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SOME DISCRETE FRACTIONAL LYAPUNOV–TYPE INEQUALITIES

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Abstract. In this work we obtain Lyapunov-type inequalities for two-point conjugate and right-focal boundary value problems depending on discrete fractional operators Δ^α, $1<\alpha\leqslant 2$.

1. Introduction

For nearly 50 years Lyapunov inequalities have been an important tool in the study of differential equations. Typical applications include bounds for eigenvalues, stability criteria for periodic differential equations, and estimates for intervals of disconjugacy (see for example [12]). The statement is as follows:

THEOREM 1.1. (cf. [14]) *If the boundary value problem*

$$\begin{aligned} y''(t)+q(t)y(t)=0, &\quad a<t<b, \\ y(a)=0=y(b), & \end{aligned} \tag{1.1}$$

has a nontrivial solution, where q is a real and continuous function, then

$$\int_a^b |q(s)|ds > \frac{4}{b-a}. \tag{1.2}$$

The author attempted and succeeded to generalize Theorem 1.1 for boundary value problems in which the classical derivative y'' is replaced by a fractional derivative D^α, namely, the Riemann–Liouville fractional derivative or the Caputo fractional derivative [8, 9]. In this work we intend to prove analogous results for fractional difference equations [16]. Taking advantage of known properties for the Green's functions of some discrete boundary value problems [3, 10] we will deduce some Lyapunov-type inequalities. In particular our results generalize the one in [5]. Moreover, we are able to formulate a Lyapunov-type theorem for a right-focal boundary value problem (this result, in the context of continuous operators is reported in the recent work [15]).

Concretizing, we will present Lyapunov-type inequalities for the fractional difference equation,

$$(\Delta^\alpha y)(t) = -q(t+\alpha-1)y(t+\alpha-1), \quad 1<\alpha\leqslant 2, \tag{1.3}$$

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coupled with one of the following boundary conditions,

$$y(\alpha-2)=0=y(\alpha+b+1), \tag{1.4}$$

or

$$y(\alpha-2)=0=\Delta y(\alpha+b), \tag{1.5}$$

where $b\in\mathbb{N}$ (cf. Section 2 for definitions of fractional operators).

We would like to close this section inviting the reader to consult some important works (and the references therein) within the promising theory of fractional difference equations [1, 2, 4, 6, 7, 10, 11].

2. Discrete fractional operators

In this section we present the reader with the definitions of *integral* and *derivative* of arbitrary order of a function defined on a discrete set. A gentle introduction to the Discrete Fractional Calculus may be found in [13].

Throughout this work and, as usual, we assume that empty sums are equal to zero.

The power function is defined by

$$x^{(y)}=\frac{\Gamma(x+1)}{\Gamma(x+1-y)},\ \text{for } x,x-y\in\mathbb{R}\setminus\{\ldots,-2,-1\}.$$

For $a\in\mathbb{R}$ we define the set $\mathbb{N}_a=\{a,a+1,a+2,\ldots\}$. Also, we use the notation $\sigma(s)=s+1$ for the shift operator and $(\Delta f)(t)=f(t+1)-f(t)$ to the forward difference operator. Moreover, $(\Delta^2 f)(t)=(\Delta\Delta f)(t)$.

For a function $f:\mathbb{N}_a\to\mathbb{R}$, the *discrete fractional sum of order* $\alpha\geqslant 0$ is defined as

$$\begin{aligned}
({}_a\Delta^0 f)(t)&=f(t),\quad t\in\mathbb{N}_a,\\
({}_a\Delta^{-\alpha} f)(t)&=\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}(t-\sigma(s))^{(\alpha-1)}f(s),\quad t\in\mathbb{N}_{a+\alpha},\ \alpha>0.
\end{aligned} \tag{2.1}$$

REMARK 2.1. Note that the operator ${}_a\Delta^{-\alpha}$ with $\alpha>0$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\alpha}$. Also observe that if $\alpha=1$, then we get the summation operator:

$$({}_a\Delta^{-1} f)(t)=\sum_{s=a}^{t-1}f(s).$$

The *discrete fractional derivative of order* $\alpha\in(1,2]$ is defined by

$$({}_a\Delta^{\alpha} f)(t)=(\Delta^2\,{}_a\Delta^{-(2-\alpha)} f)(t),\quad t\in\mathbb{N}_{a+2-\alpha}.$$

REMARK 2.2. Note that if $\alpha=2$, then the fractional derivative is just the second order forward difference operator.

3. Main results

We divide this section into two subsections for the convenience of the reader: in the first one we study the conjugate boundary value problem while in the second one we obtain the results for the right-focal boundary value problem.

3.1. Conjugate boundary value problem

Let us start presenting the Green function for the discrete fractional boundary value problem (1.3)–(1.4) [3]:

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{t^{(\alpha-1)}(\alpha+b+1-\sigma(s))^{(\alpha-1)}}{(\alpha+b+1)^{(\alpha-1)}} - (t-\sigma(s))^{(\alpha-1)}, & s < t-\alpha+1 \leqslant b+1, \\ \frac{t^{(\alpha-1)}(\alpha+b+1-\sigma(s))^{(\alpha-1)}}{(\alpha+b+1)^{(\alpha-1)}}, & t-\alpha+1 \leqslant s \leqslant b+1. \end{cases}$$

Atici *et al.* proved in [3] that

$$\max_{t\in[\alpha-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}} G(t,s) = G(s+\alpha-1,s), \text{ for } s \in [0,b+1]_{\mathbb{N}_0}.$$

Now we want to find the maximum of $G(s+\alpha-1,s)$ over s. To do that, let us start by applying the difference operator to it. Specifically:

$$\begin{aligned}
&\Delta G(s+\alpha-1,s) \\
&= \Delta\left(\frac{(s+\alpha-1)^{(\alpha-1)}(\alpha+b+1-\sigma(s))^{(\alpha-1)}}{(\alpha+b+1)^{(\alpha-1)}}\right) \\
&= \frac{1}{(\alpha+b+1)^{(\alpha-1)}}\Delta\left(\frac{\Gamma(s+\alpha)}{\Gamma(s+1)}\frac{\Gamma(\alpha+b-s+1)}{\Gamma(b-s+2)}\right) \\
&= \frac{1}{(\alpha+b+1)^{(\alpha-1)}}\left(\frac{\Gamma(s+1+\alpha)}{\Gamma(s+2)}\frac{\Gamma(\alpha+b-s)}{\Gamma(b-s+1)} - \frac{\Gamma(s+\alpha)}{\Gamma(s+1)}\frac{\Gamma(\alpha+b-s+1)}{\Gamma(b-s+2)}\right) \\
&= \frac{((s+\alpha)(b-s+1)-(\alpha+b-s)(s+1))}{(\alpha+b+1)^{(\alpha-1)}}\frac{\Gamma(s+\alpha)}{\Gamma(s+2)}\frac{\Gamma(\alpha+b-s)}{\Gamma(b-s+2)} \\
&= \frac{b(\alpha-1)+2s(1-\alpha)}{(\alpha+b+1)^{(\alpha-1)}}\frac{\Gamma(s+\alpha)}{\Gamma(s+2)}\frac{\Gamma(\alpha+b-s)}{\Gamma(b-s+2)}.
\end{aligned}$$

We conclude that $G(s+\alpha-1,s)$ is increasing for $s < \frac{b}{2}$ and decreasing for $s > \frac{b}{2}$, which means that

$$\max_{s\in[0,b+1]_{\mathbb{N}_0}} G(s+\alpha-1,s) = G\left(\frac{b}{2}+\alpha-1,\frac{b}{2}\right), \text{ if } b \text{ is even,}$$

$$\max_{s\in[0,b+1]_{\mathbb{N}_0}} G(s+\alpha-1,s) = G\left(\frac{b+1}{2}+\alpha-1,\frac{b+1}{2}\right), \text{ if } b \text{ is odd.}$$

We are now able to formulate a Lyapunov-type inequality for the conjugate boundary value problem.

THEOREM 3.1. *If the following discrete fractional boundary value problem (DFBVP)*

$$(\Delta^{\alpha} y)(t) + q(t+\alpha-1)y(t+\alpha-1) = 0, \quad t \in [0, b+1]_{\mathbb{N}_0},$$
$$y(\alpha-2) = 0 = y(\alpha+b+1),$$

has a nontrivial solution, then

$$\sum_{s=0}^{b+1} |q(s+\alpha-1)| > 4\Gamma(\alpha)\frac{\Gamma(b+\alpha+2)\Gamma^2(\frac{b}{2}+2)}{(b+2\alpha)(b+2)\Gamma^2(\frac{b}{2}+\alpha)\Gamma(b+3)}, \textit{ if } b \textit{ is even,}$$

or

$$\sum_{s=0}^{b+1} |q(s+\alpha-1)| > \Gamma(\alpha)\frac{\Gamma(b+\alpha+2)\Gamma^2(\frac{b+3}{2})}{\Gamma(b+3)\Gamma^2(\frac{b+1}{2}+\alpha)}, \textit{ if } b \textit{ is odd.}$$

Proof. Let $\mathscr{B}$ be the Banach space of functions endowed with norm $\|y\| = \max_{t\in[\alpha-2,\alpha+b+1]_{\mathbb{N}_{\alpha-2}}} |y(t)|$.

It follows from [3, Theorem 3.1] that a solution to the DFBVP satisfies the equation

$$y(t) = \sum_{s=0}^{b+1} G(t,s)q(s+\alpha-1)y(s+\alpha-1).$$

Hence,

$$\|y\| \leqslant \max_{t\in[\alpha-2,\alpha+b+1]_{\mathbb{N}_{\alpha-2}}} \sum_{s=0}^{b+1} |G(t,s)||q(s+\alpha-1)|\|y\|,$$

or, equivalently,

$$1 \leqslant \max_{t\in[\alpha-2,\alpha+b+1]_{\mathbb{N}_{\alpha-2}}} \sum_{s=0}^{b+1} |G(t,s)||q(s+\alpha-1)|.$$

Appealing now to the properties of the Green function G shown above and noting that

$$G\left(\frac{b}{2}+\alpha-1, \frac{b}{2}\right) = \frac{1}{4}\frac{(b+2\alpha)(b+2)\Gamma^2(\frac{b}{2}+\alpha)\Gamma(b+3)}{\Gamma(\alpha)\Gamma(b+\alpha+2)\Gamma^2(\frac{b}{2}+2)},$$

and

$$G\left(\frac{b+1}{2}+\alpha-1, \frac{b+1}{2}\right) = \frac{1}{\Gamma(\alpha)}\frac{\Gamma(b+3)\Gamma^2(\frac{b+1}{2}+\alpha)}{\Gamma(b+\alpha+2)\Gamma^2(\frac{b+3}{2})},$$

then the result follows immediately. □

3.2. Right-focal boundary value problem

The Green function for the boundary value problem (1.3)–(1.5) is (see [10]):

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}\frac{\Gamma(b+3)t^{(\alpha-1)}(\alpha+b-\sigma(s))^{(\alpha-2)}}{\Gamma(\alpha+b+1)}-(t-\sigma(s))^{(\alpha-1)}, & s<t-\alpha+1\leqslant b+1,\\ \frac{\Gamma(b+3)t^{(\alpha-1)}(\alpha+b-\sigma(s))^{(\alpha-2)}}{\Gamma(\alpha+b+1)}, & t-\alpha+1\leqslant s\leqslant b+1.\end{cases}$$

Moreover, Goodrich [10] proved that

$$\max_{t\in[\alpha-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}} G(t,s)=G(s+\alpha-1,s),\ \text{for } s\in[0,b+1]_{\mathbb{N}_0}.$$

Again we want to find the maximum of $G(s+\alpha-1,s)$ over $s\in[0,b+1]_{\mathbb{N}_0}$. The (discrete) derivative of G is, after some calculations,

$$\Delta G(s+\alpha-1,s)=(b(\alpha-1)+s(3-2\alpha)+1)f(s),$$

with $f(s)>0$ for all $s\in[0,b+1]_{\mathbb{N}_0}$. Now, if $\alpha<\frac{3}{2}$ then

$$h(s)=b(\alpha-1)+s(3-2\alpha)+1,$$

is increasing and since $h(0)=b(\alpha-1)+1>0$, then we conclude that $G(s+\alpha-1,s)$ is increasing. On the other hand, if $\alpha\geqslant\frac{3}{2}$ then h is decreasing but nevertheless positive since $h(b)=b(\alpha-1)+b(3-2\alpha)+1=b(2-\alpha)+1>0$. In conclusion, $G(s+\alpha-1,s)$ is increasing for all s. Therefore,

$$\max_{s\in[0,b+1]_{\mathbb{N}_0}} G(s+\alpha-1,s)=G(b+\alpha,b+1)=\Gamma(\alpha-1)(b+2).$$

We finally state the result (without proof as it is analogous to the proof of Theorem 3.1) for the right-focal fractional boundary value problem.

THEOREM 3.2. *If the right-focal discrete fractional boundary value problem*

$$(\Delta^{\alpha}y)(t)+q(t+\alpha-1)y(t+\alpha-1)=0,\quad t\in[0,b+1]_{\mathbb{N}_0},$$
$$y(\alpha-2)=0=\Delta y(\alpha+b),$$

has a nontrivial solution, then

$$\sum_{s=0}^{b+1}|q(s+\alpha-1)|>\frac{1}{\Gamma(\alpha-1)(b+2)}.$$

EXAMPLE. As mentioned in the Introduction the Lyapunov inequalities might be used to obtain bounds for the eigenvalues of certain boundary value problems. As a simple application, consider the right-focal boundary value problem in Theorem 3.2 and let the function q be constant, i.e. $q=\lambda\in\mathbb{R}$. By definition, an eigenvalue is a real number for which there is a nontrivial solution to the boundary value problem. Therefore, an eigenvalue of the BVP

$$\begin{aligned}(\Delta^{\alpha}y)(t)+\lambda y(t+\alpha-1)=0, \quad & t\in[0,b+1]_{\mathbb{N}_0},\\ y(\alpha-2)=0=\Delta y(\alpha+b), &\end{aligned}$$

must necessarily satisfy the following inequality

$$|\lambda|>\frac{1}{\Gamma(\alpha-1)(b+2)^2}.$$

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