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MAXIMAL SOLUTIONS TO FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. When do fractional differential equations have maximal solutions? This note discusses this question in the following way. Firstly, a comparison theorem is formulated that involves fractional differential inequalities. Secondly, a sequence of approximative problems involving polynomials is analyzed and it is shown that there is a subsequence of solutions whose limit is the maximal solution to the original problem of interest. In particular, the interval of existence for the maximal solution is the optimal length, aligning with best practice in the local theory of existence of solutions. We achieve this through an application of the Arzela–Ascoli Theorem and our aforementioned comparison result. A YouTube video by the author designed to complement this work is available at `http://tinyurl.com/MaxFracDE`.

1. Introduction

This note discusses the question:

When do fractional differential equations have maximal solutions?

Although the area of fractional differential equations is hundreds of years old, it has formed a very lively line of inquiry for mathematicians throughout the past decade. However, some of the basic qualitative and quantitative theory is still to be fully developed [9, p. 285]. This paper fills a gap in the literature by advancing the state of knowledge regarding maximal (and minimal) solutions to nonlinear fractional differential equations.

Let $q \in (0,1)$. We denote the (Riemann–Liouville) fractional differential operator of order q by D^q, which is defined via

$$D^q[y](t) := \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} y(s)\, ds.$$

Consider the initial value problem (IVP)

$$D^q[x - x(0)] = f(t,x), \qquad x(0) = A. \tag{1.1}$$

where $f : R \to \mathbb{R}$ and f is continuous on the rectangle

$$R := \{(t,u) \in \mathbb{R}^2 : t \in [0,b],\ |u - A| \leqslant B\}$$

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and b and B are positive constants. The left–hand side of (1.1) is known as the Caputo derivative of x of order $q \in (0,1)$ with the notation ${}^{C}D^{q}(x) := D^{q}[x - x(0)]$ sometimes used. This particular form was suggested by Caputo [1] in response to a need for improved accuracy in modelling the initial conditions of phenomena.

Early and significant advancements to our understanding of the qualitative properties of solutions to fractional differential equations appear in [2, 3, 5, 6, 7, 9, 10, 11, 12, 13, 14] and [4, Sec. 3.5, pp. 198–212]. The ideas within those works include the application of fixed–point and sequential methods to yield existence, uniqueness and approximation of solutions.

In particular, in [5], the authors gave sufficient continuity conditions under which maximal and minimal solutions to (1.1) exist on intervals of type

$$\left[0, \min\left\{b, \left[\frac{B\Gamma(q+1)}{2N+B}\right]^{1/q}\right\}\right]$$

where N is a positive constant (to be defined a little later). However, it is well–known [2, Theorem 2.1] that the optimal interval for local existence theory for solutions to (1.1) is of the larger form

$$\left[0, \min\left\{b, \left[\frac{B\Gamma(q+1)}{N}\right]^{1/q}\right\}\right]. \tag{1.2}$$

Thus, there is a need to improve the result for maximal and minimal solutions in (1.1) to align it with best–practice in the literature.

This note is organized as follows. A comparison theorem is formulated that involves fractional differential inequalities. In contrast to [5], we apply Weierstrass' polynomial approximation theorem to yield a sequence of polynomial–approximative problems to (1.1). It is shown that there is a subsequence of solutions whose limit is the maximal solution to the original problem of interest (1.1) on the larger and optimal interval (1.2). We achieve this through an application of the earlier comparison result and the Arzela–Ascoli Theorem.

2. Preliminaries

To comprehend the notation used and to keep the paper somewhat self–contained, this brief section contains some preliminary definitions and associated notation.

A solution to the IVP (1.1) on an interval I is defined to be a function $x : I \subseteq [0,b] \to \mathbb{R}$ such that the points $(t, x(t))$ lie in R for all $t \in I \subseteq [0,b]$ and $x(t)$ satisfies: (1.1) for all $t \in I \subseteq [0,b]$.

Instead of directly dealing with the problem (1.1), we will be interested in an equivalent integral equation in the proof of our main result. The following lemma is well known, see [5].

LEMMA 2.1. *If $f : R \to \mathbb{R}$ is continuous then the initial value problem* (1.1) *is equivalent to the integral equation*

$$x(t) = A + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds. \tag{2.1}$$

3. Comparison result

We now present a theorem that enables us to compare two solutions to related fractional IVPs. The ideas rely on fractional differential inequalities and a Lipschitz condition (3.2) with Lipschitz constant $L > 0$.

THEOREM 3.1. (Comparison theorem) *Let $w : R \to \mathbb{R}$ satisfy the Lipschitz condition*

$$|w(t,u) - w(t,v)| \leqslant L|u - v|, \qquad \text{for all } (t,u),\ (t,v) \in R \tag{3.2}$$

for some constant $L > 0$ and let $x = x(t)$ satisfy

$$D^q[x - x(0)] \leqslant w(t,x), \qquad x(0) = A \tag{3.3}$$

on some interval $I := [0,a] \subseteq [0,b]$. If $y = y(t)$ satisfies

$$D^q[y - y(0)] = w(t,y), \qquad y(0) = \alpha \tag{3.4}$$

on I with $A \leqslant \alpha$ then $x(t) \leqslant y(t)$ for all $t \in I$.

Proof of Theorem 3.1. Assume there is a point $t_1 \in I := [0,a] \subseteq [0,b]$ such that $x(t_1) > y(t_1)$. Define r via

$$r(t) := x(t) - y(t), \quad \text{for all } t \in [0,a]$$

so that $r(t_1) > 0$. From the assumption in Theorem 3.1 of $A \leqslant \alpha$, we have $r(0) \leqslant 0$. Let t_0 denote the greatest value in $[0,a]$ such that $r(t_0) = 0$.

For all $t \in [t_0, t_1]$ we have $r(t) \geqslant 0$ with $r(t) > 0$ for all $t \in (t_0, t_1]$.

Now for all $t \in [t_0, t_1]$ we have

$$\begin{aligned} D^q[r](t) = D^q[r - r(t_0)](t) &= D^q[x - x(t_0)](t) - D^q[y - y(t_0)](t) \\ &\leqslant w(t, x(t)) - w(t, y(t)) \\ &\leqslant L[x(t) - y(t)] \\ &= Lr(t). \end{aligned}$$

where we have applied (3.3), (3.4) and (3.2).

Hence we now have produced a fractional IVP involving a linear fractional inequality, namely

$$D^q r \leqslant Lr, \quad r(t_0) = 0 \tag{3.5}$$

for all $t \in [t_0, t_1]$. Now, in a similar spirit to Lemma 2.1 (with equalities replaced by inequalities and 0 replaced by t_0), the problem (3.5) can be equivalently recast in the integral form

$$r(t) \leqslant \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} L r(s)\, ds, \quad \text{for all } t \in [t_0, t_1].$$

It follows from the Gronwall–Reid–Bellman inequality for fractional differential equations [9, Lemma 3.1] that r satisfies

$$r(t) \leqslant 0, \quad \text{for all } t \in [t_0, t_1].$$

However, this contradicts $r(t) > 0$ for all $t \in (t_0, t_1]$. Thus we see that there is no point $t_1 \in [0, a]$ such that $x(t_1) > y(t_1)$. We thus conclude $x \leqslant y$ on $[0, a]$ as required. □

REMARK 3.2. A similar comparison result to Theorem 3.1 follows if we impose (3.2) and reverse each of the remaining inequalities contained in the assumptions of Theorem 3.1, with the conclusion being $y \leqslant x$ on I.

REMARK 3.3. The Lipschitz condition (3.2) will be satisfied, for example, if $f = f(t,x)$ has a continuous partial derivative with respect to x on R, which is satisfied for many kinds of functions, for example, polynomials [15]. Thus, Theorem 3.1 applies to a wide range of problems.

4. Application to maximal solutions

We now discuss the question raised at the beginning of the paper:

When do fractional differential equations have maximal solutions?

Let $N > 0$ be a constant that bounds f on R, that is,

$$|f(t,p)| \leqslant N, \quad \text{for all } (t,p) \in R.$$

The following result shows that under the assumption of continuity on f, the problem (1.1) will admit a maximal solution on an optimal interval of existence.

THEOREM 4.1. *If $f : R \to \mathbb{R}$ is continuous then the fractional IVP*

$$D^q[x - x(0)] = f(t,x), \qquad x(0) = A \tag{4.6}$$

has a maximal solution x_M in the sense that

$$x \leqslant x_M \quad \text{on } [0,\beta]$$
$$\beta := \min\left\{ b, \left[\frac{B\Gamma(q+1)}{N}\right]^{1/q} \right\}$$

for all solutions x on $[0,\beta]$.

Proof of Theorem 4.1. Choose a sequence of positive constants ε_n that converges to zero. Since f is continuous on R the approximation theorem of Weierstrass [8, Theorem 7.24] ensures we can choose a sequence of polynomials $P_n = P_n(t,x)$ such that for each n

$$|P_n(t,u) - f(t,u)| < \varepsilon_n, \quad \text{for all } (t,u) \in R.$$

Each P_n is Lipschitz on R with corresponding Lipschitz constants L_n.

We may assume $P_n \geqslant f$ on R and that N is a joint bound on f and each P_n on R.

By [7, Theorem 3.1], for each n, the sequence of fractional IVPs

$$D^q[x_n - x_n(0)] = P_n(t,x_n), \qquad x_n(0) = A. \tag{4.7}$$

has a unique solution $x_n = x_n(t)$ on $[0,\beta]$.

Now, we show x_n has a uniformly convergent subsequence by applying the Arzela–Ascoli theorem [8, Theorem 7.25]. This requires us to show that x_n is uniformly bounded and equicontinuous on $[0,\beta]$.

Firstly, the uniform bound on x_n on $[0,\beta]$ follows from the previous conclusion that each x_n solves (4.7) and so $(t,x_n(t)) \in R$ for each n and all $t \in [0,\beta]$. That is, for all $t \in I$ and each n we have $|x_n(t) - A| \leqslant B$.

Secondly, the uniform equicontinuity of x_n follows from the following calculations.

Let $t_1,t_2 \in [0,\beta]$. If $t_1 \leqslant t_2$ then

$$\begin{aligned}
&|x_n(t_1) - x_n(t_2)| \\
&= \frac{1}{\Gamma(q)} \left| \int_0^{t_1} (t-s)^{q-1} P_n(s,x_n(s))\, ds \right. \\
&\quad \left. - \int_0^{t_2} (t-s)^{q-1} P_n(s,x_n(s))\, ds \right| \\
&= \left| \frac{1}{\Gamma(q)} \int_0^{t_1} [(t_1-s)^{q-1} - (t_2-s)^{q-1}] P_n(s,x_n(s))\, ds \right. \\
&\quad \left. - \frac{1}{\Gamma(q)} \int_{t_2}^{t_1} (t_2-s)^{q-1} P_n(s,x_n(s))\, ds \right| \\
&\leqslant [2(t_2-t_1)^q + t_1^q - t_2^q] \frac{N}{\Gamma(q+1)} \\
&\leqslant 2(t_2-t_1)^q \frac{N}{\Gamma(q+1)} \\
&< \varepsilon
\end{aligned}$$

provided $0 < t_2 - t_1 < \delta(\varepsilon)$ with $\delta(\varepsilon) := [\varepsilon\Gamma(q+1)/2N]^{1/q}$. If $t_2 \leqslant t_1$ then a similar situation occurs. Hence our x_n are uniformly equicontinuous on $[0,\beta]$.

By the Arzela–Ascoli Theorem, there is a (continuous) function $x_M = x_M(t)$ which is the (uniform) limit of a subsequence x_{n_k} of our x_n on $[0,\beta]$.

We now show that the above limit function x_M does indeed solve our original IVP (1.1) on $[0,\beta]$.

For $t \in [0,\beta]$, consider

$$\begin{aligned}
&\left| x_{n_k}(t) - [A + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x_{n_k}(s))\, ds] \right| \\
&= \frac{1}{\Gamma(q)} \left| \int_0^t (t-s)^{q-1} |P_{n_k}(s, x_{n_k}(s)) - f(s, x_{n_k}(s))|\, ds \right| \\
&< \frac{1}{\Gamma(q)} \left| \int_0^t (t-s)^{q-1} \varepsilon_{n_k}\, ds \right| \\
&\leqslant \varepsilon_{n_k} \frac{\beta^q}{\Gamma(q+1)}.
\end{aligned}$$

Taking limits above, we see that for each $t \in [0,\beta]$,

$$\left| x_M(t) - [A + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x_M(s))\, ds] \right| \leqslant 0.$$

Hence we conclude that

$$x_M(t) = A + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x_M(s))\, ds, \quad \text{for all } t \in [0,\beta]$$

and in view of Lemma 2.1, our limit function x_M is a solution to (1.1) on $[0,\beta]$.

Now, let x be any solution to (1.1). We claim $x \leqslant x_M$ on $[0,\beta]$ and so the limit function x_M will be the maximal solution.

Since $f \leqslant P_{n_k}$ on R we see that x satisfies

$$D^q[x - x(0)] \leqslant P_{n_k}(t,x), \qquad x(0) = A.$$

Also, since x_{n_k} satisfies

$$D^q[x_{n_k} - x_{n_k}(0)] = P_{n_k}(t, x_{n_k}), \qquad x_{n_k}(0) = A$$

we know from Theorem 3.1 that

$$x \leqslant x_{n_k}.$$

In the limit we thus obtain

$$x \leqslant x_M.$$

There is one and only one maximal solution x_M to (1.1) due to the following argument. Assume there are two maximal solutions x_M and y_M. Since x_M and y_M are also solutions to (1.1) we have $x_M \leqslant y_M$ and $y_M \leqslant x_M$ on $[0,\beta]$. Hence we see that $x_M \equiv y_M$. Hence x_M is *the* maximal solution. □

REMARK 4.2. We can also show the existence of a minimal solution to (1.1) that lies below all other solutions on $[0,\beta]$ in a similar way to the proof of Theorem 4.1 by appealing to the remark following Theorem 3.1. The details are omitted for the sake of brevity.

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