Fractional Differential Calculus
Volume 5, Number 2 (2015), 183–198
doi:10.7153/fdc-05-16

EXISTENCE AND NONEXISTENCE OF POSITIVE SOLUTIONS FOR A SYSTEM OF NONLINEAR SINGULAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we establish sufficient conditions for the existence and nonexistence of positive solutions to the following nonlinear fractional differential system

$$\begin{cases} D^{\alpha}u(t)+a(t)f(t,u,v)=0 \text{ in } (0,1), \\ D^{\beta}v(t)+b(t)g(t,u,v)=0 \text{ in } (0,1), \\ u(0)=0,\ u(1)=0,\ u'(0)=0, \\ v(0)=0,\ v(1)=0,\ v'(0)=0, \end{cases} \tag{P}$$

where $2<\alpha,\beta\leqslant 3$, $a,b\in\mathscr{C}((0,1),[0,+\infty))$ and the functions f,g belong to $\mathscr{C}([0,1]\times[0,+\infty)\times[0,+\infty),[0,+\infty))$ and satisfy some appropriate conditions. Our analysis relies on Krasnoselskii fixed point theorem. Some examples are given to illustrate our results.

1. Introduction

Fractional differential equations and systems arise in variety engineering and scientific disciplines as the mathematical modeling in systems and processes in many fields. For example, in the field of physics such as a non Markovian diffusion process with memory [15], charge transport in amorphous semi-conductors [17], propagations of mechanical waves in viscoelastics media [13] and also in the fields of electromagnetic, acoustic, viscoelasticity, electrochemistry, economic, signal and image processing, control theory, etc. For details, see [4,7,8,9,12,14,17] and the references therein.

Recently, many authors have dealt with the existence of positive solutions of nonlinear fractional differential equations and systems by the use of the techniques of nonlinear analysis (fixed-point theorems, Leray-Shauder theory, Adomian decomposition method, etc), see [1,2,5,18–22] and the references therein. Namely, El-Shahed [5] discussed the existence and nonexistence of positive solutions for the following nonlinear fractional boundary value problem

$$\begin{cases} D^{\alpha}u(t)+\lambda a(t)f(u(t))=0 \text{ in } (0,1), \\ u(0)=u'(0)=u'(1)=0, \end{cases}$$

where D^{α} is the standard Riemann-Liouville fractional derivative of order α, $2<\alpha<3$ (see Definition 2.2), λ is a positive parameter, $a:\ (0,1)\to[0,+\infty)$ is continuous function with $\int_0^1 a(t)dt>0$ and $f\in\mathscr{C}([0,+\infty),[0,+\infty))$.

Mathematics subject classification (2010): 34A08, 34B18, 34B27, 47H10.
Keywords and phrases: Nonlinear fractional differential system, positive solution, Green's function, fixed point theorem.

Paper FDC-05-16

Motivated by the work of El-Shahed [5], Zhao et al. [22] established some results of existence and nonexistence of positive solutions for the following problem

$$\begin{cases} D^{\alpha}u(t) - f(t,u(t)) = 0 \text{ in } (0,1), \\ u(0) = u'(0) = u'(1) = 0, \end{cases}$$

where $2 < \alpha \leqslant 3$. The authors treated the problem for both cases of singular and nonsingular nonlinearity f. They used the lower and upper solution method and Leggett-Williams fixed point theorem to state their results.

Yu and Jiang [21] examined the existence of positive solutions for the following problem

$$\begin{cases} D^{\alpha}u(t) + f(t,u(t)) = 0 \text{ in } (0,1), \\ u(0) = u'(0) = u(1) = 0, \end{cases}$$

where $2 < \alpha \leqslant 3$, $f \in \mathscr{C}([0,1] \times [0,+\infty), (0,+\infty))$. The authors obtained some existence criteria for one or two positive solutions for the singular and nonsingular boundary value problem by the means of the Kranoselskii fixed point theorem and the mixed monotone method.

Recently, Feng et al. [6] studied the following nonlinear singular fractional problem with integral boundary conditions

$$\begin{cases} D^{\alpha}u(t) + a(t)f(t,u(t)) = 0 \text{ in } (0,1), \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u(1) = \int_0^1 h(t)u(t)dt, \end{cases} \tag{1.1}$$

where $n-1 < \alpha \leqslant n$, $n \geqslant 3$, $a : (0,1) \to [0,+\infty)$ is continuous, $h \in L^1([0,1])$ is nonnegative and $f \in \mathscr{C}([0,1] \times [0,+\infty), [0,+\infty))$. The authors developed some properties of the Green's function of the problem (1.1) and exploit them to establish some existence results.

In [19], Wang et al considered the existence and the uniqueness of a positive solution to nonzero boundary values problem for a coupled system of nonlinear fractional differential equations

$$\begin{cases} D^{\alpha}u(t) + f(t,v) = 0 \text{ in } (0,1), \\ D^{\beta}v(t) + g(t,u) = 0 \text{ in } (0,1), \\ u(0) = 0,\ u(1) = au(\zeta), \\ v(0) = 0,\ v(1) = bv(\zeta), \end{cases}$$

where $1 < \alpha, \beta < 2$, $0 \leqslant a, b \leqslant 1$, $\zeta \in (0,1)$ and $f, g \in \mathscr{C}([0,1] \times [0,+\infty), [0,+\infty))$.

Motivated by the above mentioned works, we investigate in this paper the existence of positive solutions for the following class of systems of nonlinear fractional differential equations

$$\begin{cases} D^{\alpha}u(t) + a(t)f(t,u,v) = 0 \text{ in } (0,1), \\ D^{\beta}v(t) + b(t)g(t,u,v) = 0 \text{ in } (0,1), \\ u(0) = 0,\ u(1) = 0,\ u'(0) = 0, \\ v(0) = 0,\ v(1) = 0,\ v'(0) = 0, \end{cases} \tag{P}$$

where $2 < \alpha, \beta \leqslant 3$, $a, b \in \mathscr{C}((0,1),[0,+\infty))$ and f, g are nonnegative continuous functions on $[0,1] \times [0,+\infty) \times [0,+\infty)$.

The paper is organized as follows. In Section 1, we recall some notions and notations, especially we give some properties of the Green's function which are needed later. Also, we state Krasnoselskii's fixed point theorem for a cone preserving operators and we prove a key lemma used in the proofs of our main results. Section 2 and Section 3 are devoted to establish some results for the existence of at least one or two positive solutions for (P), respectively. In the last Section, we give some sufficient conditions for nonexistence of positive solutions for (P).

2. Preliminaries

In this section, we recall some results and present some lemmas that will be used to prove our main results.

DEFINITION 2.1. (See $[10,16]$) The Riemann-Liouville fractional integral of order $\alpha > 0$ for a measurable function $f : (0,+\infty) \to \mathbb{R}$ is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \ \ t > 0,$$

where Γ is the Euler Gamma function, provided that the right-hand side is pointwise defined on $(0,+\infty)$.

DEFINITION 2.2. (See $[10,16]$) The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a measurable function $f : (0,+\infty) \to \mathbb{R}$ is defined as

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)\, ds = \left(\frac{d}{dt}\right)^n I^{n-\alpha} f(t),$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the real number α, provided that the right-hand side is pointwise defined on $(0,+\infty)$.

LEMMA 2.3. (See $[3]$) *Let $\alpha > 0$ and n be the smallest integer greater than or equal to α. Let $u \in \mathscr{C}(0,1) \cap L^1([0,1])$. Then*

(i) $D^{\alpha} I^{\alpha} u = u$.

(ii) For $\lambda > \alpha - 1$, $D^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha}$. *Moreover, we have* $D^{\alpha} t^{\alpha-m} = 0$, $m = 1, 2, \ldots, n$.

(iii) $D^{\alpha} u(t) = 0$ *if and only if* $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1, \ldots, n$.

(iv) Assume that $D^{\alpha} u \in \mathscr{C}(0,1) \cap L^1([0,1])$, *then we have*

$$I^{\alpha} D^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n},$$

$c_i \in \mathbb{R}$, $i = 1, \ldots, n$.

LEMMA 2.4. *Let $y \in \mathscr{C}([0,1])$ be a given function and $2 < \alpha \leqslant 3$, then the unique solution of the fractional differential equation*

$$D^{\alpha} u(t) + y(t) = 0 \ \ in \ (0,1), \tag{2.1}$$

subject to the boudary conditions

$$u(0) = 0,\ u(1) = 0,\ u'(0) = 0, \tag{2.2}$$

is given by

$$u(t) = \int_0^1 G_{\alpha}(t,s)\, y(s)\, ds, \tag{2.3}$$

where

$$G_{\alpha}(t,s) = \frac{(t(1-s))^{\alpha-1} - \left((t-s)^{+}\right)^{\alpha-1}}{\Gamma(\alpha)}, \quad for \ \ t,s \in [0,1]. \tag{2.4}$$

$G_{\alpha}(t,s)$ *is called the Green's function of the boundary value problem* (2.1)–(2.2). *Here, for* $x \in \mathbb{R},\ x^{+} = \max(x,0)$.

Proof. By Lemma 2.3, the fractional differential equation (2.1) is equivalent to

$$u(t) = -I^{\alpha} y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}, \tag{2.5}$$

where $c_1,\ c_2,\ c_3 \in \mathbb{R}$.

The boundary conditions (2.2) imply that $c_2 = c_3 = 0$ and $c_1 = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds$. Thus, substituting the values of $c_1,\ c_2,\ c_3$ into (2.5) gives

$$u(t) = \int_0^t \frac{t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + \int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds.$$

So the solution u satisfies (2.3). $\square$

The following properties of the Green's function play an important role in this paper.

PROPOSITION 2.5. ([6]) *The function G_{α} defined by* (2.4) *satisfies*

- (i) *G_{α} is nonnegative continuous function on $[0,1] \times [0,1]$ and $G_{\alpha}(t,s) > 0$, for all $t,s \in (0,1)$.*
- (ii) *For all $t \in [0,1],\ s \in (0,1)$, we have*

$$G_{\alpha}(t,s) \leqslant K_{\alpha}(s),$$

where

$$K_{\alpha}(s) := \frac{1}{\Gamma(\alpha)} \left((\tau(s)(1-s))^{\alpha-1} - (\tau(s) - s)^{\alpha-1} \right), \tag{2.6}$$

and

$$\tau(s) = \frac{s}{1-(1-s)^{\frac{\alpha-1}{\alpha-2}}}.$$

(*iii*) *Let* $\theta \in \left(0, \frac{1}{2}\right)$. *Then, there exists* $\gamma_\alpha > 0$ *such that*

$$\min_{t \in [\theta, 1-\theta]} G_\alpha(t,s) \geqslant \gamma_\alpha K_\alpha(s), \; \forall s \in [0,1].$$

REMARK 2.6. The constant γ_α satisfies the following properties

(*i*) $0 < \gamma_\alpha < 1$.

(*ii*) $\theta^{\alpha-1} \geqslant \gamma_\alpha$.

For details, see the proof of Proposition 2.9 in [6].

LEMMA 2.7. ([11]) *Let* P *be a cone of a real Banach space* E, *and* Ω_1, Ω_2 *two bounded open balls of* E *centered at the origin with* $\overline{\Omega_1} \subset \Omega_2$. *Suppose that* $T : P \cap \left(\overline{\Omega_2} \backslash \Omega_1\right) \longrightarrow P$ *is completely continuous operator such that either*

(*i*) $\|Tx\| \leqslant \|x\|$, $x \in P \cap \partial\Omega_1$ *and* $\|Tx\| \geqslant \|x\|$, $x \in P \cap \partial\Omega_2$, *or*

(*ii*) $\|Tx\| \geqslant \|x\|$, $x \in P \cap \partial\Omega_1$ *and* $\|Tx\| \leqslant \|x\|$, $x \in P \cap \partial\Omega_2$

holds. Then T *has a fixed point in* $P \cap \left(\overline{\Omega_2} \backslash \Omega_1\right)$.

Let $E = \mathscr{C}([0,1]) \times \mathscr{C}([0,1])$, then E is a Banach space endowed with the norm $\|(u,v)\| = \|u\| + \|v\|$, where $\|u\| = \sup_{t \in [0,1]} |u(t)|$.

We define the cone $P \subset E$ by

$$P = \{(u,v) \in E : u(t) \geqslant 0, \; v(t) \geqslant 0, \; t \in [0,1]\}.$$

Let $\theta \in \left(0, \frac{1}{2}\right)$, and set $J_\theta = [\theta, 1-\theta]$. We consider the set $\Omega \subset P$ given by

$$\Omega = \left\{ (u,v) \in P : \min_{t \in J_\theta} u(t) \geqslant \gamma_\alpha \|u\|, \; \min_{t \in J_\theta} v(t) \geqslant \gamma_\beta \|v\| \right\}.$$

Due to the concavity of the function *minimum* it is easy to verify that Ω is nonempty closed convex set, so Ω is a cone of E. Let

$$\Omega_r = \{(u,v) \in \Omega : \|(u,v)\| \leqslant r\}, \quad \partial\Omega_r = \{(u,v) \in \Omega : \|(u,v)\| = r\}, \quad r > 0. \tag{2.7}$$

The space $\mathbb{R}^2$ is endowed with the norm $\|.\|_1$ given by $\|(u,v)\|_1 = |u| + |v|$ for any $(u,v) \in \mathbb{R}^2$. So, for each $r > 0$, we note $\overline{B_r} = \left\{(u,v) \in \mathbb{R}^2, \|(u,v)\|_1 \leqslant r\right\}$ and $\overline{B_r}^c = \left\{(u,v) \in \mathbb{R}^2, \|(u,v)\|_1 > r\right\}$.

Next, we define the operator $T : E \longrightarrow E$ as follows

$$(T(u,v))(t) = (T_1(u,v), T_2(u,v))(t), \quad t \in [0,1], \tag{2.8}$$

where

$$T_1(u,v)(t) = \int_0^1 G_\alpha(t,s) a(s) f(s, u(s), v(s)) ds$$

and

$$T_2(u,v)(t) = \int_0^1 G_\beta(t,s) b(s) g(s,u(s),v(s)) ds.$$

In the sequel we need the following notations

$$\sigma_\alpha^\theta = \int_\theta^{1-\theta} a(t) K_\alpha(t) dt \tag{2.9}$$

and

$$\sigma_\beta^\theta = \int_\theta^{1-\theta} b(t) K_\beta(t) dt \tag{2.10}$$

where $\theta \in \left[0, \frac{1}{2}\right]$ and K_α, K_β are defined in Proposition 2.5 by (2.6).

In the remainder of the paper, we adopt the following hypotheses:

(H_1) $a,b \in \mathscr{C}((0,1),[0,+\infty))$, $a(t) \neq 0$ and $b(t) \neq 0$ on any subinterval of $(0,1)$ and $0 < \sigma_\alpha^0, \sigma_\beta^0 < +\infty$.

(H_2) $f,g \in \mathscr{C}([0,1] \times [0,+\infty) \times [0,+\infty),[0,+\infty))$.

(H_3) There exist $t_1,t_2 \in (0,1)$ such that $f(t_1,u,v) > 0$ and $g(t_2,u,v) > 0$ for each $u,v \in (0,+\infty)$.

REMARK 2.8. In the special case when the functions f and g are nondecreasing with respect to the second and the third variable on $(0,1)$, (H_3) can be replaced by

(H_3') There exist $t_1,t_2 \in (0,1)$ such that $f(t_1,0,0) > 0$ and $g(t_2,0,0) > 0$.

LEMMA 2.9. *Let $(H_1),(H_2)$ and (H_3) hold. Then $(u,v) \in \mathscr{C}([0,1]) \times \mathscr{C}([0,1])$ is a solution of the boundary value problem (P) if and only if $(u,v) \in \mathscr{C}([0,1]) \times \mathscr{C}([0,1])$ is a solution of the integral equations*

$$u(t) = \int_0^1 G_\alpha(t,s) a(s) f(s,u(s),v(s)) ds, \tag{2.11}$$

$$v(t) = \int_0^1 G_\beta(t,s) b(s) g(s,u(s),v(s)) ds. \tag{2.12}$$

That is (u,v) is a fixed point of the operator T defined by (2.8).

Proof. The proof is immediate from Lemma 2.4, so we omit it. □

We call $G(t,s) = \left(G_\alpha(t,s), G_\beta(t,s)\right)$ the Green's function of the problem (P).

Now, we state the following lemma which will be used in the proofs of our main results.

LEMMA 2.10. *Let (H_1) and (H_2) hold. Then $T : \Omega \longrightarrow \Omega$ is completely continuous.*

Proof. Since the functions G_α, G_β, f, g are nonnegative and continuous and using (H_1), we conclude that $T : P \longrightarrow P$ is continuous.

Let $(u,v) \in \Omega$, then by Proposition 2.5 (ii) and (iii), we have for all $t \in J_\theta$

$$\begin{aligned} T_1(u,v)(t) &\geqslant \gamma_\alpha \int_0^1 K_\alpha(s)a(s)f(s,u(s),\ v(s))ds \\ &\geqslant \gamma_\alpha \max_{\tau\in[0,1]} \int_0^1 G_\alpha(\tau,s)a(s)f(s,u(s),\ v(s))ds \\ &= \gamma_\alpha \|T_1(u,v)\|. \end{aligned}$$

Similarly, we have for all $t \in J_\theta$

$$T_2(u,v)(t) \geqslant \gamma_\beta \|T_2(u,v)\|.$$

Therefore, $T(\Omega) \subset \Omega$.

Now, let S be a bounded set of Ω, then there exists a positive constant $M > 0$ such that $\|(u,v)\| \leqslant M$, for all $(u,v) \in S$.

Let

$$M_1 := \max_{t\in[0,1],(u,v)\in\overline{B_M}} f(t,u,v)$$

and

$$M_2 := \max_{t\in[0,1],(u,v)\in\overline{B_M}} g(t,u,v)$$

From the hypothesis (H_1) and Proposition 1.5 (ii), we have for all $t \in [0,1]$ and $u,v \in S$

$$T_1(u,v)(t) \leqslant \int_0^1 K_\alpha(s)a(s)f(s,u(s),v(s))ds \leqslant M_1\sigma_\alpha^0.$$

Similarly,

$$T_2(u,v)(t) \leqslant \int_0^1 K_\beta(s)b(s)g(s,u(s),v(s))ds \leqslant M_2\sigma_\beta^0.$$

So

$$\|T(u,v)\| \leqslant M_1\sigma_\alpha^0 + M_2\sigma_\beta^0.$$

Hence, $T(S)$ is uniformly bounded.

Now, let us prove that $T(S)$ is equicontinuous on $[0,1]$. Using Proposition 2.5 (i), we deduce that the function G_α is uniformly continuous on $[0,1]\times[0,1]$. Thus, for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for any $t_1, t_2 \in [0,1]$ satisfying $|t_2 - t_1| < \delta$ and for all $s \in [0,1]$,

$$|G_\alpha(t_2,s) - G_\alpha(t_1,s)| < \frac{\varepsilon}{M_1\sigma_\alpha^0 2}.$$

Then, for $u,v \in S$, we have

$$|T_1(u,v)(t_2) - T_1(u,v)(t_1)| < \frac{\varepsilon}{2}. \tag{2.13}$$

Similarly,

$$\left|T_2(u,v)(t_2) - T_2(u,v)(t_1)\right| < \frac{\varepsilon}{2}. \tag{2.14}$$

Therefore, by (2.13) and (2.14) we obtain

$$d\left(T(u,v)(t_2), T(u,v)(t_1)\right) < \varepsilon,$$

where d is the distance associated to the norm $\|\,.\,\|_1$ on $\mathbb{R}^2$.

Thus, $T(S)$ is equicontinuous. So, by Ascoli's theorem $T(S)$ is relatively compact in E. So $T : \Omega \longrightarrow \Omega$ is completely continuous. This completes the proof. $\square$

Hereinafter, we use the following notations

$$f^{\delta} = \limsup_{\|(u,v)\| \longrightarrow \delta} \left(\max_{t\in[0,1]} \frac{f(t,u,v)}{u+v} \right) \tag{2.15}$$

and

$$f_{\delta} = \liminf_{\|(u,v)\| \longrightarrow \delta} \left(\min_{t\in[0,1]} \frac{f(t,u,v)}{u+v} \right), \tag{2.16}$$

where $\delta = 0$ or $+\infty$.

3. Existence of positive solution

In this section, we set $\gamma = \min\left(\gamma_\alpha, \gamma_\beta\right)$ and we recall that the constants σ_α^θ and σ_β^θ are respectively defined by (2.9) and (2.10).

THEOREM 3.1. *Suppose that conditions $(H_1), (H_2)$ and (H_3) hold. In addition, suppose that one of the following conditions is satisfied*

(A_1) (i) $f_0 > \dfrac{1}{\gamma\gamma_\alpha\sigma_\alpha^\theta}$ *and* $g_0 > \dfrac{1}{\gamma\gamma_\beta\sigma_\beta^\theta}$.

(ii) $f^\infty < \dfrac{1}{2\sigma_\alpha^0}$ *and* $g^\infty < \dfrac{1}{2\sigma_\beta^0}$.

(A_2) *There exist two constants r, R with $0 < r \leqslant R$ such that the functions f and g are nondecreasing with respect to the second and the third variable on $[0,R]$ and satisfying for all $t \in [0,1]$*

(i) $f\left(t, \gamma_\alpha\|u\|, \gamma_\beta\|v\|\right) > \dfrac{r}{\gamma_\alpha\sigma_\alpha^\theta}$ *and* $g\left(t, \gamma_\alpha\|u\|, \gamma_\beta\|v\|\right) > \dfrac{r}{\gamma_\beta\sigma_\beta^\theta}$.

(ii) $f(t,R,R) < \dfrac{R}{2\sigma_\alpha^0}$ *and* $g(t,R,R) < \dfrac{R}{2\sigma_\beta^0}$.

Then problem (P) has at least one positive solution.

Proof. Let T be the operator defined by (2.8).

Case 1. Suppose that condition (A_1) holds, then we deduce by $(A_1)(i)$ that there exists $r_1 > 0$ such that

$$f(t,u,v) > (f_0 - \varepsilon_1)(u+v), \ \ t \in [0,1], \ \ (u,v) \in \overline{B_{r_1}}$$

for ε_1 chosen such that $\gamma\gamma_\alpha\sigma_\alpha^\theta(f_0 - \varepsilon_1) \geqslant 1$.

Let Ω_{r_1} be the set defined by (2.7) for $r = r_1$. Then, for all $t \in J_\theta$, $(u,v) \in \partial\Omega_{r_1}$, we have

$$\begin{aligned}
T_1(u,v)(t) &\geqslant \gamma_\alpha \int_0^1 K_\alpha(s)a(s)f(s,u(s),v(s))ds \\
&\geqslant \gamma_\alpha(f_0 - \varepsilon_1)\int_\theta^{1-\theta} K_\alpha(s)a(s)(u(s)+v(s))\,ds \\
&\geqslant \gamma_\alpha(f_0 - \varepsilon_1)\int_\theta^{1-\theta} K_\alpha(s)a(s)\left(\gamma_\alpha\|u\| + \gamma_\beta\|v\|\right)ds \\
&\geqslant \gamma\gamma_\alpha(f_0 - \varepsilon_1)\|(u,v)\|\int_\theta^{1-\theta} K_\alpha(s)a(s)ds \\
&\geqslant \|(u,v)\|.
\end{aligned}$$

So

$$\|T_1(u,v)\| \geqslant \min_{t\in J_\theta} T_1(u,v)(t) \geqslant \|(u,v)\|. \tag{3.1}$$

Similarly, we prove

$$\|T_2(u,v)\| \geqslant \|(u,v)\|. \tag{3.2}$$

Hence, from (3.1) and (3.2), we conclude that for each $(u,v) \in \Omega \cap \partial\Omega_{r_1}$

$$\|T(u,v)\| \geqslant 2\|(u,v)\| \geqslant \|(u,v)\|. \tag{3.3}$$

Now, considering $(A_1)(ii)$, there exists $R_1 > 0$ such that

$$f(t,u,v) < (f^\infty + \varepsilon_2)(u+v), \ \ t \in [0,1], \ \ (u,v) \in \overline{B_{R_1}}^c$$

where ε_2 satisfies $2\sigma_\alpha^0(f^\infty + \varepsilon_2) < 1$.

Let

$$M := \max_{t\in[0,1],(u,v)\in\overline{B_{R_1}}} f(t,u,v).$$

Then for each $t \in [0,1]$ and $(u,v) \in \mathbb{R}^2$, we have

$$f(t,u,v) \leqslant M + (f^\infty + \varepsilon_2)(u+v).$$

Choose $r_2 > \max\left\{2r_1, R_1, M\sigma_\alpha^0\left(\frac{1}{2} - \sigma_\alpha^0\left(f^\infty + \varepsilon_2\right)\right)^{-1}\right\}$, then for $(u,v) \in \partial\Omega_{r_2}$ and for all $t \in [0,1]$ we have

$$\begin{aligned}
T_1(u,v)(t) &\leqslant \int_0^1 K_\alpha(s)a(s)f(s,u(s),v(s))ds \\
&\leqslant \int_0^1 K_\alpha(s)a(s)\left(M + \left(f^\infty + \varepsilon_2\right)\left(u(s)+v(s)\right)\right)ds \\
&= M\sigma_\alpha^0 + \left(f^\infty + \varepsilon_2\right)\int_0^1 K_\alpha(s)a(s)\left(u(s)+v(s)\right)ds \\
&\leqslant \frac{r_2}{2} - r_2\sigma_\alpha^0\left(f^\infty + \varepsilon_2\right) + \left(f^\infty + \varepsilon_2\right)\int_0^1 K_\alpha(s)a(s)ds\,\|(u,v)\| \\
&\leqslant \frac{1}{2}\|(u,v)\|. \qquad (3.4)
\end{aligned}$$

Analogously, we prove

$$\|T_2(u,v)\| \leqslant \frac{1}{2}\|(u,v)\|. \qquad (3.5)$$

Hence, from (3.4) and (3.5), we conclude that for each $(u,v) \in \Omega \cap \partial\Omega_{r_2}$

$$\|T(u,v)\| \leqslant \|(u,v)\|. \qquad (3.6)$$

Thus by using Lemme 2.9 and applying Lemma 2.7 to (3.3) and (3.6), we conclude that the boundary value problem (P) has a nonnegative solution $(u,v) \in \Omega$ with $r_1 \leqslant \|(u,v)\| \leqslant r_2$.

Case 2. Suppose that condition (A_2) is satisfied and let $(u,v) \in \Omega \cap \partial\Omega_r$, then we have for all $t \in J_\theta$

$$u(t) \geqslant \gamma_\alpha \|u\| \text{ and } v(t) \geqslant \gamma_\beta \|v\|.$$

Therefore, by using $(A_2)(i)$, we obtain for all $t \in J_\theta$ and $(u,v) \in \partial\Omega_r$

$$\begin{aligned}
T_1(u,v)(t) &\geqslant \gamma_\alpha \int_0^1 K_\alpha(s)a(s)f(s,u(s),v(s))ds \\
&\geqslant \gamma_\alpha \int_\theta^{1-\theta} K_\alpha(s)a(s)f\left(t,\gamma_\alpha\|u\|,\gamma_\beta\|v\|\right)ds \\
&\geqslant \gamma_\alpha \frac{r}{\gamma_\alpha \sigma_\alpha^\theta}\int_\theta^{1-\theta} K_\alpha(s)a(s)ds \\
&= r = \|(u,v)\|
\end{aligned}$$

So

$$\|T_1(u,v)\| \geqslant \|(u,v)\|. \qquad (3.7)$$

By the same manner, we obtain

$$\|T_2(u,v)\| \geqslant \|(u,v)\|. \qquad (3.8)$$

Hence, from (3.7) and (3.8), we conclude that for each $(u,v) \in \Omega \cap \partial\Omega_r$

$$\|T(u,v)\| \geqslant \|(u,v)\|. \tag{3.9}$$

Next, using the monotonicity of the function f, we have for all $t \in [0,1]$ and for $(u,v) \in \partial\Omega_R$,

$$\begin{aligned} T_1(u,v)(t) &\leqslant \int_0^1 K_\alpha(s)a(s)f(s,u(s),v(s))ds \\ &\leqslant \int_0^1 K_\alpha(s)a(s)f(s,R,R)ds \\ &\leqslant \frac{R}{2\sigma_\alpha^0}\int_0^1 K_\alpha(s)a(s)ds \\ &= \frac{R}{2} = \frac{1}{2}\|(u,v)\|. \end{aligned} \tag{3.10}$$

Similarly, we prove

$$\|T_2(u,v)\| \leqslant \frac{1}{2}\|(u,v)\|. \tag{3.11}$$

Hence, from (3.10) and (3.11), we conclude that for each $(u,v) \in \Omega \cap \partial\Omega_R$

$$\|T(u,v)\| \leqslant \|(u,v)\|. \tag{3.12}$$

Thus by applying Lemma 2.7 to (3.9) and (3.12), we deduce that the boundary value problem (P) has a nonnegative solution $(u,v) \in \Omega$ with $r \leqslant \|(u,v)\| \leqslant R$.

Now, let us prove that (u,v) is a positive solution for the the problem (P) that is $u(t) > 0$ and $v(t) > 0$ for each $t \in (0,1)$.

In the contrary case we can find $t^* \in (0,1)$ such that $u(t^*) = 0$ or $v(t^*) = 0$. We will suppose that $u(t^*) = 0$. Since $u(t) \geqslant 0$, we have

$$u(t^*) = 0 = \int_0^1 G_\alpha(t^*,s)a(s)f(s,u(s),v(s))ds \geqslant 0.$$

Since the functions G_α, a and f are nonnegative and continuous, we obtain

$$G_\alpha(t^*,s)a(s)f(s,u(s),v(s)) = 0 \quad a.e.(s).$$

According the assumption (H_1) and the fact that G_α is positive on $(0,1)$, we deduce that

$$f(s,u(s),v(s)) = 0 \quad a.e.(s).$$

Now, by the hypothesis (H_3) and the continuity of the function f we deduce that there exists a subset $K \subset (0,1)$ with $\mu(K) > 0$ where μ is the Lebesgue measure on $[0,1]$ such that $f(t,u(t), v(t)) > 0$ on K and this is a contradiction. This ends the proof. $\square$

EXAMPLE 3.2. Consider the system of nonlinear differential equations

$$\begin{cases} D^{\frac{5}{2}}u(t)+\frac{1}{\sqrt{t}}\left(1+t+u^{\frac{1}{3}}v^{\frac{1}{2}}\right)=0 \text{ in } (0,1), \\ D^{\frac{5}{2}}v(t)+\frac{1}{\sqrt{t}}(t+u^{\frac{1}{3}}v^{\frac{1}{3}})=0 \text{ in } (0,1), \\ u(0)=0,\ u(1)=0,\ u'(0)=0, \\ v(0)=0,\ v(1)=0,\ v'(0)=0. \end{cases} \tag{3.13}$$

Then $\alpha=\beta=\frac{5}{2}$, $a(t)=b(t)=\frac{1}{\sqrt{t}}$, $f(t,u,v)=\left(1+t+u^{\frac{1}{3}}v^{\frac{1}{2}}\right)$, $g(t,u,v)=(t+u^{\frac{1}{3}}v^{\frac{1}{3}})$. It is easy to verify that the conditions (H_1), (H_2) and (H_3) hold. In addition, we have $f^\infty=\limsup\limits_{\|(u,v)\|\longrightarrow+\infty}\max\limits_{t\in[0,1]}\frac{f(t,u,v)}{u+v}=\lim\limits_{\|(u,v)\|\longrightarrow+\infty}\frac{2+u^{\frac{1}{3}}v^{\frac{1}{2}}}{u+v}=0$ and $g^\infty=0$. On the other hand, we have $f_0=\liminf\limits_{\|(u,v)\|\longrightarrow 0}\min\limits_{t\in[0,1]}\frac{f(t,u,v)}{u+v}=\lim\limits_{\|(u,v)\|\longrightarrow 0}\frac{1+u^{\frac{1}{3}}v^{\frac{1}{2}}}{u+v}=+\infty$ and $g_0=+\infty$. Therefore the hypothesis (A_1) is satisfied. Theorem 3.1 yields that the problem (3.13) admits a positive solution.

THEOREM 3.3. *Suppose that conditions (H_1), (H_2) and (H_3) hold. In addition, suppose that the following condition is satisfied:*

(A_3) (i) $f^0<\frac{1}{2\sigma_\alpha^0}$ *and* $g^0<\frac{1}{2\sigma_\beta^0}$.

(ii) $f_\infty>\frac{1}{\gamma\gamma_\alpha\sigma_\alpha^\theta}$ *and* $g_\infty>\frac{1}{\gamma\gamma_\beta\sigma_\beta^\theta}$.

Then problem (P) has at least one positive solution.

Proof. The proof is analogous to that of Theorem 3.1, so we omit it. □

EXAMPLE 3.4. Consider the system of nonlinear differential equations

$$\begin{cases} D^{\frac{5}{2}}u(t)+\frac{2}{\sqrt{t}}(u+v)(u+v+2-t)=0 \text{ in } (0,1), \\ D^{\frac{5}{2}}v(t)+\frac{1}{\sqrt{t}}(u+v)(u+v+1-t)=0 \text{ in } (0,1), \\ u(0)=0,\ u(1)=0,\ u'(0)=0, \\ v(0)=0,\ v(1)=0,\ v'(0)=0. \end{cases} \tag{3.14}$$

Then $\alpha=\beta=\frac{5}{2}$, $a(t)=\frac{2}{\sqrt{t}}$, $b(t)=\frac{1}{\sqrt{t}}$, $f(t,u,v)=(u+v)(u+v+2-t)$, $g(t,u,v)=(u+v)(u+v+1-t)$. It is easy to verify that the conditions (H_1), (H_2) and (H_3) hold. In addition, we have $f^0=\limsup\limits_{\|(u,v)\|\to 0}\max\limits_{t\in[0,1]}\frac{f(t,u,v)}{u+v}=\lim\limits_{\|(u,v)\|\to 0}(u+v+2)=2$ and $g^0=1$. Now, by a simple computation, we obtain $\sigma_\alpha^0\approx 0.20962$ and $\sigma_\beta^0\approx 0.10481$.

So we have $2\sigma_\alpha^0 f^0 \approx 0.83848 < 1$ and $2\sigma_\beta^0 g^0 \approx 0.94472 < 1$, that is $(A_3)\,(i)$ is verified. Next, we have $f_\infty = \liminf\limits_{\|(u,v)\| \longrightarrow +\infty} \min\limits_{t\in[0,1]} \dfrac{f(t,u,v)}{u+v} = \lim\limits_{\|(u,v)\| \longrightarrow +\infty} (u+v+1) = +\infty$ and $g_\infty = +\infty$. Therefore hypothesis (A_3) is satisfied. Then by Theorem 3.3 the problem (3.14) admits a positive solution.

4. Existence of multiple positive solutions

Now, we discuss the multiplicity of positive solutions for system (P).

THEOREM 4.1. *Assume (H_1), (H_2) and (H_3). Suppose that the following assumptions hold*

(A_4) (i) $f_0 > \dfrac{1}{\gamma\gamma_\alpha \sigma_\alpha^\theta}$ *and* $g_0 > \dfrac{1}{\gamma\gamma_\beta \sigma_\beta^\theta}$.

(ii) $f_\infty > \dfrac{1}{\gamma\gamma_\alpha \sigma_\alpha^\theta}$ *and* $g_\infty > \dfrac{1}{\gamma\gamma_\beta \sigma_\beta^\theta}$.

(A_5) *There exists $r > 0$ such that* $\max_{t\in[0,1],(u,v)\in\overline{B_r}} f(t,u,v) < \dfrac{r}{2\sigma_\alpha^0}$ *and* $\max_{t\in[0,1],(u,v)\,\in\overline{B_r}} g(t,u,v) < \dfrac{r}{2\sigma_\beta^0}$.

Then problem (P) has at least two positive solutions (u_1,v_1) and (u_2,v_2) which satisfy

$$0 < \|(u_1,v_1)\| < r < \|(u_2,v_2)\|.$$

Proof. By $(A_4)\,(i)$, there exists $0 < r_1 < r$ such that for each $t \in [0,1]$ and $(u,v) \in \overline{B_{r_1}}$,

$$f(t,u,v) > (f_0 - \varepsilon_1)(u+v),$$

where ε_1 is chosen such that $\gamma\gamma_\alpha\sigma_\alpha^\theta (f_0 - \varepsilon_1) \geqslant 1$. So using the same manner to prove (3.3), we obtain

$$\|T(u,v)\| \geqslant \|(u,v)\|, \quad (u,v) \in \partial\Omega_{r_1}. \tag{4.1}$$

Now by $(A_4)\,(ii)$, there exists $r_2 > r$ such that for each $t \in [0,1]$ and $(u,v) \in \overline{B_{r_2}}^c$,

$$f(t,u,v) > (f_\infty - \varepsilon_2)(u+v),$$

where ε_2 satisfies $\gamma\gamma_\alpha\sigma_\alpha^\theta (f_\infty - \varepsilon_2) \geqslant 1$. So using the same manner to prove (3.3), we obtain

$$\|T(u,v)\| \geqslant \|(u,v)\|, \quad (u,v) \in \partial\Omega_{r_2}. \tag{4.2}$$

On the other hand, by hypothesis (A_5), we have for all $t \in [0,1]$ and $(u,v) \in \partial\Omega_r$

$$\begin{aligned} T_1(u,v)(t) &\leqslant \int_0^1 K_\alpha(s)a(s)f(s,u(s),v(s))ds \\ &< \frac{r}{2\sigma_\alpha}\int_0^1 K_\alpha(s)a(s)ds = \frac{1}{2}\|(u,v)\|. \end{aligned} \tag{4.3}$$

Similarly, we prove

$$\|T_2(u,v)\| < \frac{1}{2}\|(u,v)\|. \tag{4.4}$$

Hence, from (4.3) and (4.4), we conclude that for each $(u,v) \in \Omega \cap \partial\Omega_r$

$$\|T(u,v)\| < r = \|(u,v)\|. \tag{4.5}$$

Thus by applying Lemma 2.7 to (4.2) and (4.5), we deduce that (P) has a positive solution $(u_1,v_1) \in \Omega \cap \left(\overline{\Omega_r}\backslash\Omega_{r_1}\right)$ with $r_1 \leqslant \|(u_1,v_1)\| < r$, and by (4.1) and (4.5), we conclude that system (P) has a positive solution $(u_2,v_2) \in \Omega \cap \left(\overline{\Omega_{r_2}}\backslash\Omega_r\right)$ with $r < \|(u_2,v_2)\| \leqslant r_2$. Thus, the boundary value problem (P) admits two positive solutions satisfying

$$r_1 \leqslant \|(u_1,v_1)\| < r < \|(u_2,v_2)\| \leqslant r_2.$$

This ends the proof. $\square$

THEOREM 4.2. *Assume $(H_1),(H_2)$ and (H_3). Suppose that the following assumptions hold*

(A_6) (i) $f^0 < \dfrac{1}{2\sigma_\alpha^0}$ *and* $g^0 < \dfrac{1}{2\sigma_\beta^0}$.

(ii) $f^\infty < \dfrac{1}{2\sigma_\alpha^0}$ *and* $g^\infty < \dfrac{1}{2\sigma_\beta^0}$.

(A_7) *There exists $R > 0$ such that* $\max_{t\in[0,1],(u,v)\in\overline{B_R}} f(t,u,v) > \frac{R}{\gamma_\alpha \sigma_\alpha^\theta}$ *and* $\max_{t\in[0,1],(u,v)\in\partial B_R} g(t,u,v) > \frac{R}{\gamma_\beta \sigma_\beta^\theta}$.

Then problem (P) has at least two positive solutions (u_1,v_1) and (u_2,v_2) which satisfy.

$$0 < \|(u_1,v_1)\| < R < \|(u_2,v_2)\|.$$

5. Nonexistence of a positive solution

Our last results correspond to the case when system (P) has no positive solution.

THEOREM 5.1. *Assume that $(H_1),(H_2)$ and (H_3) hold and $f(t,u,v) < \dfrac{1}{2\sigma_\alpha^0}(u+v)$, $g(t,u,v) < \dfrac{1}{2\sigma_\beta^0}(u+v)$, for all $t \in [0,1]$, $u > 0$, $v > 0$. Then the boundary value problem (P) has no positive solution.*

Proof. Assume the contrary, that problem (P) admits a positive solution (u,v). Then, by Lemma 2.9, (u,v) satisfy (2.11) and (2.12). By Proposition 2.5 (ii) we have

$$\begin{aligned}\|u\| &= \max_{t\in[0,1]}\int_0^1 G_\alpha(t,s)a(s)f(s,u(s),v(s))ds\\ &\leqslant \int_0^1 K_\alpha(s)a(s)f(s,u(s),v(s))ds\\ &< \frac{1}{2\sigma_\alpha^0}\int_0^1 K_\alpha(s)a(s)(u+v)(s)\,ds\\ &\leqslant \frac{1}{2}\|(u,v)\|.\end{aligned}$$

Similarly, we obtain

$$\|v\| < \frac{1}{2}\|(u,v)\|.$$

So, $\|(u,v)\| = \|u\|+\|v\| < \|(u,v)\|$, which is a contradiction. The proof is complete. $\square$

EXAMPLE 5.2. The following boundary value problem

$$\begin{cases} D^{\frac{5}{2}}u+\frac{1}{\sqrt{t}}\frac{\left((u+v)^2+(u+v)\right)(2+\sin u+\cos v)}{1.2(u+v)+2}=0 \text{ in } (0,1),\\ D^{\frac{7}{3}}v+\frac{1}{\sqrt{t}}(u+v)\left(\frac{5}{2}+\sin(u+v)\right)=0 \text{ in } (0,1),\\ u(0)=0,\ u(1)=0,\ u'(0)=0,\\ v(0)=0,\ v(1)=0,\ v'(0)=0,\end{cases} \tag{5.1}$$

has no positive solution. In fact, set $\alpha=\frac{5}{2}$, $\beta=\frac{7}{3}$, $a(t)=b(t)=\frac{1}{\sqrt{t}}$, $f(t,u,v)=\frac{\left((u+v)^2+(u+v)\right)(2+\sin u+\cos v)}{1.2(u+v)+2}$ and $g(t,u,v)=(u+v)\left(\frac{5}{2}+\sin(u+v)\right)$. It is easy to verify that the conditions $(H_1),(H_2)$ and (H_3) hold. A simple calculation shows that $\sigma^0_{\frac{5}{2}}\approx 0.10481$ and $\sigma^0_{\frac{7}{3}}\approx 0.13946$. So we have $2\sigma^0_{\frac{5}{2}}f(t,u,v)<0.9957(u+v)$ and $2\sigma^0_{\frac{7}{3}}g(t,u,v)<0.97622(u+v)$. Thus Theorem 5.1 is valid.

THEOREM 5.3. *Assume that (H_1), (H_2) and (H_3) hold and $f(t,u,v)>\frac{1}{\gamma\gamma_\alpha\sigma_\alpha^\theta}(u+v)$, $g(t,u,v)>\frac{1}{\gamma\gamma_\beta\sigma_\beta^\theta}(u+v)$, for all $t\in[0,1]$, $u>0$, $v>0$. Then the boundary value problem (P) has no positive solution.*

Acknowledgements. The author thanks the referees for their valuable suggestions and comments.

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(Received October 8, 2014)

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