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OPIAL–TYPE INEQUALITIES FOR FRACTIONAL INTEGRAL OPERATOR INVOLVING MITTAG—LEFFLER FUNCTION

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Abstract. In this paper we give generalization of Opial-type inequalities by using generalized fractional integral operator involving generalized Mittag–Leffler function. We deduce some results which already have been proved. Also we consider n-exponential convexity of some non-negative differences of inequalities involving Mittag-Leffler function and deduce their exponential convexity and log-convexity.

1. Introduction and preliminaries

Fractional calculus refers to integration or differentiation of fractional order. Several mathematicians contributed to this subject over the years. People like Liouville, Riemann, and Weyl made major contributions to the theory of fractional calculus. The story on the fractional calculus continued with contributions from Fourier, Abel, Lacroix, Leibniz, Grunwald and Letnikov. For a historical survey the reader may see [15, 16, 18].

Fractional integral inequalities are useful in establishing the uniqueness of solutions for certain fractional partial differential equations. They also provide upper and lower bounds for the solutions of fractional boundary value problems. These considerations have led various researchers in the field of integral inequalities to explore certain extensions and generalizations by involving fractional calculus operators (see, [1, 2, 8, 6, 14, 25, 26]).

In [3, 4, 9, 10, 11] we have established Opial-type integral inequalities for different kinds of fractional derivatives and fractional integral operators for example Riemann–Liouville, Caputo, Canvati etc. In this paper we give Opial-type integral inequalities for fractional integral operator containing a generalized Mittag–Leffler function in the kernel [21]. Definition of this generalized fractional integral operator containing Mittag–Leffler function is as follows.

DEFINITION 1.1. Let α,β,k,l,γ be positive real numbers and $\omega\in\mathbb{R}$. Then the generalized fractional integral operator containing Mittag–Leffler function $\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}$ for a real-valued continuous function f is defined by:

$$(\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}f)(x)=\int_a^x(x-t)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(x-t)^{\alpha})f(t)dt, \tag{1.1}$$

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where the function $E^{\gamma,\delta,k}_{\alpha,\beta,l}$ is generalized Mittag–Leffler function defined as

$$E^{\gamma,\delta,k}_{\alpha,\beta,l}(t)=\sum_{n=0}^{\infty}\frac{(\gamma)_{kn}}{\Gamma(\alpha n+\beta)}\frac{t^n}{(\delta)_{ln}}, \tag{1.2}$$

and $(a)_n$ is the Pochhammer symbol: $(a)_n=a(a+1)\dots(a+n-1)$, $(a)_0=1$.

If $\delta=l=1$ in (1.1), then integral operator $\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}$ reduces to an integral operator containing generalized Mittag–Leffler function $E^{\gamma,1,k}_{\alpha,\beta,1}$ introduced by Srivastava and Tomovski in [22]. Along $\delta=l=1$ in addition if $k=1$ (1.1) reduces to an integral operator defined by Prabhakar in [25] containing Mittag–Leffler function $E^{\gamma}_{\alpha,\beta}$. Let $e^{\gamma}_{\alpha,\beta}(t)=t^{\beta-1}E^{\gamma}_{\alpha,\beta}(\omega t^{\alpha})$. For $\omega=0$ in (1.1), integral operator $\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}$ would correspond essentially to the right-handed Riemann–Liouville fractional integral operator (see, [21]),

$$I^{\beta}f(x)=\frac{1}{\Gamma(\beta)}\int_a^x(x-t)^{\beta-1}f(t)dt,\ \beta>0.$$

We define a variant of Sobolev space:

$$W^{m,1}[a,b]=\left\{f\in L^1[a,b]:\frac{d^m}{dt^m}f\in L^1[a,b]\right\}.$$

DEFINITION 1.2. (Prabhakar derivative [12]) Let $f\in L^1[0,b]$, $0<t<b\leqslant\infty$, and $f*e^{-\gamma}_{\alpha,m-\beta,\omega}\in W^{m,1}[0,b]$, $m=[\beta]$ then the Prabhakar derivative is defined by following relation

$$\left(\mathbf{D}^{\gamma}_{\alpha,\beta,\omega,0+}f\right)(t)=\frac{d^m}{dt^m}\varepsilon^{-\gamma}_{\alpha,m-\beta,\omega,0+}f(t).$$

DEFINITION 1.3. (Caputo-Prabhakar derivative [12]) Let $f\in L^1[0,b]$, $0<t<b\leqslant\infty$, and $f*e^{-\gamma}_{\alpha,m-\beta,\omega}\in W^{m,1}[0,b]$, $m=[\beta]$ then the Caputo-Prabhakar derivative for $f\in AC^m[0,b]$ is defined by following relation

$$\begin{aligned}\left({}^{C}D^{\gamma}_{\alpha,\beta,\omega,0+}f\right)(t)&=\varepsilon^{-\gamma}_{\alpha,m-\beta,\omega,0+}\frac{d^m}{dt^m}f(t)\\&=\left(\mathbf{D}^{\gamma}_{\alpha,\beta,\omega,0+}f\right)(t)-\sum_{k=0}^{m-1}t^{k-\alpha}E^{-\gamma}_{\alpha,k-\beta+1}(\omega t^{\alpha})f^{(k)}(0+).\end{aligned}$$

REMARK 1.4. Let $\beta>0$ and $f\in AC^m[0,b]$, $0<t<b\leqslant\infty$, then

$$\left({}^{C}D^{\gamma}_{\alpha,\beta,\omega,0+}f\right)(t)=\mathbf{D}^{\gamma}_{\alpha,\beta,\omega,0+}\left(f(t)-\sum_{k=0}^{m-1}\frac{t^k}{k!}f^{(k)}(0+)\right).$$

In 1960. Opial established the following integral inequality [23].

Let $x(t) \in C^{(1)}[0,h]$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then

$$\int_0^h |x(t)x'(t)|dt \leqslant \frac{h}{4}\int_0^h \left(x'(t)\right)^2 dt, \tag{1.3}$$

where constant $\frac{h}{4}$ is the best possible.

Opial's inequality (1.3) is studied extensively by many researchers. It recognizes as fundamental result in the theory of differential equations (see the monograph [1]). Following theorems include generalizations of Opial's inequality and for it we need next characterization: for detail see [24, page 234–238]. We say that a function $u : [a,b] \longrightarrow \mathbb{R}$ belongs to the class $U(v,K)$ if it admits the representation

$$u(x) = \int_a^x K(x,t)v(t)\,dt$$

where v is a continuous function and K is an arbitrary non-negative kernel such that $v(x) > 0$ implies $u(x) > 0$ for every $x \in [a,b]$. We also assume that all integrals under consideration exist and are finite. The following theorem is given in [17] (also see [1, p. 89] and [24, p. 236]).

THEOREM 1.5. *Let $u_1 \in U(v_1,K)$, $u_2 \in U(v_2,K)$ and $v_2(x) > 0$ for every $x \in [a,b]$. Further, let $\phi(u)$ be convex, non-negative and increasing for $u \geqslant 0$, $f(u)$ be convex for $u \geqslant 0$ and $f(0) = 0$. If f is differentiable function and $M = \max K(x,t)$, then*

$$M\int_a^b v_2(t)\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right) f'\left(u_2(t)\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right)\right)dt \leqslant f\left(M\int_a^b v_2(t)\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right)dt\right).$$

Extension of above result stated in the following theorem [5].

THEOREM 1.6. *With same assumptions as in Theorem 1.5 we have*

$$\begin{aligned} &M\int_a^b v_2(t)\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right) f'\left(u_2(t)\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right)\right)dt \\ &\leqslant f\left(M\int_a^b v_2(t)\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right)dt\right) \\ &\leqslant \frac{1}{b-a}\int_a^b f\left(M(b-a)v_2(t)\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right)\right)dt. \end{aligned}$$

In [24, p. 236]), also the following result is proved.

THEOREM 1.7. *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Let $u \in U(v,K)$ where $(\int_a^x (K(x,t))^p\,dt)^{\frac{1}{p}} \leqslant M$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_a^b |u(x)|^{1-q}\phi'(|u(x)|)|v(x)|^q\,dx \leqslant \frac{q}{M^q}\phi\left(M\left(\int_a^b |v(x)|^q\,dx\right)^{\frac{1}{q}}\right). \tag{1.4}$$

The reverse of above inequality holds if $\phi(x^{\frac{1}{q}})$ is concave.

Properties of non-negative difference of inequality (1.4) are studied in [9]. In the following result we have extension of inequality (1.4) (see, [4]).

THEOREM 1.8. *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Let $u \in U(v,K)$ where $(\int_a^x (K(x,t))^p\, dt)^{\frac{1}{p}} \leqslant M$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\begin{aligned}
&\int_a^b |u(x)|^{1-q} \phi'(|u(x)|)|v(x)|^q\, dx \\
&\leqslant \frac{q}{M^q}\, \phi\Big(M\Big(\int_a^b |v(x)|^q\, dx\Big)^{\frac{1}{q}}\Big) \\
&\leqslant \frac{q}{M^q\,(b-a)} \int_a^b \phi\left((b-a)^{\frac{1}{q}} M|v(x)|\right) dx. \qquad (1.5)
\end{aligned}$$

The reverse of above inequality holds if $\phi(x^{\frac{1}{q}})$ is concave.

In [3] we denote the non-negative difference of extreme terms in the above inequality by $\Psi_\phi(u,v)$, as follows:

$$\begin{aligned}
\Psi_\phi(u,v) = &\frac{q}{M^q\,(b-a)} \int_a^b \phi\left((b-a)^{\frac{1}{q}} M|v(x)|\right) dx \\
&- \int_a^b |u(x)|^{1-q} \phi'(|u(x)|)|v(x)|^q\, dx, \qquad (1.6)
\end{aligned}$$

and among other properties of above functional we have proved the following results.

THEOREM 1.9. *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Let $u \in U(v,K)$ where $(\int_a^x (K(x,t))^p\, dt)^{\frac{1}{p}} \leqslant M$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\phi \in C^2(I)$, where $I \subseteq (0,\infty)$ is compact interval, then there exists $\xi \in I$ such that the following equality holds*

$$\begin{aligned}
\Psi_\phi(u,v) = &\frac{\xi\,\phi''(\xi) - (q-1)\,\phi'(\xi)}{2\,q\,\xi^{2q-1}} \\
&\times \left((b-a)M^q \int_a^b |v(x)|^{2q}\, dx - 2\int_a^b |u(x)|^q\, |v(x)|^q\, dx\right). \qquad (1.7)
\end{aligned}$$

THEOREM 1.10. *Let $\phi_1, \phi_2 : [0,\infty) \longrightarrow \mathbb{R}$ be differentiable functions such that for $q > 1$ the function $\phi_i(x^{\frac{1}{q}})$ is convex and $\phi_i(0) = 0$, $i = 1,2$. Let $u \in U(v,K)$ where $(\int_a^x (K(x,t))^p\, dt)^{\frac{1}{p}} \leqslant M$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\phi_1, \phi_2 \in C^2(I)$, where $I \subseteq (0,\infty)$ is compact interval and*

$$(b-a)M^q \int_a^b |v(x)|^{2q}\, dx - 2\int_a^b |u(x)|^q |v(x)|^q dx \neq 0,$$

then there exists an $\xi \in I$ *such that we have*

$$\frac{\Psi_{\phi_1}(u,v)}{\Psi_{\phi_2}(u,v)} = \frac{\xi\,\phi_1''(\xi) - (q-1)\,\phi_1'(\xi)}{\xi\,\phi_2''(\xi) - (q-1)\,\phi_2'(\xi)}, \tag{1.8}$$

provided the denominators are not equal to zero.

In Section 2 we give Opial-type integral inequalities and related results using generalized fractional integral operator involving generalized Mittag–Leffler function. As special cases some results of [3, 4, 5] are deduced. In Section 3 exponential convexity of non-negative differences of last term in (1.5) with other two terms is given.

2. Inequalities for generalized fractional integral involving generalized Mittag–Leffler function

We need the following lemma [21].

LEMMA 2.1. *Series given in (1.2) is absolutely convergent for all values of* t *provided* $k < l+\alpha$.

The following results appear as generalizations of Opial-type integral inequalities for Riemann–Liouville fractional integral.

THEOREM 2.2. *Let* u_1, u_2 *and* ϕ, f *be same as in Theorem 1.5, also let* $\alpha, \beta, k, l, \gamma, \omega > 0$ *such that* $k < l+\alpha$ *and* $\beta > 1$, *then we have*

$$\begin{aligned}
&E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-a)^{\alpha})(b-a)^{\beta-1} \\
&\quad\times \int_a^b v_2(x)\phi\left(\left|\frac{v_1(x)}{v_2(x)}\right|\right) f'\left((\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}v_2)(x)\,\phi\left(\left|\frac{(\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}v_1)(x)}{(\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}v_2)(x)}\right|\right)\right)dx \\
&\leqslant f\left(E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-a)^{\alpha})(b-a)^{\beta-1}\int_a^b v_2(x)\phi\left(\left|\frac{v_1(x)}{v_2(x)}\right|\right)dx\right).
\end{aligned} \tag{2.1}$$

Proof. Let us define the followings

$$K(x,t) := \begin{cases} (x-t)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(x-t)^{\alpha}), & a \leqslant t \leqslant x; \\ 0, & x < t \leqslant b, \end{cases}$$

$$u_1(x) := (\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}v_1)(x) = \int_a^x (x-t)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(x-t)^{\alpha})v_1(t)\,dt, \tag{2.2}$$

$$u_2(x) := (\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}v_2)(x) = \int_a^x (x-t)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(x-t)^{\alpha})v_2(t)\,dt. \tag{2.3}$$

One can observe that

$$\sum_{n=0}^{\infty}\frac{(\gamma)_{kn}\omega^n(x-t)^{n\alpha}}{\Gamma(\alpha n+\beta)(\delta)_{nl}} \leqslant \sum_{n=0}^{\infty}\frac{(\gamma)_{kn}\omega^n(b-a)^{n\alpha}}{\Gamma(\alpha n+\beta)(\delta)_{nl}} = E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-a)^{\alpha})$$

and Lemma 2.1 ensure the existence of $E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-a)^{\alpha})$ for $k<l+\alpha$, further $(x-t)^{\beta-1}\leqslant(x-a)^{\beta-1}\leqslant(b-a)^{\beta-1}$ for $\beta>1$. Then

$$K(x,t)\leqslant E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-a)^{\alpha})(b-a)^{\beta-1},\ \ \beta>1,\ \ k<l+\alpha. \tag{2.4}$$

Therefore we can take $M=E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-a)^{\alpha})(b-a)^{\beta-1}$, functions u_1,u_2 as in (2.2) and (2.3), in Theorem 1.5 to get (2.1). □

In the following we give the extension of Theorem 2.2.

THEOREM 2.3. *With same assumptions as in Theorem 2.2 we have*

$$\begin{aligned}
&E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-a)^{\alpha})(b-a)^{\beta-1}\\
&\qquad\times\int_a^b v_2(x)\phi\left(\left|\frac{v_1(x)}{v_2(x)}\right|\right)f'\left((\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}v_2)(x)\phi\left(\left|\frac{(\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}v_1)(x)}{(\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}v_2)(x)}\right|\right)\right)dx\\
&\leqslant f\left(E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-a)^{\alpha})(b-a)^{\beta-1}\int_a^b v_2(x)\phi\left(\left|\frac{v_1(x)}{v_2(x)}\right|\right)dx\right)\\
&\leqslant\frac{1}{b-a}\int_a^b f\left(E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-a)^{\alpha})(b-a)^{\beta-1}v_2(x)\phi\left(\left|\frac{v_1(x)}{v_2(x)}\right|\right)\right)dx.
\end{aligned}\tag{2.5}$$

Proof. Proof is similar to the proof of Theorem 2.2, here we use Theorem 1.6 instead of Theorem 1.5. □

REMARK 2.4. If $k=l=\delta=1$, then we get Opial-type inequalities for integral operator introduced by Prabharkar in [25]. If $k=l=1$ and $\omega=0$ in (2.1), then we get result for Reimann–Liouville fractional integral and using it in (2.5) we get [5, Corollary 3.2].

THEOREM 2.5. *Let u and ϕ be same as in Theorem 1.7. Also, let $\alpha,\beta,k,l,\gamma,\omega>0$ such that $k<l+\alpha$ and $\beta>1$, with $q>1$. Then we have*

$$\begin{aligned}
&\int_a^b |(\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}v)(x)|^{1-q}\phi'(|(\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}v)(x)|)|v(x)|^q\,dx\\
&\leqslant\frac{q(b-a)^{1-q\beta}}{(E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-a)^{\alpha}))^q}\\
&\quad\times\phi\left(E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-a)^{\alpha})(b-a)^{\beta-\frac{1}{q}}\left(\int_a^b|v(x)|^q dx\right)^{\frac{1}{q}}\right)\\
&\leqslant\frac{q(b-a)^{-q\beta}}{(E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-a)^{\alpha}))^q}\int_a^b\phi\left(E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-a)^{\alpha})(b-a)^{\beta}|v(x)|\right)dx,
\end{aligned}$$

the reverse of above inequality holds if $\phi(x^{\frac{1}{q}})$ is concave.

Proof. Let us define the followings

$$K(x,t) := \begin{cases} (x-t)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(x-t)^{\alpha}), & a \leqslant t \leqslant x; \\ 0, & x < t \leqslant b, \end{cases}$$

$$u(x) := (\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}v)(x) = \int_a^x (x-t)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(x-t)^{\alpha})v(t)\,dt\,. \tag{2.6}$$

From (2.4), for $\alpha > 0$ and $\beta > 1$, $k < l+\alpha$ we have

$$K(x,t) \leqslant E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-a)^{\alpha})(b-a)^{\beta-1},$$

from which we can have

$$\left(\int_a^x (K(x,t))^p\,dt\right)^{\frac{1}{p}} \leqslant E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-a)^{\alpha})(b-a)^{\beta-\frac{1}{q}},$$

here we use $\frac{1}{p}+\frac{1}{q}=1$. Therefore we can take $M = E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-a)^{\alpha})(b-a)^{\beta-\frac{1}{q}}$ and function u defined by (2.6) in Theorem 1.8 and get required inequality. □

REMARK 2.6. If $k=l=\delta=1$, then we get Opial-type inequalities for integral operator introduced by Prabharkar in [25]. If $k=l=1$ and $\omega=0$ in Theorem 2.5, then we get extension of (1.4) for Riemann–Liouville fractional integral given in [4, Theorem 3.1].

By using $\phi(x) = x^{p+q}$ we have the following result.

COROLLARY 2.7. *Let $\alpha,\beta,k,l,\gamma,\omega > 0$ such that $k < l+\alpha$ and $\beta > 1$, with $\frac{1}{p}+\frac{1}{q}=1$ and $v \in L_1[a,b]$. Then following inequalities hold*

$$\begin{aligned}
&\int_a^b |(\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}v)(x)|^p\,|v(x)|^q\,dx \\
&\leqslant q(b-a)^{p\left(\beta-\frac{1}{q}\right)}(E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-a)^{\alpha}))^p\left(\int_a^b |v(x)|^q dx\right)^{\frac{p+q}{q}} \\
&\leqslant q(b-a)^{p\beta}(E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-a)^{\alpha}))^p\int_a^b |v(x)|^{p+q}dx.
\end{aligned}$$

REMARK 2.8. If $k=l=\delta=1$, then we get result for integral operator introduced by Prabharkar in [25]. If $k=l=1$ and $\omega=0$ in above inequality, then we get [4, Corollary 3.2].

COROLLARY 2.9. *Let* $v \in AC^m[0,x]$, $x \in [a,b]$, *u and* ϕ *be same as in Theorem* 1.7. *Let* $\alpha > 0$, $m-\beta > 1$, $m = [\beta]$, $\gamma \in \mathbb{R}$, $\omega > 0$, *with* $q > 1$. *Then we have*

$$\begin{aligned}
&\int_a^b \left| \left({}^C D^{\gamma}_{\alpha,\beta,\omega,0+} v \right)(x) \right|^{1-q} \phi' \left(\left| \left({}^C D^{\gamma}_{\alpha,\beta,\omega,0+} v \right)(x) \right| \right) |v(x)|^q \, dx \\
&\leqslant \frac{q}{(b-a)^{q(m-\beta)-1} E^{-\gamma}_{\alpha,m-\beta}\left(\omega (b-a)^{\alpha}\right)} \\
&\quad \times \phi \left((b-a)^{(m-\beta)-\frac{1}{q}} E^{-\gamma}_{\alpha,m-\beta}\left(\omega (b-a)^{\alpha}\right) \left(\int_a^b \left| v^{(m)}(x) \right|^q dx \right)^{1/q} \right) \\
&\leqslant \frac{q}{(b-a)^{q(m-\beta)} E^{-\gamma}_{\alpha,m-\beta}\left(\omega (b-a)^{\alpha}\right)} \\
&\quad \times \int_a^b \phi \left((b-a)^{(m-\beta)} E^{-\gamma}_{\alpha,m-\beta}\left(\omega (b-a)^{\alpha}\right) \left| v^{(m)}(x) \right| \right) dx,
\end{aligned}$$

the reverse of above inequality holds if $\phi(x^{\frac{1}{q}})$ *is concave.*

Following Remark 1.4 we obtain the following inequalities for Prabhakar derivative.

COROLLARY 2.10. *Let* $v \in AC^m[0,x]$, $x \in [a,b]$, *u and* ϕ *be same as in Theorem* 1.7. *Let* $\alpha > 0$, $m-\beta > 1$, $m = [\beta]$, $\gamma \in \mathbb{R}$, $\omega > 0$, *with* $q > 1$. *If* $v^{(k)}(0+) = 0$, $k = 0,1,2,\dots m-1$, *then we have*

$$\begin{aligned}
&\int_a^b \left| \left(\mathbf{D}^{\gamma}_{\alpha,\beta,\omega,0+} v \right)(x) \right|^{1-q} \phi' \left(\left| \left(\mathbf{D}^{\gamma}_{\alpha,\beta,\omega,0+} v \right)(x) \right| \right) |v(x)|^q \, dx \\
&\leqslant \frac{q}{(b-a)^{q(m-\beta)-1} E^{-\gamma}_{\alpha,m-\beta}\left(\omega (b-a)^{\alpha}\right)} \\
&\quad \times \phi \left((b-a)^{(m-\beta)-\frac{1}{q}} E^{-\gamma}_{\alpha,m-\beta}\left(\omega (b-a)^{\alpha}\right) \left(\int_a^b \left| v^{(m)}(x) \right|^q dx \right)^{1/q} \right) \\
&\leqslant \frac{q}{(b-a)^{q(m-\beta)} E^{-\gamma}_{\alpha,m-\beta}\left(\omega (b-a)^{\alpha}\right)} \\
&\quad \times \int_a^b \phi \left((b-a)^{(m-\beta)} E^{-\gamma}_{\alpha,m-\beta}\left(\omega (b-a)^{\alpha}\right) \left| v^{(m)}(x) \right| \right) dx.
\end{aligned}$$

The reverse of the above inequalities hold when $\phi\left(x^{\frac{1}{q}}\right)$ *is concave.*

Now we give generalizations of Theorem 1.9 and Theorem 1.10.

THEOREM 2.11. *With same assumptions as in Theorem 1.9, let* $\alpha,\beta,k,l,\gamma,\omega>0$ *such that* $k<l+\alpha$ *and* $\beta>1$*, with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $v\in L_1[a,b]$*. Then there exists* $\xi\in I$ *such that the following equality holds*

$$\begin{aligned}
&\Psi_{\phi}((\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}v)(x),v(x))\\
&=\frac{\xi\,\phi''(\xi)-(q-1)\,\phi'(\xi)}{2\,q\,\xi^{2q-1}}(E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-a)^{\alpha}))^{q}\\
&\quad\times(b-a)^{q\beta}\int_a^b|v(x)|^{2q}\,dx-2\int_a^b|(\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}v)(x)|^{q}\,|v(x)|^{q}\,dx.
\end{aligned}$$

Proof. From the proof of Theorem 2.5 we have for $\beta>1,k<l+\alpha$

$$\left(\int_a^x(K(x,t))^p\,dt\right)^{\frac{1}{p}}\leqslant E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-a)^{\alpha})(b-a)^{\beta-\frac{1}{q}}.$$

Using the function $u=(\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}v)(x)$ and $M=E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-a)^{\alpha})(b-a)^{\beta-\frac{1}{q}}$ in Theorem 1.9 we get required equality. □

THEOREM 2.12. *With same assumptions as in Theorem 1.10, let* $\alpha,\beta,k,l,\gamma,\omega>0$ *such that* $k<l+\alpha$ *and* $\beta>1$*, with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $v\in L_1[a,b]$*. Then there exists* $\xi\in I$ *such that the following equality holds*

$$\frac{\Psi_{\phi_1}((\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}v)(x),v(x))}{\Psi_{\phi_2}((\varepsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}v)(x),v(x))}=\frac{\xi\,\phi_1''(\xi)-(q-1)\,\phi_1'(\xi)}{\xi\,\phi_2''(\xi)-(q-1)\,\phi_2'(\xi)},\tag{2.7}$$

provided the denominators are not equal to zero.

Proof. Proof is similar to the poof of Theorem 2.11, here we use Theorem 1.10 instead of Theorem 1.9. □

REMARK 2.13. If $k=l=\delta=1$, then we get results for integral operator introduced by Prabharkar in [25]. If $k=l=1$ and $\omega=0$, then we get results for Riemann–Liouville fractional integral given in [3, Theorem 3.1, Theorem 3.2].

3. Method of exponential convexity

Following definitions and properties of exponentially convex functions comes from [7], also [13, 19]. Throughout we consider I is an interval in $\mathbb{R}$.

DEFINITION 3.1. A function $\psi\colon I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} \xi_i \xi_j \psi\left(\frac{x_i + x_j}{2}\right) \geqslant 0$$

holds for all choices $\xi_i \in \mathbb{R}$ and $x_i \in I$, $i = 1,\dots,n$.

A function $\psi\colon I \to \mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

REMARK 3.2. It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leqslant n$.

By definition of positive semi-definite matrices and some basic linear algebra we have the following proposition.

PROPOSITION 3.3. *If ψ is an n-exponentially convex in the Jensen sense, then the matrix $\left[\psi\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{k}$ is a positive semi-definite matrix for all $k \in \mathbb{N}$, $k \leqslant n$. Particularly,* $\det\left[\psi\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{k} \geqslant 0$ *for all $k \in \mathbb{N}$, $k \leqslant n$.*

DEFINITION 3.4. A function $\psi\colon I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

A function $\psi\colon I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

REMARK 3.5. It is known (and easy to show) that $\psi : I \to (0,\infty)$ is a log-convex in the Jensen sense if and only if

$$\alpha^2 \psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2\psi(y) \geqslant 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a function is log-convex in the Jensen-sense if and only if it is 2-exponentially convex in the Jensen sense.

Also, using basic convexity theory it follows that a function is log-convex if and only if it is 2-exponentially convex.

Next we need divided differences, commonly used when dealing with functions that have different degree of smoothness.

DEFINITION 3.6. The second order divided difference of a function $f\colon I\to\mathbb{R}$ at mutually different points $y_0,y_1,y_2\in I$ is defined recursively by

$$\begin{aligned} [y_i;f] &= f(y_i), \quad i=0,1,2 \\ [y_i,y_{i+1};f] &= \frac{f(y_{i+1})-f(y_i)}{y_{i+1}-y_i}, \quad i=0,1 \\ [y_0,y_1,y_2;f] &= \frac{[y_1,y_2;f]-[y_0,y_1;f]}{y_2-y_0}. \end{aligned} \tag{3.1}$$

REMARK 3.7. The value $[y_0,y_1,y_2;f]$ is independent of the order of the points y_0, y_1 and y_2. This definition may be extended to include the case in which some or all the points coincide. Namely, taking the limit $y_1\to y_0$ in (3.1), we get

$$\lim_{y_1\to y_0}[y_0,y_1,y_2;f]=[y_0,y_0,y_2;f]=\frac{f(y_2)-f(y_0)-f'(y_0)(y_2-y_0)}{(y_2-y_0)^2},\ y_2\neq y_0$$

provided that f' exists, and furthermore, taking the limits $y_i\to y_0$, $i=1,2$ in (3.1), we get

$$\lim_{y_2\to y_0}\lim_{y_1\to y_0}[y_0,y_1,y_2;f]=[y_0,y_0,y_0;f]=\frac{f''(y_0)}{2}$$

provided that f'' exists.

Motivated by inequalities in (1.5) we define the following functionals, non-negative differences of last term with other two terms as follows:

$$\Psi_{1\phi_s}(u,v)=\frac{q}{M^q\,(b-a)}\int_a^b\phi\left((b-a)^{\frac{1}{q}}M|v(x)|\right)dx-\frac{q}{M^q}\,\phi\left(M\left(\int_a^b|v(x)|^q\,dx\right)^{\frac{1}{q}}\right) \tag{3.2}$$

$$\Psi_{2\phi_s}(u,v)=\frac{q}{M^q\,(b-a)}\int_a^b\phi\left((b-a)^{\frac{1}{q}}M|v(x)|\right)dx-\int_a^b|u(x)|^{1-q}\,\phi'(|u(x)|)|v(x)|^q\,dx. \tag{3.3}$$

We use a method of producing n-exponentially convex and exponentially convex functions given in [19], to prove the n-exponential convexity for the functionals $\Psi_{i\phi}(u,v)$, $i=1,2$ defined in (3.2), (3.3).

THEOREM 3.8. *Let J be an interval in $\mathbb{R}$ and $\Upsilon=\{\phi_s : s\in J\}$ be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s\mapsto[y_0,y_1,y_2;F_{\phi_s}]$ is n-exponentially convex in the Jensen sense on J for every three mutually different points $y_0,y_1,y_2\in I$, where $F_{\phi_s}(y)=\phi_s(y^{\frac{1}{q}})$. Let $\Psi_{i\phi}(u,v)$, $i=1,2$ be functionals defined in (3.2), (3.3). Then $s\mapsto\Psi_{i\phi_s}(u,v)$, $i=1,2$ are n-exponentially convex functions in the Jensen sense on J. If the functions $s\mapsto\Psi_{i\phi_s}(u,v)$, $i=1,2$ are also continuous on J, then are n-exponentially convex on J.*

Proof. See the proof of Theorem 5.11 in [3]. □

COROLLARY 3.9. *Let J be an interval in $\mathbb{R}$ and $\Upsilon = \{\phi_s : s \in J\}$ be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto [y_0, y_1, y_2; F_{\phi_s}]$ is exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$, where $F_{\phi_s}(y) = \phi_s(y^{\frac{1}{q}})$. Let $\Psi_{i\phi}(u,v)$, $i = 1,2$ be functionals defined in (3.2), (3.3). Then functions $s \mapsto \Psi_{i\phi_s}(u,v)$, $i = 1,2$ are exponentially convex function in the Jensen sense on J. If the functions $s \mapsto \Psi_{i\phi_s}(u,v)$, $i = 1,2$ are continuous on J, then are exponentially convex on J.*

Let us denote means for $\phi_s, \phi_p \in \Omega$ by

$$\mu_{s,p}(\Psi_i, \Omega) = \begin{cases} \left(\frac{\Psi_{i\phi_s}(u,v)}{\Psi_{i\phi_p}(u,v)}\right)^{\frac{1}{s-p}} & , s \neq p, \\ \exp\left(\frac{\frac{d}{ds}\Psi_{i\phi_s}(u,v)}{\Psi_{i\phi_s}(u,v)}\right), & s = p, \end{cases} \tag{3.4}$$

for $i = 1,2$.

THEOREM 3.10. *Let J be an interval in $\mathbb{R}$ and $\Omega = \{\phi_s : s \in J\}$ be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto [y_0, y_1, y_2; F_{\phi_s}]$ is 2-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$, where $F_{\phi_s}(y) = \phi_s(y^{\frac{1}{q}})$. Let $\Psi_{i\phi}(u,v)$, $i = 1,2$ be functionals defined in (3.2), (3.3). Then the following statements hold:*

(i) If the functions $s \mapsto \Psi_{i\phi}(u,v)$, $i = 1,2$ are continuous on J, then are 2-exponentially convex functions on J. If the functions $s \mapsto \Psi_{i\phi_s}(u,v)$, $i = 1,2$ are additionally positive, then are also log-convex on J, and for $r,s,t \in J$ such that $r < s < t$, we have

$$\left(\Psi_{i\phi_s}(u,v)\right)^{t-r} \leqslant \left(\Psi_{i\phi_r}(u,v)\right)^{t-s} \left(\Psi_{i\phi_t}(u,v)\right)^{s-r} \quad i = 1,2. \tag{3.5}$$

(ii) If the functions $s \mapsto \Psi_{i\phi_s}(u,v)$, $i = 1,2$ are positive and differentiable on J, then for every $s,p,r,t \in J$, such that $s \leqslant r$ and $p \leqslant t$, we have

$$\mu_{s,p}(\Psi_i, \Omega) \leqslant \mu_{r,t}(\Psi_i, \Omega), \quad i = 1,2. \tag{3.6}$$

Proof. See the proof of Theorem 5.13 in [3]. □

REMARK 3.11. The results from Theorem 3.8, Corollary 3.9 and Theorem 3.10 still hold when two of the points $y_0, y_1, y_2 \in I$ coincide, for a family of differentiable functions ϕ_s such that the function $s \mapsto [y_0, y_1, y_2; F_{\phi_s}]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense). Furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs can be obtained by recalling Remark 3.7 and suitable characterization of convexity.

4. Concluding remarks

In this last section we are interested to give remarks on proved results in the sense that they are extended results to fractional calculus, and also similar results can be obtained taking other non negative differences as functionals.

REMARK 4.1. Opial and Opial-type inequalities have many applications in differential calculus (see references), of course have equal importance in fractional differential calculus. Here in this paper given results are generalizations of Opial-type inequalities for fractional differential calculus.

REMARK 4.2. As we prove the n-exponential convexity of the functionals $\Psi_{i\phi_s}(u,v),\ i=1,2$ obtained from the inequalities given in (1.5), similarly we can define and prove the n-exponential convexity of functionals obtained from the inequalities given for fractional integral operators involving ML-functions but here we omit the details.

Some of the estimates can be applied for proving existence and uniqueness of some linear and nonlinear fractional differential equations containing Caputo, Prabhakar, Caputo-Prabhakar derivative operators [12] which is a focus of our next research.

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