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EXISTENCE AND MULTIPLICITY RESULTS FOR THE BOUNDARY VALUE PROBLEM OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we devote to investigation of the existence of positive solutions for the boundary value problem of nonlinear fractional differential equations

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t))=0, \ \ 0<t<1, \\ u(0)=u'(0)=\cdots u^{(n-2)}(0)=D_{0+}^{\beta}u(1), \end{cases}$$

where D_{0+}^{α}, D_{0+}^{β} are the standard Riemann-Liouville fractional derivative, $n-1<\alpha\leqslant n$, $n-2\leqslant\beta\leqslant n-1$, $n\geqslant 3$. By means of constructing an exact cone of the Banach space and fixed-point theorem, some new multiplicity results for the boundary value problem are obtained. The interest is that we establish the theorems of the existence of infinitely many positive solutions.

1. Introduction

In this paper, we are concerned with positive solutions for the boundary value problem of nonlinear fractional differential equations

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t))=0, \ \ 0<t<1, \\ u(0)=u'(0)=\cdots u^{(n-2)}(0)=D_{0+}^{\beta}u(1), \end{cases} \tag{1.1}$$

where D_{0+}^{α}, D_{0+}^{β} are the standard Riemann-Liouville fractional derivative, $n-1<\alpha\leqslant n$, $n-2\leqslant\beta\leqslant n-1$, $n\geqslant 3$, and $f:[0,1]\times[0,\infty)\to[0,\infty)$ is continuous.

Fractional differential equations are mathematical apparatus for simulation of process and phenomena observed in the fields of control theory, physics, chemistry, biotechnologies, industrial robotics, engineering, etc. So there have been quite a few results on properties of their solutions [1–3, 7–8, 10–11, 13]. Recently, there are some papers investigating the existence and multiplicity of solutions for boundary value problem of fractional differential equations [4–6, 9, 12, 14, 16–18]. For example, in [5], Bai and Lu investigated the boundary value problem of the following fractional dierential equations

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t))=0, \ \ 0<t<1, \ 1<\alpha\leqslant 2, \\ u(0)=u(1)=0, \end{cases}$$

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where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. Jiang and Yuan generalized the results of the above equation [16].

In [14], Kaufmann and Mboumi studied the existence of positive solutions of nonlinear fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t)+a(t)f(u(t))=0, \ \ 0<t<1, \ 1<\alpha\leqslant 2, \\ u(0)=u'(1)=0, \end{cases}$$

In [18], Liang and Zhang used lower and upper solution method and fixed point theorem to show the existence and non-existence of positive solutions of nonlinear fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t)+a(t)f(u(t))=0, \ \ 0<t<1, \ 3<\alpha\leqslant 4, \\ u(0)=u'(0)=u''(0)=u'(1)=0. \end{cases}$$

In the present paper, we derive the corresponding Green function. Consequently BVP (1.1) is reduced to an equivalent Fredholm integral equation. Next, applying the properties of Green function, we construct the exact cone of the Banach space. Finally, by using fixed-point theorems, existence and multiplicity results for the BVP (1.1) are obtained. The interest is that we establish the theorems of the existence of infinitely many positive solutions. Meanwhile, some examples are given to illustrate the effect of these theorems.

2. Preliminary results

DEFINITION 2.1. ([7]) The fractional integral of order $\alpha>0$ of a function $y:(0,\infty)\to R$ is given by

$$I_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds$$

provided the right integral converges.

DEFINITION 2.2. ([7]) The standard Riemann-Liouville fractional derivative of order $\alpha>0$ of a continuous function $y:(0,\infty)\to R$ is given by

$$D_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t (t-s)^{n-\alpha-1}y(s)ds$$

where $n=[\alpha]+1$, provided the right integral converges.

LEMMA 2.1. ([7]) *Assume that $u\in C(0,1)\cap L(0,1)$ with a fractional derivative of order $\alpha>0$ that belongs to $C(0,1)\cap L(0,1)$. Then*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t)=u(t)+C_1t^{\alpha-1}+C_2t^{\alpha-2}+\cdots+C_Nt^{\alpha-N}$$

for some $C_i\in R$, $i=1,2,\ldots,N$, where N is the smallest integer greater than or equal to α.

LEMMA 2.2. *Assume that $g(t) \in L[0,1]$ and α, β are two constants such that $\alpha > n-1 \geqslant \beta \geqslant n-2$, $n \geqslant 3$. Then*

$$D_{0^+}^{\beta} \int_0^t (t-s)^{\alpha-1} g(s)ds = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} g(s)ds$$

The proof of Lemma 2.2 is similar to that of Lemma 2,2 in [18], here we omit it.

LEMMA 2.3. *Let $g(t) \in L[0,1]$ and $n-1 < \alpha \leqslant n$, $n-2 \leqslant \beta \leqslant n-1$, $n \geqslant 3$, the unique solution of*

$$\begin{cases} D_{0^+}^{\alpha} u(t) + g(t) = 0, \ 0 < t < 1, \\ u(0) = u'(0) = \cdots u^{(n-2)}(0) = D_{0^+}^{\beta} u(1) = 0, \end{cases} \tag{2.1}$$

is

$$u(t) = \int_0^t G(t,s)g(s)ds,$$

where

$$G(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, \ 0 \leqslant s \leqslant t \leqslant 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, \ 0 \leqslant t \leqslant s \leqslant 1 \end{cases} \tag{2.2}$$

Proof. Applying Lemma 2.1, the Eq. (2.1) is equivalent to the integral equation

$$u(t) = -I_{0^+}^{\alpha} u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}$$

for some $C_i \in R$, $i = 1, 2, \ldots, n$. Consequently, the general solution of Eq. (2.1) is

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}.$$

From $u(0) = u'(0) = \cdots u^{(n-2)}(0) = D_{0^+}^{\beta} u(1)$, we get $C_2 = \cdots = C_n$. Thus

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds + C_1 t^{\alpha-1}.$$

From Lemma 2.2, we get

$$D_{0^+}^{\beta} u(t) = -\frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} g(s)ds + C_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1}.$$

In view of the boundary condition $D_{0^+}^{\beta} u(1) = 0$, we conclude that

$$C_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} g(s)ds.$$

Therefore, the unique solution of BVP (2.1) is

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-\beta-1} g(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} (t-s)^{\alpha-\beta-1} g(s)ds \\ &= \int_0^1 G(t,s)g(s)ds. \quad \square \end{aligned}$$

LEMMA 2.4. *The function $G(t,s)$ satisfies the following conditions*
(i) $G(t,s)$ is continuous on $[0,1]\times[0,1]$;
(ii) $G(t,s)>0$ for any $s,t\in[0,1]$;
(iii) For any $s,t\in[0,1]$, $\frac{t^{\alpha-1}w(s)}{\Gamma(\alpha)}\leqslant G(t,s)\leqslant\frac{w(s)}{\Gamma(\alpha)}$, where $w(s)=(1-s)^{\alpha-\beta-1}-(1-s)^{\alpha-1}$.

Proof. It is easy to check that (i) holds. So we will prove that (ii) and (iii) hold. If $0\leqslant s\leqslant t\leqslant 1$, let $h(t,s)=(1-s)^{\alpha-\beta-1}-(1-\frac{s}{t})^{\alpha-1}$, then $G(t,s)=\frac{t^{\alpha-1}h(t,s)}{\Gamma(\alpha)}$. Since $\frac{\partial h(t,s)}{\partial t}=-(\alpha-1)(1-\frac{s}{t})^{\alpha-2}\frac{s}{t^2}\leqslant 0$, so $h(t,s)$ is decreasing on $[s,1]$ with respect to t. Then $h(t,s)\geqslant h(1,s)=(1-s)^{\alpha-\beta-1}-(1-s)^{\alpha-1}>0$. Which implies (ii) holds. Furthermore, we have for $0\leqslant s\leqslant t\leqslant 1$

$$G(t,s)=\frac{t^{\alpha-1}h(t,s)}{\Gamma(\alpha)}\geqslant\frac{t^{\alpha-1}h(1,s)}{\Gamma(\alpha)}=\frac{t^{\alpha-1}[(1-s)^{\alpha-\beta-1}-(1-s)^{\alpha-1}]}{\Gamma(\alpha)}=\frac{t^{\alpha-1}w(s)}{\Gamma(\alpha)}.$$

If $0\leqslant t\leqslant s\leqslant 1$. It is easy to see that

$$G(t,s)\geqslant\frac{t^{\alpha-1}[(1-s)^{\alpha-\beta-1}-(1-s)^{\alpha-1}]}{\Gamma(\alpha)}.$$

On the other hand, If $0\leqslant s\leqslant t\leqslant 1$, we get

$$\frac{dG(t,s)}{dt}=\frac{(\alpha-1)[t^{\alpha-2}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-2}]}{\Gamma(\alpha)}=\frac{(\alpha-1)[\frac{(t-ts)^{\alpha-2}}{(1-s)^{\beta-1}}-(t-s)^{\alpha-2}]}{\Gamma(\alpha)}$$
$$\geqslant\frac{(\alpha-1)[\frac{(t-s)^{\alpha-2}}{(1-s)^{\beta-1}}-(t-s)^{\alpha-2}]}{\Gamma(\alpha)}\geqslant 0.$$

Then $G(t,s)$ is increasing with respect to t for $0\leqslant s\leqslant t\leqslant 1$. So

$$G(t,s)\leqslant G(1,s)\leqslant\frac{(1-s)^{\alpha-\beta-1}-(1-s)^{\alpha-1}}{\Gamma(\alpha)}=\frac{w(s)}{\Gamma(\alpha)}.$$

If $0\leqslant t\leqslant s\leqslant 1$, we also get

$$G(t,s)\leqslant\frac{s(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}=\frac{[1-(1-s)](1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\leqslant\frac{[1-(1-s)^{\beta-1}](1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}$$
$$=\frac{(1-s)^{\alpha-\beta-1}-(1-s)^{\alpha-1}}{\Gamma(\alpha)}.$$

Thus, we conclude that (iii) holds. □

LEMMA 2.5. ([15]) *Let Q be a cone of a Banach space E, and Ω, Ω' are open subsets of E with $0\in\Omega\subset\overline{\Omega}\subset\Omega'$. Suppose that $A:Q\to Q$ is a completely continuous operator such that one of the following two conditions is satisfied:*
(i) $\|Ax\|\leqslant\|x\|$ for $x\in Q\cap\partial\Omega$ and $\|Ax\|\geqslant\|x\|$ for $x\in Q\cap\partial\Omega'$.
(ii) $\|Ax\|\geqslant\|x\|$ for $x\in Q\cap\partial\Omega$ and $\|Ax\|\leqslant\|x\|$ for $x\in Q\cap\partial\Omega'$.
Then, A has a fixed point $x\in Q\cap\overline{\Omega'}\setminus\Omega$.

2. Main results

In this section, we establish the theorems of positive solutions for BVP (1.1). For convenience, let us list the following assumptions.

(H_1) $0 \leqslant \lim_{u\to 0^+} \sup_{t\in[0,1]} \frac{f(t,u)}{u} \leqslant M_1$, where $M_1 = [\max_{0\leqslant t\leqslant 1} \int_0^1 G(t,s)ds]^{-1}$;

(H_2) There exists $\gamma \in [0, \frac{2\alpha-2}{3\alpha-\beta-2}]$ such that $\lim_{u\to +\infty} \inf_{t\in[\gamma,1]} \frac{f(t,u)}{u} \leqslant \frac{\Gamma(\alpha)(\alpha-\beta)}{M(\alpha,\beta)}$, where $M(\alpha,\beta) = \frac{(2\alpha-2)^{2\alpha-2}(\alpha-\beta)^{\alpha-\beta}}{(3\alpha-\beta-2)^{3\alpha-\beta-2}}$;

(H_3) There exists $\gamma \in [0, \frac{2\alpha-2}{3\alpha-\beta-2}]$ such that $\lim_{u\to 0^+} \inf_{t\in[\gamma,1]} \frac{f(t,u)}{u} \leqslant \frac{\Gamma(\alpha)(\alpha-\beta)}{M(\alpha,\beta)}$;

(H_4) There exist a number $r > 0$ and a function $h(t) \in C(0,1)$ such that for $t \in [0,1]$, $0 < u \leqslant r$, $f(t,u) \leqslant h(t)$, and $\max_{0\leqslant t\leqslant 1} \int_0^1 G(t,s)h(s)ds \leqslant r$;

(H_5) $0 \leqslant \lim_{u\to +\infty} \sup_{t\in[0,1]} \frac{f(t,u)}{u} \leqslant M_1$;

(H_6) There exists $\gamma \in [0, \frac{2\alpha-2}{3\alpha-\beta-2}]$ such that for $t \in [\gamma,1]$, $\gamma^{\alpha-1}\nu \leqslant u \leqslant \nu$, $f(t,u) \geqslant \frac{\Gamma(\alpha)(\alpha-\beta)\nu}{M(\alpha,\beta)}$.

Let the Banach space $E = C[0,1]$ be endowed norm $u = \sup_{0\leqslant t\leqslant 1} |u(t)|$. For $\forall c > 0$, we define $\Omega_c = \{u \in E : \|u\| \leqslant c\}$. Let P_γ be the cone $P_\gamma = u \in E : u(t) \geqslant 0,\ t \in [0,1]$, $\min_{\gamma\leqslant t\leqslant 1} u(t) \geqslant \gamma^{\alpha-1}\|u\|\}$, where $\gamma \in [0, \frac{2\alpha-2}{3\alpha-\beta-2}]$. Suppose that u is a solution of BVP (1.1), then

$$u(t) = \int_0^t G(t,s)f(s,u(s))ds, \quad 0 \leqslant t \leqslant 1.$$

Dene an operator $A : P_\gamma \to E$ as follows

$$(Au)(t) = \int_0^t G(t,s)f(s,u(s))ds.$$

Clearly, the fixed points of the operator A are the solutions of the BVP (1.1).

LEMMA 3.1. *The operator A is defined by (3.1). Then $A(P_\gamma) \subset P_\gamma$ and A is completely continuous.*

Proof. It follows from Lemma 2.4 that

$$\|Au\| \leqslant \frac{1}{\Gamma(\alpha)} \int_0^1 w(s)f(s,u(s))ds,$$
$$(Au)(t) \geqslant \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 w(s)f(s,u(s))ds,$$

Then

$$\min_{\gamma\leqslant t\leqslant 1} (Au)(t) \geqslant \gamma^{\alpha-1}\|Au\|.$$

So we have $A(p_\gamma) \subset P_\gamma$. Next, in view of nonnegativeness of $G(t,s), f(t,u)$ and continuity of $f(t,u)$ with respect to u, we can see that $A : P_\gamma \to P_\gamma$ is continuous. Let

$\Omega \in P_\gamma$. It is easy to show that $A(\Omega)$ is uniformly bounded and is equicontinuity. Thus, the operator A is completely continuous. $\square$

THEOREM 3.1. *Suppose that* (H_1), (H_2) *hold. Then BVP (1.1) has at least one positive solution.*

Proof. We first prove that there exist $\gamma_0 \in [0,1]$ and $R_{\gamma_0} > 0$ such that for $u \in P_{\gamma_0} \cap \partial\Omega_{R_{\gamma_0}}$

$$\|Au\| \geqslant \|u\|.$$

Since condition (H_2) holds, there exist $\varepsilon > 0$ and $N > 0$ such that for $t \in [\gamma, 1]$ and $u \geqslant N$

$$f(t,u) \geqslant \left(\frac{\Gamma(\alpha)(\alpha-\beta)}{M(\alpha,\beta)} + \varepsilon\right) u. \tag{3.2}$$

Choose $R_\gamma \geqslant \frac{N}{\gamma^{\alpha-1}}$. For $u \in P_\gamma \cap \partial\Omega_{R_\gamma}$, we obtain

$$\min_{\gamma \leqslant t \leqslant 1} (Au)(t) \geqslant \gamma^{\alpha-1}\|u\| = \gamma^{\alpha-1} R_\gamma \geqslant N. \tag{3.3}$$

Then, it follows from (3.2) and (3.3) that for $u \in P_\gamma \cap \partial\Omega_{R_\gamma}$

$$\begin{aligned}
(Au)(\gamma) &= \int_0^t G(\gamma,s) f(s,u(s))ds \geqslant \int_\gamma^t G(\gamma,s) f(s,u(s))ds \\
&\geqslant \left(\frac{\Gamma(\alpha)(\alpha-\beta)}{M(\alpha,\beta)} + \varepsilon\right) \frac{1}{\Gamma(\alpha)} \gamma^{\alpha-1}\|u\| \int_\gamma^t \gamma^{\alpha-1}(1-s)^{\alpha-\beta-1} ds \\
&\geqslant \frac{1}{M(\alpha,\beta)} \gamma^{2\alpha-2}(1-\gamma)^{\alpha-\beta}\|u\|.
\end{aligned}$$

Thus

$$\|Au\| \geqslant \frac{1}{M(\alpha,\beta)} \gamma^{2\alpha-2}(1-\gamma)^{\alpha-\beta}\|u\|, \ \ u \in P_\gamma \cap \partial\Omega_{R_\gamma}. \tag{3.4}$$

Let $\varphi(\gamma) = \gamma^{2\alpha-2}(1-\gamma)^{\alpha-\beta}$. It follows that when $\gamma = \gamma_0 = \frac{2\alpha-2}{3\alpha-\beta-2}$, $\varphi(\gamma_0) = \max_{0 \leqslant \gamma \leqslant 1} \varphi(\gamma)$ $= M(\alpha,\beta)$. Then taking $\gamma = \gamma_0$, we have

$$\|Au\| \geqslant \|u\|, \ \ u \in P_{\gamma_0} \cap \partial\Omega_{R_{\gamma_0}}.$$

In view of (H_1), we know that there exist $\varepsilon \in (0, M_1)$ and $0 < r_1 < R_{\gamma_0}$ such that for $t \in [0,1]$, $0 \leqslant u \leqslant r_1$

$$f(t,u) \leqslant (M_1 - \varepsilon)u,$$

Then for $u \in P_{\gamma_0} \cap \partial\Omega_{r_1}$

$$(Au)(t) = \int_0^t G(\gamma,s) f(s,u(s))ds \leqslant (M_1-\varepsilon) r_1 \int_0^t G(t,s)ds \leqslant (M_1-\varepsilon)\frac{r_1}{M_1} \leqslant r_1 = \|u\|.$$

So, we have

$$\|Au\| \leqslant \|u\|, \ \ u \in P_{\gamma_0} \cap \partial\Omega_{r_1}.$$

Consequently, by Lemma 2.5, BVP (1.1) has at least one positive solution in $P_{\gamma_0} \cap (\overline{\Omega}_{\gamma_0} \setminus \Omega_{r_1})$. $\square$

THEOREM 3.2. *Suppose that* (H_3), (H_4) *hold. Then BVP (1.1) has at least one positive solution.*

Proof. Similar to the proof of Theorem 3.1, we now show that there exist $\gamma_0 \in [0,1]$ and $< r_{\gamma_0} < r$ such that for $u \in P_{\gamma_0} \cap \partial\Omega_{r_{\gamma_0}}$

$$\|Au\| \geqslant \|u\|.$$

Indeed, since condition (H_3) holds, there exist $\varepsilon > 0$ and $L > 0$ such that

$$f(t,u) \geqslant \left(\frac{\Gamma(\alpha)(\alpha-\beta)}{M(\alpha,\beta)} + \varepsilon\right)u, \ \ 0 < u \leqslant L.$$

Choosing $r_\gamma = \min\{\frac{r}{2}, L\}$, for $u \in P_\gamma \cap \partial\Omega_{r_\gamma}$, we have $\min\limits_{\gamma\leqslant t\leqslant 1} u(t) \geqslant \gamma^{\alpha-1} r_\gamma$, $u(t) \leqslant r_\gamma \leqslant L$. It follows that

$$\begin{aligned}(Au)(\gamma) &= \int_0^1 G(\gamma,s)f(s,u(s))ds \geqslant \int_\gamma^1 G(\gamma,s)f(s,u(s))ds \\ &\geqslant \left(\frac{\Gamma(\alpha)(\alpha-\beta)}{M(\alpha,\beta)} + \varepsilon\right)\frac{r_\gamma \gamma^{2\alpha-2}}{\Gamma(\alpha)}\int_\gamma^t (1-s)^{\alpha-\beta-1}ds \\ &\geqslant \frac{1}{M(\alpha,\beta)}\gamma^{2\alpha-2}(1-\gamma)^{\alpha-\beta} r_\gamma.\end{aligned}$$

Thus

$$\|Au\| \geqslant \frac{1}{M(\alpha,\beta)}\gamma^{2\alpha-2}(1-\gamma)^{\alpha-\beta}\|u\|, \ \ u \in P_\gamma \cap \partial\Omega_{r_\gamma}.$$

Taking $\gamma = \gamma_0 = \frac{2\alpha-2}{3\alpha-\beta-2}$, we have

$$\|Au\| \geqslant \|u\|, \ \ u \in P_{\gamma_0} \cap \partial\Omega_{r_{\gamma_0}}.$$

For $u \in P_{\gamma_0} \cap \partial\Omega_r$, we have $0 \leqslant u \leqslant r$. It follows from condition (H_4) that

$$(Au)(t) = \int_0^1 G(\gamma,s)f(s,u(s))ds \leqslant \min_{0\leqslant t\leqslant 1}\int_0^1 G(t,s)h(s)ds \leqslant r = \|u\|.$$

That is for $u \in P_{\gamma_0} \cap \partial\Omega_r$

$$\|Au\| \leqslant \|u\|.$$

So, by Lemma 2.5, BVP (1.1) has at least one positive solution in $P_{\gamma_0} \cap (\overline{\Omega}_{r_{\gamma_0}} \setminus \Omega_r)$. □

THEOREM 3.3. *Suppose that* (H_5), (H_6) *hold. Then BVP (1.1) has at least one positive solution.*

Proof. Suppose that (H_6) holds, similar to the proof of the Theorem 3.1, taking $\gamma = \gamma_0 = \frac{2\alpha-2}{3\alpha-\beta-2}$, we have for $u \in P_{\gamma_0} \cap \partial\Omega_\nu$

$$\|Au\| \geqslant \|u\|.$$

Since condition (H_5) holds, there exist $\varepsilon \in (0, M_1)$ and $N_1 > 0$ such that for $u \geqslant N_1$ and $t \in [0,1]$

$$f(t,u) \leqslant (M_1 - \varepsilon)u. \tag{3.5}$$

If $\max\limits_{0 \leqslant t \leqslant 1} f(t,u)$ is bounded for $u \in [0,+\infty)$, that is to say that for all $u \in [0,+\infty)$ and $t \in [0,1]$

$$f(t,u) \leqslant L_1.$$

Let $\nu_1 = \max\{N_1, \nu, \frac{L_1}{M_1}\}$. For $u \in P_{\gamma_0} \cap \partial\Omega_{\nu_1}$, from the above inequality, we have

$$(Au)(t) = \int_0^t G(t,s)f(s,u(s))ds \leqslant L_1 \int_0^t G(\gamma,s)ds \leqslant \frac{L_1}{M_1} \leqslant \nu_1 = \|u\|.$$

If $\max\limits_{0 \leqslant t \leqslant 1} f(t,u)$ is unbounded for $u \in [0,+\infty)$, then there exists $\nu_2 > \max\{N_1, \nu\}$ such that for $u \in [0, \nu_2]$ and $t \in [0,1]$

$$f(t,u) \leqslant \max_{0 \leqslant t \leqslant 1} f(t, \nu_2). \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\begin{aligned}(Au)(t) &= \int_0^1 G(\gamma,s)f(s,u(s))ds \leqslant \int_0^1 G(t,s) \max_{0 \leqslant t \leqslant 1} f(t,\nu_2)ds \\ &\leqslant (M_1 - \varepsilon)\nu_2 \int_0^1 G(t,s)ds \leqslant (M_1 - \varepsilon)\frac{\nu_2}{M_1} \leqslant \nu_2 = \|u\|.\end{aligned}$$

Thus, we can see that there exists $\overline{\nu} \geqslant \nu$ such that for $u \in P_{\gamma_0} \cap \partial\Omega_{\overline{\nu}}$

$$\|Au\| \leqslant |u\|.$$

Applying Lemma 2.5, BVP (1.1) has at least one positive solution in $P_{\gamma_0} \cap (\overline{\Omega_{\overline{\nu}}} \setminus \Omega_\nu)$. $\square$

THEOREM 3.4. *Suppose that (H_1) and (H_3) hold. Then BVP (1.1) has at least one positive solutions.*

THEOREM 3.5. *Suppose that (H_2) and (H_4) hold. Then BVP (1.1) has at least one positive solutions.*

THEOREM 3.6. *Suppose that (H_1) and (H_6) hold. Then BVP (1.1) has at least one positive solutions.*

THEOREM 3.7. *Suppose that (H_3) and (H_5) hold. Then BVP (1.1) has at least one positive solutions.*

THEOREM 3.8. *Suppose that (H_4) and (H_6) hold. Then BVP (1.1) has at least one positive solutions.*

Furthermore, from the Theorems 3.1-3.8, we have the multiplicity results for BVP (1.1) as follows:

THEOREM 3.9. *Suppose that* (H_2), (H_3) *and* (H_4) *hold. Then BVP (1.1) has at least two positive solutions.*

Proof. Suppose that (H_3) and (H_4) hold, by Theorem 3.2, we see that BVP (1.1) has at least one positive solution in $P_{\gamma_0} \cap (\overline{\Omega}_{r_{\gamma_0}} \setminus \Omega_r)$. Since (H_2) and (H_4) hold, according to the proof of Theorem 3.1 and theorem 3.2, we conclude that BVP (1.1) has at least one positive solution in $P_{\gamma_0} \cap (\overline{\Omega}_{R_{\gamma_0}} \setminus \Omega_{r_{\gamma_0}})$. Hence, BVP (1.1) has at least two positive solutions in $P_{\gamma_0} \cap (\overline{\Omega}_{R_{\gamma_0}} \setminus \Omega_{r_{\gamma_0}}) \cap (\overline{\Omega}_{r_{\gamma_0}} \setminus \Omega_r)$. $\square$

THEOREM 3.10. *Suppose that* (H_3), (H_4) *and* (H_5) *hold. Then BVP (1.1) has at least two positive solutions.*

THEOREM 3.11. *Suppose that* (H_4), (H_5) *and* (H_6) *hold, and* $\gamma^{\alpha-1}R > r$. *Then BVP (1.1) has at least two positive solutions.*

THEOREM 3.12. *Suppose that* (H_2), (H_3), (H_4) *and* (H_6) *hold, and* $\gamma^{\alpha-1}R > r$. *Then BVP (1.1) has at least three positive solutions.*

REMARK 3.1. If we replace the condition (H_2), (H_3) with $(H_2'$ $\lim\limits_{u\to+\infty} \inf\limits_{t\in[\gamma,1]} \frac{f(t,u)}{u} = \infty$, $(H_3'$ $\lim\limits_{u\to 0^+} \inf\limits_{t\in[\gamma,1]} \frac{f(t,u)}{u} = \infty$, then the conclusions of the above theorems 3.2–3.5, 3,7, 3.9–3.10, 3.12 are valid.

REMARK 3.2. We often take $\gamma = \gamma_0 = \frac{2\alpha-2}{3\alpha-\beta-2}$ to show the effect of the above theorems in practice.

Finally, we obtain the following two theorems of infinitely many positive solutions for BVP (1.1).

THEOREM 3.13. *Suppose that there exist sequences* $r_n, R_n > 0$ *such that*

$$0 < r_1 < R_1 < r_2 < R_2 < \cdots < r_n < R_n < \cdots,$$

and $\frac{1}{\gamma_0^{\alpha-1}} r_n < R_n < r_{n+1}\gamma_0^{\alpha_1}$, $n = 1, 2, \ldots$ *satisfy*

(H_7) $f(t,u) > \frac{\Gamma(\alpha)(\alpha-\beta)}{M(\alpha,\beta)} R_n$ *for* $t \in [\gamma_0, 1]$ *and* $\gamma_0^{\alpha_1} R_n \leqslant u \leqslant R_n$.

(H_8) *there exist* $h_n, \overline{h}_n \in C[0,1]$ *such that*

$$f(t,u) \leqslant h_n(t), \quad for \quad t \in [0,\gamma_0], \ \ 0 \leqslant u \leqslant r_n,$$
$$f(t,u) \leqslant \overline{h}_n(t), \quad for \quad t \in [\gamma_0,1], \ \ \gamma_0 r_n \leqslant u \leqslant r_n,$$

and

$$\max_{0\leqslant t\leqslant \gamma_0} \int_0^1 G(t,s)h_n(s)ds + \max_{\gamma_0\leqslant t\leqslant 1} \int_0^1 G(t,s)\overline{h}_n(s)ds \leqslant r_n,$$

where $\gamma_0 = \frac{2\alpha-2}{3\alpha-\beta-2}$. *Then BVP (1.1) has innitely many positive solutions.*

Proof. Suppose that (H_7) holds, from the proof of the above theorems, we have for $u \in P_{\gamma_0} \cap \partial\Omega_{R_n}$

$$\|Au\| \geqslant |u\|.$$

Since (H_8) holds, for $u \in P_{\gamma_0} \cap \partial\Omega_{r_n}$, we have $\min_{\gamma_0 \leqslant t \leqslant 1} u(t) \geqslant \gamma_0^{\alpha-1} r_n, u(t) \leqslant r_n$. Then for $u \in P_{\gamma_0} \cap \partial\Omega_{r_n}$

$$
\begin{aligned}
(Au)(t) &= \int_0^1 G(t,s)f(s,u(s))ds = \int_0^{\gamma_0} G(t,s)f(s,u(s))ds + \int_{\gamma_0}^1 G(t,s)f(s,u(s))ds \\
&\leqslant \int_0^{\gamma_0} G(t,s)h_n(s)ds + \int_{\gamma_0}^1 G(t,s)\overline{h}_n(s)ds \\
&\leqslant \max_{0 \leqslant t \leqslant \gamma_0} \int_0^{\gamma_0} G(t,s)h_n(s)ds + \max_{\gamma_0 \leqslant t \leqslant 1} \int_{\gamma_0}^1 G(t,s)\overline{h}_n(s)ds \leqslant r_n = \|u\|.
\end{aligned}
$$

Hence, by Lemma 2.5, we see that BVP (1.1) has infinitely many positive solutions. □

THEOREM 3.14. *Suppose that there exist sequences $r_n, R_n > 0$ such that*

$$0 < r_1 < R_1 < r_2 < R_2 < \cdots < r_n < R_n < \cdots,$$

and $\frac{1}{\gamma_0^{\alpha-1}} r_n < R_n < r_{n+1}\gamma_0^{\alpha_1}$, $n = 1, 2, \dots$ satisfy

(H_9) *$f(t,u) > \frac{\Gamma(\alpha)(\alpha-\beta)}{M(\alpha,\beta)} r_n$ for $t \in [\gamma_0, 1]$ and $\gamma_0^{\alpha_1} r_n \leqslant u \leqslant r_n$.*

(H_{10}) *there exist $h_n', \overline{h}_n' \in C[0,1]$ such that*

$$
\begin{aligned}
f(t,u) &\leqslant h_n'(t), \quad for \quad t \in [0, \gamma_0], \;\; 0 \leqslant u \leqslant R_n, \\
f(t,u) &\leqslant \overline{h}_n'(t), \quad for \quad t \in [\gamma_0, 1], \;\; \gamma_0 R_n \leqslant u \leqslant R_n,
\end{aligned}
$$

and

$$\max_{0 \leqslant t \leqslant \gamma_0} \int_0^1 G(t,s)h_n'(s)ds + \max_{\gamma_0 \leqslant t \leqslant 1} \int_0^1 G(t,s)\overline{h}_n'(s)ds \leqslant R_n,$$

where $\gamma_0 = \frac{2\alpha-2}{3\alpha-\beta-2}$. Then BVP (1.1) has infinitely many positive solutions.

EXAMPLE 3.1. Consider the problem

$$
\begin{cases}
D_{0+}^{\frac{5}{2}} u(t) + u^2(\frac{1}{2}\sin t + 1) = 0, \;\; 0 < t < 1, \\
u(0) = u'(0) = D_{0+}^{\frac{3}{2}} u(1) = 0.
\end{cases}
\tag{3.7}
$$

where $\alpha = \frac{5}{2}$, $\beta = \frac{3}{2}$, $f(t,u) = u^2(\frac{1}{2}\sin t + 1)$, $\limsup_{u \to 0^+} \frac{f(t,u)}{u} = 0$, $\liminf_{u \to +\infty} \frac{f(t,u)}{u} = \infty$ for $t \in [0,1]$. We see that condition (H_1) and condition (H_2) hold. Applying Theorem 3.1, we conclude that BVP (3.7) has at least one solution.

EXAMPLE 3.2. Consider the problem

$$
\begin{cases}
D_{0+}^{\frac{7}{2}} u(t) + f(t,u) = 0, \;\; 0 < t < 1, \\
u(0) = u'(0) = u''(0) = D_{0+}^{\frac{5}{2}} u(1) = 0.
\end{cases}
\tag{3.8}
$$

where $\alpha = \frac{7}{2}$, $\beta = \frac{5}{2}$,

$$f(t,u) = \begin{cases} \frac{t}{5} + 4u^2, & (t,u) \in [0,1] \times [0,1], \\ \frac{t}{5} + u^2 + u + 2, & (t,u) \in [0,1] \times (1,+\infty). \end{cases}$$

Taking $\gamma_0 = \frac{5}{6}$, for $t \in [\frac{5}{6},1]$, it is easy to see that (H_2) and (H_3) hold. Choose $R = \frac{1}{5}$, $v = 60$, we can check that $\gamma^{\alpha-1} v > R$ and (H_4), (H_6) hold. By Theorem 3.12, BVP (3.8) has at least three positive solutions.

EXAMPLE 3.3. Consider the problem

$$\begin{cases} D_{0+}^{\frac{7}{2}} u(t) + a(t)g(u) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = D_{0+}^{\frac{5}{2}} u(1) = 0. \end{cases} \tag{3.9}$$

where $\alpha = \frac{7}{2}$, $\beta = \frac{5}{2}$,

$$a(t) = \begin{cases} \dfrac{t}{300}, & t \in \left[0, \dfrac{5}{6}\right], \\ \dfrac{719t}{60} - \frac{599}{60}, & t \in \left[\dfrac{5}{6}, 1\right], \end{cases}$$

$$g(u) = \begin{cases} \dfrac{u}{6}, & u \in \left[0, 8\gamma_0^{\frac{3}{2}}\right], \\ \dfrac{u}{6}, & u \in \left[8^n\gamma_0^{\frac{3}{2}}, 8^n\right], \\ \dfrac{1080\gamma_0^{\frac{3}{2}} - 1}{6(3\gamma_0^{\frac{3}{2}} - 1)}(u - 8^n) + \frac{1}{6} \times 8^n, & u \in \left[8^n, \ 3 \times 8^n\gamma_0^{\frac{3}{2}}\right], \\ 60u, & u \in \left[3 \times 8^n\gamma_0^{\frac{3}{2}}, 3 \times 8^n\right], \\ \dfrac{540 - 4\gamma_0^{\frac{3}{2}}}{3(8\gamma_0^{\frac{3}{2}} - 3)}(3 \times 8^n - u) + 180 \times 8^n, & u \in \left[3 \times 8^n, \times 8^{n+1}\gamma_0^{\frac{3}{2}}\right], \ n = 1,2,3,\ldots, \end{cases}$$

$\gamma_0 = \frac{5}{6}$, $f(t,u) = a(t)g(u)$. Set $r_n = 8^n$, $R_n = 3 \times 8^n$, $n = 1,2,\ldots$. Then

$$0 < r_1 < R_1 < r_2 < R_2 < \cdots < r_n < R_n < \cdots,$$

and $\frac{1}{\gamma_0^{\alpha-1}} r_n < R_n < r_{n+1}\gamma_0^{\alpha_1}$, $n = 1,2,\ldots$. It is obvious that (H_7) and (H_8) hold. Thus by Theorem 3.13, BVP (3.9) has infinitely many positive solutions.

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