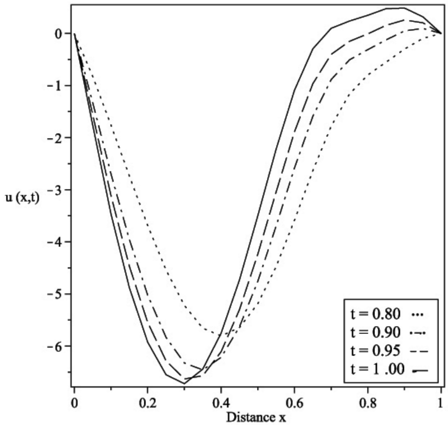

Figure 1.

6. Conclusion

In this paper, a finite difference method for the space-time fractional wave equation in a bounded domain have been described and demonstrated, we prove that this method is unconditionally stable and convergent. The technique can be applied to solve fractional-order differential equation.

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