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NONLOCAL BOUNDARY VALUE PROBLEMS FOR HYBRID FRACTIONAL DIFFERENTIAL EQUATIONS AND INCLUSIONS OF HADAMARD TYPE

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Abstract. This paper investigates the existence of solutions for nonlocal boundary value problems of nonlinear hybrid fractional differential equations and inclusions of Hadamard type. We make use of fixed point theorems due to Dhage [7], [8] to obtain the existence results. We also discuss some examples for illustration of the main results.

1. Introduction

In this paper, we study the existence of solutions for boundary value problems of hybrid fractional differential equations and inclusions of Hadamard type with nonlocal conditions. As a first problem, we consider

$$\begin{cases} D^{\alpha}\left(\dfrac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), \;\; 1 \leqslant t \leqslant e, \;\; 1 < \alpha \leqslant 2, \\ x(1) = 0, \;\; x(e) = m(x), \end{cases} \tag{1}$$

where D^{α} is the Hadamard fractional derivative, $f \in C([1,e]\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $g : C([1,e]\times\mathbb{R},\mathbb{R})$ and $m : C([1,e],\mathbb{R}) \to \mathbb{R}$.

The second problem is concerned with the case when hybrid part of Hadamard type fractional differential equation involves Hadamard integral for a given nonlinear function, and is given by

$$\begin{cases} D^{\alpha}\left(\dfrac{x(t)}{f(t,x(t)) + \dfrac{1}{\Gamma(\beta)}\displaystyle\int_1^t \left(\ln\frac{t}{s}\right)^{\beta-1}\frac{q(s,x(s))}{s}ds}\right) = g(t,x(t)), \; 1 \leqslant t \leqslant e, \\ x(1) = 0, \;\; x(e) = m(x), \end{cases} \tag{2}$$

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where $1 < \alpha \leqslant 2,\ \beta > 0,\ f,\ g \in C([1,e] \times \mathbb{R}, \mathbb{R})$ are such that

$$f(t,x(t)) + \frac{1}{\Gamma(\beta)} \int_1^t \left(\ln \frac{t}{s}\right)^{\beta-1} \frac{g(s,x(s))}{s} ds \neq 0, \quad \forall (t,x) \in [1,e] \times \mathbb{R}.$$

Here we emphasize that the integral in (2) is known as Hadamard integral to be defined later. In the third problem, we study the multivalued case of the problem (1) given by

$$\begin{cases} D^{\alpha}\left(\dfrac{x(t)}{f(t,x(t))}\right) \in F(t,x(t)), \quad 1 \leqslant t \leqslant e, \quad 1 < \alpha \leqslant 2, \\ \qquad\qquad x(1) = 0, \quad x(e) = m(x), \end{cases} \tag{3}$$

where $F : [1,e] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map, $\mathscr{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

It is well known that the nonlocal conditions are regarded as more plausible than the standard initial conditions for the description of some physical phenomena. In the above problems, $m(x)$ may be understood as $m(x) = \sum_{\nu=1}^{p} \alpha_\nu x(t_\nu)$ where $\alpha_\nu, \nu = 1,\ldots,p$, are given constants and $0 < t_1 < \ldots < t_p \leqslant 1$. For more details we refer to the work by Byszewski [5].

Fractional differential equations have attracted the attentions of many researchers working in a variety of disciplines, due to the development and applications of these equations in many fields such as engineering, mathematics, physics, chemistry, etc. For more details, see ([3, 12]). However, it has been noticed that most of the work on the topic is concerned with Riemann-Liouville or Caputo type fractional differential equations. Besides these fractional derivatives, another kind of fractional derivatives found in the literature is the fractional derivative due to Hadamard introduced in 1892 [10], which differs from the aforementioned derivatives in the sense that the kernel of the integral in the definition of Hadamard derivative contains logarithmic function of arbitrary exponent. A detailed description of Hadamard fractional derivative and integral can be found in [11, 12].

In [9], the authors initiated the study on hybrid differential equations. Sun *et al.* [14] studied the existence of solutions for a Riemann-Liouville type fractional boundary value problem. Recently, an initial-value problem for hybrid Hadamard fractional differential equations is discussed in [1].

This paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel. In Section 3, we present the existence theorems for the problems (1) and (2) while the existence of solutions for the problem (3) is discussed in Section 3. Our results rely on fixed point theorems in Banach algebras due to Dhage [7], [8] under Lipschitz and Carathéodory conditions.

For some recent work on hybrid fractional differential equations, we refer to [2], [4], [15] and the references cited therein.

2. Preliminaries

This section is devoted to the preliminary concepts of fractional calculus and multivalued maps.

2.1. Fractional calculus

DEFINITION 2.1. [12] The Hadamard fractional integral of order q for a function $g:(1,\infty)\to\mathbb{R}$ is defined as

$$I^q g(t)=\frac{1}{\Gamma(q)}\int_1^t\left(\ln\frac{t}{s}\right)^{q-1}\frac{g(s)}{s}ds,\ \ q>0,\ \ t>1.$$

DEFINITION 2.2. [12] The Hadamard derivative of fractional order q for a function $g:(1,\infty)\to\mathbb{R}$ is defined as

$$D^q g(t)=\frac{1}{\Gamma(n-q)}\left(t\frac{d}{dt}\right)^n\int_1^t\left(\ln\frac{t}{s}\right)^{n-q-1}\frac{g(s)}{s}ds,\ \ n-1<q<n,\ \ n=[q]+1,$$

where $[q]$ denotes the integer part of the real number q.

LEMMA 2.3. *Given $y\in C([1,e],\mathbb{R})$, the integral solution of boundary value problem*

$$\begin{cases} D^\alpha\left(\dfrac{x(t)}{f(t,x(t))}\right)=y(t),\ \ 0<t<1, \\ x(1)=0,\ \ x(e)=m(x) \end{cases} \tag{4}$$

is given by

$$\begin{aligned} x(t)=f(t,x(t))\Bigg(&\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{y(s)}{s}ds \\ &+(\ln t)^{\alpha-1}\left[\frac{m(x)}{f(e,m(x))}-\frac{1}{\Gamma(\alpha)}\int_1^e\left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{y(s)}{s}ds\right]\Bigg),\quad t\in[1,e]. \end{aligned}$$

Proof. As argued in [12], the solution of Hadamard differential equation in (4) can be written as

$$x(t)=f(t,x(t))\left(\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{y(s)}{s}ds+c_1(\ln t)^{\alpha-1}+c_2(\ln t)^{\alpha-2}\right), \tag{5}$$

where $c_1,c_2\in\mathbb{R}$ are arbitrary constants. Using the boundary conditions given in (4), we find that

$$c_2=0,\quad c_1=\frac{m(x)}{f(e,m(x))}-\frac{1}{\Gamma(\alpha)}\int_1^e\left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{y(s)}{s}ds.$$

Substituting the values of c_1, c_2 in (5), we get

$$x(t) = f(t,x(t))\left(\frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{y(s)}{s}ds\right.$$
$$\left.+(\ln t)^{\alpha-1}\left[\frac{m(x)}{f(e,m(x))} - \frac{1}{\Gamma(\alpha)}\int_1^e \left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{y(s)}{s}ds\right]\right),\ t\in[1,e]. \quad \square$$

2.2. Multi-valued analysis

Let us recall some basic definitions on multi-valued maps [6].

For a normed space $(X,\|\cdot\|)$, let $\mathscr{P}_{cl}(X) = \{Y \in \mathscr{P}(X) : Y \text{ is closed}\}$, $\mathscr{P}_b(X) = \{Y \in \mathscr{P}(X) : Y \text{ is bounded}\}$, $\mathscr{P}_{cp}(X) = \{Y \in \mathscr{P}(X) : Y \text{ is compact}\}$, and $\mathscr{P}_{cp,cv}(X) = \{Y \in \mathscr{P}(X) : Y \text{ is compact and convex}\}$. A multi-valued map $G : X \to \mathscr{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}}G(x)$ is bounded in X for all $\mathbb{B} \in \mathscr{P}_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathscr{N}_0$ of x_0 such that $G(\mathscr{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathscr{P}_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : [0;1] \to \mathscr{P}_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y,G(t)) = \inf\{|y-z| : z \in G(t)\}$$

is measurable.

Let $C([1,e],\mathbb{R})$ denote a Banach space of continuous functions from $[1,e]$ into $\mathbb{R}$ with the norm $\|x\| = \sup_{t\in[1,e]}|x(t)|$. Let $L^1([1,e],\mathbb{R})$ be the Banach space of measurable functions $x : [1,e] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_1^e |x(t)|dt$.

For each $y \in C([1,e],\mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([1,e],\mathbb{R}) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in [1,e]\}.$$

DEFINITION 2.4. A multivalued map $F : [1,e]\times\mathbb{R} \to \mathscr{P}(\mathbb{R})$ is said to be Carathéodory if

(*i*) $t \longmapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;

(*ii*) $x \longmapsto F(t,x)$ is upper semicontinuous for almost all $t \in [1,e]$;

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) there exists a function $g \in L^1([1,e],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leqslant g(t)$$

for all $x \in \mathbb{R}$ and for a.e. $t \in [1,e]$.

2.3. Fixed point theorems and a useful lemma

The following fixed point theorems due to Dhage [7], [8] are fundamental in the proof of our main results for single and multivalued cases respectively.

LEMMA 2.5. ([7]) *Let S be a non-empty, closed convex and bounded subset of the Banach algebra X; let $A : X \to X$ and $B : S \to X$ be two operators such that:*

(a) A is Lipschitzian with a Lipschitz constant k,

(b) B is completely continuous,

(c) $x = AxBy \Rightarrow x \in S$ for all $y \in S$, and

(d) $Mk < 1$, where $M = \|B(S)\| = \sup\{\|B(x)\| : x \in S\}$.

Then the operator equation $x = AxBx$ has a solution.

LEMMA 2.6. ([8]) *Let X be a Banach algebra and let $A : X \to X$ be a single valued and $B : X \to \mathscr{P}_{cp,cv}(X)$ be a multi-valued operator satisfying:*

(a) A is single-valued Lipschitz with a Lipschitz constant k,

(b) B is compact and upper semi-continuous,

(c) $2Mk < 1$, where $M = \|B(X)\|$.

Then either

(i) the operator inclusion $x \in AxBx$ has a solution, or

(ii) the set $\mathscr{E} = \{u \in X | \mu u \in AuBu,\ \mu > 1\}$ is unbounded.

We also use the following lemma in the sequel.

LEMMA 2.7. ([13]) *Let X be a Banach space. Let $F : [1,e] \times \mathbb{R} \to \mathscr{P}_{cp,cv}(X)$ be an L^1-Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([1,e],X)$ to $C([1,e],X)$. Then the operator*

$$\Theta \circ S_F : C([1,e],X) \to \mathscr{P}_{cp,cv}(C([1,e],X)),\ \ x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([1,e],X) \times C([1,e],X)$.

3. Existence results-the single valued case

THEOREM 3.1. *Assume that*

(H_1) *the function* $f:[1,e]\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$ *is continuous and there exists a bounded function* ϕ, *with bound* $\|\phi\|$, *such that* $\phi(t)>0$, *for* $t\in[1,e]$ *and*

$$|f(t,x(t))-f(t,y(t))|\leqslant\phi(t)|x(t)-y(t)|,\quad \textit{for } t\in[1,e] \textit{ and for all } x,y\in\mathbb{R};$$

(H_2) *there exists a constant* $M_1>0$ *such that* $\left|\dfrac{m(x)}{f(e,m(x))}\right|\leqslant M_1$;

(H_3) *there exist a function* $p\in C([1,e],\mathbb{R}^+)$ *and a continuous nondecreasing function* $\Omega:[0,\infty)\to(0,\infty)$ *such that*

$$|g(t,x(t))|\leqslant p(t)\Omega(\|x\|),\quad t\in[1,e],\ \textit{and for all } x\in\mathbb{R};$$

(H_4) *there exists a number* $r>0$ *such that*

$$r\geqslant\frac{F_0\left[\dfrac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r)+M_1\right]}{1-\|\phi\|\left[\dfrac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r)+M_1\right]},\tag{6}$$

where

$$\|\phi\|\left[\frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r)+M_1\right]<1,$$

and $F_0=\sup_{t\in[1,e]}|f(t,0)|$.

Then the boundary value problem (1) has at least one solution on $[1,e]$.

Proof. Set $X=C([1,e],\mathbb{R})$ and define a subset S of X as follows:

$$S=\{x\in X:\|x\|\leqslant r\},$$

where r satisfies the inequality (6).

Clearly S is closed, convex and bounded subset of the Banach space X. By Lemma 2.3, the boundary value problem (1) is equivalent to the integral equation

$$\begin{aligned}x(t)=f(t,x(t))\Bigg(&\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{g(s,x(s))}{s}ds\\&+(\ln t)^{\alpha-1}\left[\frac{m(x)}{f(e,m(x))}-\frac{1}{\Gamma(\alpha)}\int_1^e\left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{g(s,x(s))}{s}ds\right]\Bigg),\quad t\in[1,e].\end{aligned}\tag{7}$$

Define two operators $\mathscr{A}: X \to X$ by

$$\mathscr{A}x(t) = f(t,x(t)), \quad t \in [1,e], \tag{8}$$

and $\mathscr{B}: S \to X$ by

$$\begin{aligned}\mathscr{B}x(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{g(s,x(s))}{s} ds \\ &\quad + (\ln t)^{\alpha-1} \left[\frac{m(x)}{f(e,m(x))} - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln \frac{e}{s}\right)^{\alpha-1} \frac{g(s,x(s))}{s} ds \right], \quad t \in [1,e].\end{aligned} \tag{9}$$

Then $x = \mathscr{A}x\mathscr{B}x$. We shall show that the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Lemma 2.5. For the sake of clarity, we split the proof into a sequence of steps.

Step 1. *We first show that $\mathscr{A}$ is Lipschitz on X, i.e. (a) of Lemma 2.5 holds.*

Let $x, y \in X$. Then by (H_1) we have

$$\begin{aligned}|\mathscr{A}x(t) - \mathscr{A}y(t)| &= |f(t,x(t)) - f(t,y(t))| \\ &\leqslant \phi(t)|x(t) - y(t)| \\ &\leqslant \|\phi\| \|x - y\|\end{aligned}$$

for all $t \in [1,e]$. Taking the supremum over the interval $[1,e]$, we obtain

$$\|\mathscr{A}x - \mathscr{A}y\| \leqslant \|\phi\| \|x - y\|$$

for all $x, y \in X$. So $\mathscr{A}$ is a Lipschitz on X with Lipschitz constant $\|\phi\|$.

Step 2. *The operator $\mathscr{B}$ is completely continuous on S, i.e. (b) of Lemma 2.5 holds.*

First we show that $\mathscr{B}$ is continuous on S. Let $\{x_n\}$ be a sequence in S converging to a point $x \in S$. Then by Lebesgue dominated convergence theorem,

$$\begin{aligned}\lim_{n\to\infty} \mathscr{B}x_n(t) &= \lim_{n\to\infty} \Bigg(\frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{g(s,x_n(s))}{s} ds \\ &\quad + (\ln t)^{\alpha-1} \left[\frac{m(x)}{f(e,m(x))} - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln \frac{e}{s}\right)^{\alpha-1} \frac{g(s,x_n(s))}{s} ds \right] \Bigg) \\ &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{\lim_{n\to\infty} g(s,x_n(s))}{s} ds \\ &\quad + (\ln t)^{\alpha-1} \left[\frac{m(x)}{f(e,m(x))} - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln \frac{e}{s}\right)^{\alpha-1} \frac{\lim_{n\to\infty} g(s,x_n(s))}{s} ds \right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{g(s,x(s))}{s}ds \\
&\quad +(\ln t)^{\alpha-1}\left[\frac{m(x)}{f(e,m(x))}-\frac{1}{\Gamma(\alpha)}\int_1^e \left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{g(s,x(s))}{s}ds\right] \\
&= \mathscr{B}x(t),
\end{aligned}
$$

for all $t\in[1,e]$. This shows that $\mathscr{B}$ is continuous os S. It is enough to show that $\mathscr{B}(S)$ is a uniformly bounded and equicontinuous set in X. First we note that

$$
\begin{aligned}
|\mathscr{B}x(t)| &= \Bigg|\frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{g(s,x(s))}{s}ds \\
&\quad +(\ln t)^{\alpha-1}\left[\frac{m(x)}{f(e,m(x))}-\frac{1}{\Gamma(\alpha)}\int_1^e \left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{g(s,x(s))}{s}ds\right]\Bigg| \\
&\leqslant \left[\frac{\|p\|\Omega(r)}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{1}{s}ds+M_1+\frac{\|p\|\Omega(r)}{\Gamma(\alpha)}\int_1^e \left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{1}{s}ds\right] \\
&= \frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r)+M_1,
\end{aligned}
$$

for all $t\in[1,e]$. Taking supremum over the interval $[1,e]$, the above inequality becomes

$$
\|\mathscr{B}x\|\leqslant\frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r)+M_1,
$$

for all $x\in S$. This shows that $\mathscr{B}$ is uniformly bounded on S.

Next we show that $\mathscr{B}$ is an equicontinuous set in X. Let $\tau_1,\tau_2\in[1,e]$ with $\tau_1<\tau_2$ and $x\in S$. Then we have

$$
\begin{aligned}
|(\mathscr{B}x)(\tau_2)-(\mathscr{B}x)(\tau_1)| &\leqslant \frac{\|p\|\Omega(r)}{\Gamma(\alpha)}\left|\int_1^{\tau_1}\left(\ln\frac{\tau_1}{s}\right)^{\alpha-1}\frac{1}{s}ds-\int_1^{\tau_2}\left(\ln\frac{\tau_2}{s}\right)^{\alpha-1}\frac{1}{s}ds\right| \\
&\quad +\frac{\|p\|\Omega(r)|(\ln\tau_2)^{\alpha-1}-(\ln\tau_1)^{\alpha-1}|}{\Gamma(\alpha)}\int_1^e\left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{1}{s}ds \\
&\leqslant \frac{\|p\|\Omega(r)}{\Gamma(\alpha)}\left|\int_1^{\tau_1}\left[\left(\ln\frac{\tau_1}{s}\right)^{\alpha-1}-\left(\ln\frac{\tau_2}{s}\right)^{\alpha-1}\right]\frac{1}{s}ds\right| \\
&\quad +\frac{\|p\|\Omega(r)}{\Gamma(\alpha)}\left|\int_{\tau_1}^{\tau_2}\left(\ln\frac{\tau_2}{s}\right)^{\alpha-1}\frac{1}{s}ds\right| \\
&\quad +\frac{\|p\|\Omega(r)|(\ln\tau_2)^{\alpha-1}-(\ln\tau_1)^{\alpha-1}|}{\Gamma(\alpha)}\int_1^e\left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{1}{s}ds.
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $x\in S$ as $t_2-t_1\to 0$. Therefore, it follows from the Arzelá-Ascoli theorem that $\mathscr{B}$ is a completely continuous operator on S.

Step 3. *Next we show that hypothesis (c) of Lemma 2.5 is satisfied.* Let $x \in X$ and $y \in S$ be arbitrary elements such that $x = \mathscr{A}x\mathscr{B}y.$ Then we have

$$\begin{aligned}
|x(t)| &= |\mathscr{A}x(t)||\mathscr{B}y(t)| \\
&= |f(t,x(t))|\Bigg|\Bigg(\frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{g(s,y(s))}{s}ds \\
&\quad +(\ln t)^{\alpha-1}\left[\frac{m(y)}{f(e,m(y))}-\frac{1}{\Gamma(\alpha)}\int_1^e \left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{g(s,y(s))}{s}ds\right]\Bigg)\Bigg| \\
&\leqslant [|f(t,x(t))-f(t,0)|+|f(t,0)|]\times\Bigg|\Bigg(\frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{g(s,y(s))}{s}ds \\
&\quad +(\ln t)^{\alpha-1}\left[\frac{m(y)}{f(e,m(y))}-\frac{1}{\Gamma(\alpha)}\int_1^e \left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{g(s,y(s))}{s}ds\right]\Bigg)\Bigg| \\
&\leqslant [\phi(t)|x(t)|+F_0]\Bigg[M_1+\frac{\|p\|\Omega(r)}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{1}{s}ds \\
&\quad +\frac{\|p\|\Omega(r)}{\Gamma(\alpha)}\int_1^e \left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{1}{s}ds\Bigg] \\
&\leqslant [\|\phi\||x(t)|+F_0]\left[\frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r)+M_1\right].
\end{aligned}$$

Thus

$$|x(t)| \leqslant \|\phi\||x(t)|\left[\frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r)+M_1\right]+F_0\left[\frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r)+M_1\right]$$

or

$$|x(t)| \leqslant \frac{F_0\left[\frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r)+M_1\right]}{1-\|\phi\|\left[\frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r)+M_1\right]}.$$

Taking supremum for $t \in [1,e],$ we obtain

$$\|x\| \leqslant \frac{F_0\left[\frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r)+M_1\right]}{1-\|\phi\|\left[\frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r)+M_1\right]} \leqslant r,$$

that is, $x \in S.$

Step 4. *Now we show that $Mk < 1$, that is, (d) of Lemma 2.5 holds.*

This is obvious by (H_3), since we have $k = \|\phi\|$ and $M = \|B(S)\| = \sup\{\|\mathscr{B}x\| : x \in S\} \leqslant \frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r) + M_1$.

Thus all the conditions of Lemma 2.5 are satisfied and hence the operator equation $x = \mathscr{A}x\mathscr{B}x$ has a solution in S. In consequence, the problem (1) has a solution on $[1,e]$. This completes the proof. □

EXAMPLE 3.2. Consider the boundary value problem

$$\begin{cases} D^{3/2}\left(\dfrac{x(t)}{|x(t)|\sin t + 1}\right) = \dfrac{1}{4}\cos x(t), \quad 1 < t < e, \\ \\ x(1) = 0, \quad x(e) = \dfrac{1}{16}\sin x(\eta), \quad \eta \in (0,1). \end{cases} \tag{10}$$

Let $f(t,x) = |x|\sin t + 1, g(t,x) = \frac{1}{4}\cos x$. Then (H_1) and (H_2) hold with $\phi(t) = 1$ and $p(t) = \frac{1}{4}$, $\Omega(r) = 1$ respectively. Since $\frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r) + M_1 = \frac{2}{3\sqrt{\pi}} + \frac{1}{16} < 1$, the boundary value problem (10) has a solution.

In the next we give a result for the boundary value problem (2). For simplicity we consider $m = 0$.

THEOREM 3.3. *Assume that (H_2) and the following conditions hold:*

(H_5) *the functions $f,q : [1,e] \times \mathbb{R} \to \mathbb{R}$ are continuous and there exist bounded functions ϕ and ψ with bounds $\|\phi\|$ and $\|\psi\|$ such that $\phi(t) > 0, \psi(t) > 0$ for $t \in [1,e]$ and $|f(t,x(t)) - f(t,y(t))| \leqslant \phi(t)|x(t) - y(t)|$, $|q(t,x(t)) - q(t,y(t))| \leqslant \psi(t)|x(t) - y(t)|$, for $t \in [1,e]$ and for all x, $y \in \mathbb{R}$;*

(H_6) *there exists a number $r > 0$ such that*

$$r \geqslant \frac{2(F_0\Gamma(\beta+1) + H_0)\|p\|\Omega(r)}{[\Gamma(\alpha+1)\Gamma(\beta+1) - 2(\|\phi\|\Gamma(\beta+1) + \|\psi\|)\|p\|\Omega(r)]}, \tag{11}$$

where $[\Gamma(\alpha+1)\Gamma(\beta+1) - 2(\|\phi\|\Gamma(\beta+1) + \|\psi\|)\|p\|\Omega(r)] > 0$, $F_0 = \sup_{t\in[1,e]}|f(t,0)|$ and $H_0 = \sup_{t\in[1,e]}|q(t,0)|$.

Then the boundary value problem (2) has at least one solution on $[1,e]$.

Proof. Setting the operator $\mathscr{A} : X \to X$ as

$$\mathscr{A}x(t) = f(t,x(t)) + \frac{1}{\Gamma(\beta)}\int_1^t \left(\ln\frac{t}{s}\right)^{\beta-1}\frac{q(s,x(s))}{s}ds, \; t \in [1,e], \tag{12}$$

the proof is similar to that of Theorem 3.1. So we omit it. □

EXAMPLE 3.4. Consider the problem (2) with $\alpha = 3/2$, $f(t,x) = (\sin x + x + 1)/\sqrt{t+3}$, $\beta = 3$, $q(t,x) = (\tan^{-1} x + \pi)/\sqrt{1+t}$, $g(t,x) = \cos x/(3+t)$, $1 < t < e$. Then $\phi(t) = 2/\sqrt{t+3}$, $\psi(t) = 1/\sqrt{t+1}$, $p(t) = 1/(3+t)$. With $\|\phi\| = 1$, $\|\psi\| = 1/\sqrt{2}$, $\|p\| = 1/4$, $\Omega(r) = 1$ and

$$\Gamma(\alpha+1)\Gamma(\beta+1) - 2(\|\phi\|\Gamma(\beta+1) + \|\psi\|)\|p\|\Omega(r) \simeq 4.622489,$$

all the conditions of Theorem 3.3 are satisfied. Hence the problem (2) with given data has at least one solution on $[1,e]$.

4. Existence result-the multivalued case

DEFINITION 4.1. A function $x \in C^2([1,e],\mathbb{R})$ is called a solution of the problem (3) if there exists a function $v \in L^1([1,e],\mathbb{R})$ with $v(t) \in F(t,x(t))$, a.e. on $[1,e]$ such that $D^\alpha\left(\dfrac{x(t)}{f(t,x(t))}\right) = v(t)$, a.e. on $[1,e]$ and $x(1) = 0$, $x(e) = m(x)$.

THEOREM 4.2. *Assume that* $(H_1),(H_2)$ *and the following conditions hold:*

(A_1) $F : [1,e] \times \mathbb{R} \to \mathscr{P}_{cp,cv}(\mathbb{R})$ *is* L^1*-Carathéodory multivalued map;*

(A_2) *there exists a continuous function* $\zeta \in C([1,e],\mathbb{R}^+)$ *such that*

$$\|F(t,x)\|_{\mathscr{P}} := \sup\{|y| : y \in F(t,x)\} \leqslant \zeta(t) \ \textit{for each} \ (t,x) \in [1,e] \times \mathbb{R};$$

(A_3) *there exists a positive real number* R *such that*

$$R > \frac{F_0\left[\dfrac{2}{\Gamma(\alpha+1)}\|\zeta\| + M_1\right]}{1 - \|\phi\|\left[\dfrac{2}{\Gamma(\alpha+1)}\|\zeta\| + M_1\right]}, \tag{13}$$

where $\|\phi\|\left[\dfrac{2}{\Gamma(\alpha+1)}\|\zeta\| + M_1\right] < \dfrac{1}{2}$, $F_0 = \sup_{t\in[1,e]} |F(t,0)|$.

Then the boundary value problem (3) has at least one solution on $[1,e]$.

Proof. Set $X = C([1,e],\mathbb{R})$. To transform the problem (3) into a fixed point problem, define an operator $\mathscr{F} : X \to \mathscr{P}(X)$ as

$$\mathscr{F}(x) = \left\{ \begin{array}{l} h \in C([1,e],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} f(t,x(t))\left(\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \right. \\ \left. +(\ln t)^{\alpha-1}\left[\dfrac{m(x)}{f(e,m(x))} - \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^e \left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds\right]\right), \end{array}\right. \end{array} \right\}$$

for $v \in S_{F,x}$. Now we define two operators $\mathscr{A} : X \to X$ by

$$\mathscr{A}x(t) = f(t,x(t)),\ t \in [1,e], \tag{14}$$

and $\mathscr{B} : X \to \mathscr{P}(X)$ by

$$\mathscr{B}(x) = \left\{ \begin{array}{l} h \in C([1,e],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \\ +(\ln t)^{\alpha-1} \left[\dfrac{m(x)}{f(e,m(x))} - \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^e \left(\ln \frac{e}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \right]. \end{array} \right. \end{array} \right\} \tag{15}$$

Observe that $\mathscr{F}(x) = \mathscr{A}x\mathscr{B}x$. We shall show that the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Lemma 2.5. For the sake of convenience, we split the proof into several steps.

Step 1. *$\mathscr{A}$ is Lipschitz on X (see Step 1 of Theorem 3.1), so (a) of Lemma 2.6 holds.*

Step 2. *The multi-valued operator $\mathscr{B}$ is compact and upper semicontinuous on X, i.e. (b) of Lemma 2.6 holds.*

First we show that $\mathscr{B}$ has convex values. Let $u_1, u_2 \in \mathscr{B}x$. Then there are $v_1, v_2 \in S_{F,x}$ such that

$$\begin{aligned} u_i(t) = {} & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{v_i(s)}{s} ds \\ & +(\ln t)^{\alpha-1} \left[\frac{m(x)}{f(e,m(x))} - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln \frac{e}{s}\right)^{\alpha-1} \frac{v_i(s)}{s} ds \right], \end{aligned}$$

$i = 1,2,\ t \in [1,e]$. For any $\theta \in [0,1]$, we have

$$\begin{aligned} & \theta u_1(t) + (1-\theta) u_2(t) \\ = {} & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{[\theta u_1(s) + (1-\theta) u_2(s)]}{s} ds \\ & +(\ln t)^{\alpha-1} \left[\frac{m(x)}{f(e,m(x))} - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln \frac{e}{s}\right)^{\alpha-1} \frac{[\theta u_1(s) + (1-\theta) u_2(s)]}{s} ds \right] \\ = {} & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{\overline{v}(s)}{s} ds \\ & +(\ln t)^{\alpha-1} \left[\frac{m(x)}{f(e,m(x))} - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln \frac{e}{s}\right)^{\alpha-1} \frac{\overline{v}(s)}{s} ds \right], \end{aligned}$$

where $\overline{v}(t) = \theta v_1(t) + (1-\theta) v_2(t) \in F(t,x(t))$ for all $t \in [1,e]$. Hence $\theta u_1(t) + (1-\theta) u_2(t) \in \mathscr{B}x$ and consequently $\mathscr{B}x$ is convex for each $x \in X$. As a result $\mathscr{B}$ defines a multi valued operator $\mathscr{B} : X \to \mathscr{P}_{cv}(X)$.

Next we show that $\mathscr{B}$ maps bounded sets into bounded sets in X. To see this, let Q be a bounded set in X. Then there exists a real number $r>0$ such that $\|x\| \leqslant r$, $\forall x \in Q$.

Now for each $h \in \mathscr{B}x$, there exists a $v \in S_{F,x}$ such that

$$\begin{aligned} h(t) = & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \\ & +(\ln t)^{\alpha-1}\left[\frac{m(x)}{f(e,m(x))} - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln \frac{e}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds\right]. \end{aligned}$$

Then for each $t \in [1,e]$, we have

$$\begin{aligned} |\mathscr{B}x(t)| = & \left| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \right. \\ & \left. +(\ln t)^{\alpha-1}\left[\frac{m(x)}{f(e,m(x))} - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln \frac{e}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds\right] \right| \\ \leqslant & \frac{2}{\Gamma(\alpha+1)} \|\zeta\| + M_1. \end{aligned}$$

This further implies that

$$\|h\| \leqslant \frac{2}{\Gamma(\alpha+1)} \|\zeta\| + M_1,$$

and so $\mathscr{B}(X)$ is uniformly bounded.

Next we show that $\mathscr{B}$ maps bounded sets into equicontinuous sets. Let Q be, as above, a bounded set and $h \in \mathscr{B}x$ for some $x \in Q$. Then there exists a $v \in S_{F,x}$ such that

$$\begin{aligned} h(t) = & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \\ & +(\ln t)^{\alpha-1}\left[\frac{m(x)}{f(e,m(x))} - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln \frac{e}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds\right], \end{aligned}$$

$t \in [1,e]$. Then for any $\tau_1, \tau_2 \in [1,e]$ we have

$$\begin{aligned} |h(\tau_2) - h(\tau_1)| \leqslant & \frac{\|\zeta\|}{\Gamma(\alpha)} \left| \int_1^{\tau_1} \left(\ln \frac{\tau_1}{s}\right)^{\alpha-1} \frac{1}{s} ds - \int_1^{\tau_2} \left(\ln \frac{\tau_2}{s}\right)^{\alpha-1} \frac{1}{s} ds \right| \\ & + \frac{\|\zeta\| |(\ln \tau_2)^{\alpha-1} - (\ln \tau_1)^{\alpha-1}|}{\Gamma(\alpha)} \int_1^e \left(\ln \frac{e}{s}\right)^{\alpha-1} \frac{1}{s} ds \end{aligned}$$

$$\leqslant \frac{\|\zeta\|}{\Gamma(\alpha)}\left|\int_1^{\tau_1}\left[\left(\ln\frac{\tau_1}{s}\right)^{\alpha-1}-\left(\ln\frac{\tau_2}{s}\right)^{\alpha-1}\right]\frac{1}{s}ds\right|$$
$$+\frac{\|\zeta\|}{\Gamma(\alpha)}\left|\int_{\tau_1}^{\tau_2}\left(\ln\frac{\tau_2}{s}\right)^{\alpha-1}\frac{1}{s}ds\right|$$
$$+\frac{\|\zeta\||(\ln\tau_2)^{\alpha-1}-(\ln\tau_1)^{\alpha-1}|}{\Gamma(\alpha)}\int_1^e\left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{1}{s}ds.$$

Obviously the right hand side of the above inequality tends to zero independently of $x\in Q$ as $t_2-t_1\to 0$. Therefore it follows by the Arzelá-Ascoli theorem that $\mathscr{B}: X\to\mathscr{P}(X)$ is completely continuous.

In our next step, we show that $\mathscr{B}$ is upper semicontinuous. It is known [6, Proposition 1.2] that $\mathscr{B}$ will be upper semicontinuous if we establish that it has a closed graph, since $\mathscr{B}$ is already shown to be completely continuous. Thus we will prove that *$\mathscr{B}$ has a closed graph.*

Let $x_n\to x_*, h_n\in\mathscr{B}(x_n)$ and $h_n\to h_*$. Then we need to show that $h_*\in\mathscr{B}$. Associated with $h_n\in\mathscr{B}(x_n)$, there exists $v_n\in S_{F,x_n}$ such that for each $t\in[1,e]$,

$$h_n(t)=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{v_n(s)}{s}ds$$
$$+(\ln t)^{\alpha-1}\left[\frac{m(x)}{f(e,m(x))}-\frac{1}{\Gamma(\alpha)}\int_1^e\left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{v_n(s)}{s}ds\right].$$

Thus it suffices to show that there exists $v_*\in S_{F,x_*}$ such that for each $t\in[1,e]$,

$$h_*(t)=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{v_*(s)}{s}ds$$
$$+(\ln t)^{\alpha-1}\left[\frac{m(x)}{f(e,m(x))}-\frac{1}{\Gamma(\alpha)}\int_1^e\left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{v_*(s)}{s}ds\right].$$

Let us consider the linear operator $\Theta: L^1([1,e],\mathbb{R})\to C([1,e],\mathbb{R})$ given by

$$f\mapsto\Theta(v)(t)=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds$$
$$+(\ln t)^{\alpha-1}\left[\frac{m(x)}{f(e,m(x))}-\frac{1}{\Gamma(\alpha)}\int_1^e\left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds\right].$$

Observe that

$$\|h_n(t)-h_*(t)\|=\left\|\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{(v_n(s)-v_*(s))}{s}ds\right.$$
$$\left.-(\ln t)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\int_1^e\left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{(v_n(s)-v_*(s))}{s}ds\right\|\to 0,\text{as } n\to\infty.$$

Thus, it follows by Lemma 2.7 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$\begin{aligned} h_*(t) = & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{v_*(s)}{s} ds \\ & + (\ln t)^{\alpha-1} \left[\frac{m(x)}{f(e,m(x))} - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln \frac{e}{s}\right)^{\alpha-1} \frac{v_*(s)}{s} ds \right], \end{aligned}$$

for some $v_* \in S_{F,x_*}$.

As a result we have that the operator $\mathscr{B}$ is compact and upper semicontinuous operator on X.

Step 3. *Now we show that* $2Mk < 1$, *i.e. (c) of Lemma* 2.6 *holds.*

This is obvious by (A_3) since we have $k = \|\phi\|$ and $M = \|B(X)\| = \sup\{|\mathscr{B}x : x \in X\} \leqslant \dfrac{2}{\Gamma(\alpha+1)} \|\zeta\| + M_1$.

Thus all the conditions of Lemma 2.6 are satisfied and a direct application of it yields that either the conclusion (i) or the conclusion (ii) holds. We show that the conclusion (ii) is not possible.

Let $u \in \mathscr{E}$ be arbitrary. Then we have for $\lambda > 1$, $\lambda u \in \mathscr{A}u(t)\mathscr{B}u(t)$. Then there exists $v \in S_{F,x}$ such that for any $\lambda > 1$, one has

$$\begin{aligned} u(t) = \lambda^{-1}[f(t,u(t)] \Bigg(& \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \\ & + (\ln t)^{\alpha-1} \left[\frac{m(u)}{f(e,m(u))} - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln \frac{e}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \right], \end{aligned}$$

for all $t \in [1,e]$. Then we have

$$\begin{aligned} |u(t)| \leqslant & \lambda^{-1} |f(t,u(t)| \Bigg(\frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{|v(s)|}{s} ds \\ & + (\ln t)^{\alpha-1} \left[\left| \frac{m(u)}{f(e,m(u))} \right| + \frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln \frac{e}{s}\right)^{\alpha-1} \frac{|v(s)|}{s} ds \right] \Bigg) \\ \leqslant & [|f(t,u(t) - f(t,0)| + |f(t,0)|] \Bigg(\frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{|v(s)|}{s} ds \\ & + (\ln t)^{\alpha-1} \left[\left| \frac{m(u)}{f(e,m(u))} \right| + \frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln \frac{e}{s}\right)^{\alpha-1} \frac{|v(s)|}{s} ds \right] \Bigg) \\ \leqslant & [\|\phi\| \|u\| + F_0] \left[\frac{2}{\Gamma(\alpha+1)} \|\zeta\| + M_1 \right]. \end{aligned}$$

Then we have

$$\|u\| \leqslant \frac{F_0\left[\frac{2}{\Gamma(\alpha+1)}\|\zeta\|+M_1\right]}{1-\|\phi\|\left[\frac{2}{\Gamma(\alpha+1)}\|\zeta\|+M_1\right]} \leqslant R.$$

Thus the condition (ii) of Theorem 2.6 does not hold by (13). Therefore the operator equation $\mathscr{A}x\mathscr{B}x$ and consequently problem (3) has a solution on $[1,e]$. This completes the proof. □

EXAMPLE 4.3. Consider the boundary value problem

$$\begin{cases} D^{3/2}\left[\dfrac{x(t)}{\frac{1}{12}e^{1-t}\tan^{-1}x+2}\right] \in F(t,x(t)), \quad 1<t<e, \\ x(1)=0, \quad x(e)=\dfrac{1}{16}\sin x(\eta), \quad 0<\eta<1, \end{cases} \tag{16}$$

where $F:[1,e]\times\mathbb{R}\to\mathscr{P}(\mathbb{R})$ is a multivalued map given by

$$t \to F(t,x) = \left[\frac{|x|^3}{10(|x|^3+3)}, \frac{|\sin x|}{3(|\sin x|+1)}+\frac{1}{3}\right].$$

By the condition (H_1), $\phi(t)=e^{1-t}/12$ with $\|\phi\|=1/12$. For $\tilde{f}\in F$, we have

$$|\tilde{f}| \leqslant \max\left(\frac{|x|^3}{10(|x|^3+3)}, \frac{|\sin x|}{3(|\sin x|+1)}+\frac{1}{3}\right) \leqslant \frac{2}{3}, \quad x\in\mathbb{R}$$

and

$$\|F(t,x)\| = \sup\{|y| : y\in F(t,x)\} \leqslant \frac{2}{3} = \zeta(t), \quad x\in\mathbb{R}.$$

Clearly

$$\|\phi\|\left[\frac{2}{\Gamma(\alpha+1)}\|\zeta\|+M_1\right] = \frac{1}{12}\left[\frac{16}{9\sqrt{\pi}}+\frac{1}{16}\right] \simeq 0.088131 < 1/2$$

and $R>0.3898789$. Hence all the conditions of Theorem 4.2 are satisfied and accordingly, the problem (16) has a solution on $[1,e]$.

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